A descent theorem in type theory

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The descent property for model toposes (Rezk)

Let **M** be a model category, let **I** be a small category and $X : \mathbf{I} \to \mathbf{M}$ be a functor with $\bar{X} :\equiv \mathrm{hocolim}_{\mathbf{I}}(X)$.

- Given a morphism $f : A \to \bar{X}$ in **M** we can define $Y : \mathbf{I} \to \mathbf{M}$ by $Y(i) :\equiv X(i) \times^h_{\bar{X}} A$. Then the canonical morphism $\bar{Y} \to A$ is a weak equivalence.
- Given $Y : \mathbf{I} \to \mathbf{M}$ and a cartesian transformation $\alpha : Y \to X$, the canonical morphism $Y(i) \to X(i) \times^h_{\bar{X}} \bar{Y}$ is a weak equivalence.

Another way of looking at this is that there is an equivalence

“cartesian transformations to X” $\simeq$ “morphisms into $\bar{X}$”

The goal is to give a translation of the descent property into type theory.

- To simplify matters a bit: we take diagrams over graphs.
  - So this is just the first step towards a more general theorem. We get to see what's possible.
  - Graphs are not restricted in homotopy level the way categories (currently) are.
- We can directly give definitions of cartesian transformations and attempt a proof of a direct translation of the descent property. However, this is not as easy as it sounds.
- Instead, we make use of the virtues of *dependent* type theory
  - Functions into the colimit are equivalently described by families over the colimit.
  - Likewise, cartesian transformations of diagrams are equivalently described by certain families of diagrams. *We just have to figure out which ones!*

Overview of the rest of the talk

- First we give a bit of theory of graphs in type theory.
- Then we describe colimits as higher inductive types.
- And finally we head for the descent theorem.

Graphs and their families and morphisms

- A graph is a pair $\langle \Gamma_0, \Gamma_1 \rangle$ where

$$\Gamma_0 : \mathsf{Type} \qquad \text{the type of } \textit{vertices}$$
$$\Gamma_1 : \Gamma_0 \to \Gamma_0 \to \mathsf{Type} \qquad \text{the type of } \textit{edges}$$

- A family of graphs A over Γ is a pair $\langle A_0, A_1 \rangle$ where

$$A_0 : \Gamma_0 \to \mathsf{Type}$$
$$A_1 : \prod_{\{i,j:\Gamma_0\}} \prod_{(q:\Gamma_1(i,j))} A_0(i) \to A_0(j) \to \mathsf{Type}.$$

- A graph morphism $f : \Delta \to \Gamma$ is a pair $\langle f_0, f_1 \rangle$ where

$$f_0 : \Delta_0 \to \Gamma_0$$
$$f_1 : \prod_{\{u,v:\Delta_0\}} \Delta_1(u,v) \to \Gamma_1(f_0(u), f_0(v)).$$

Sums and fibers of graphs

- Suppose A is a family of graphs over Γ. Define the graph $\Sigma(\Gamma, A)$ by

$$\Sigma(\Gamma, A)_0 :\equiv \textstyle\sum_{(i:\Gamma_0)} A_0(i)$$
$$\Sigma(\Gamma, A)_1(\langle i, x\rangle, \langle j, y\rangle) :\equiv \textstyle\sum_{(q:\Gamma_1(i,j))} A_1(q, x, y).$$

  There's a graph morphism $\pi_1 : \Sigma(\Gamma, A) \to \Gamma$ for the projection.

- Suppose $f : \Delta \to \Gamma$ is a morphism of graphs. Define the family fib_f over Γ by

$$(\mathsf{fib}_f)_0(i) :\equiv \mathsf{fib}_{f_0}(i)$$
$$(\mathsf{fib}_f)_1(q, \langle u, \alpha\rangle, \langle v, \beta\rangle) :\equiv \mathsf{fib}_{f_1(u,v)}(\langle \alpha^{-1}, \beta^{-1}\rangle_*(q))$$

The object classifier of Graphs

- The object classifier for graphs U is defined by

$$\mathsf{U}_0 :\equiv \mathsf{Type}$$
$$\mathsf{U}_1(X, Y) :\equiv X \to Y \to \mathsf{Type}$$

- Therefore a family of graphs over Γ is exactly a morphism $\Gamma \to \mathsf{U}$ of graphs.

Theorem

For any graph Γ there is an equivalence

$$\left(\sum_{(\Delta:\mathsf{Graph})} \Delta \to \Gamma\right) \simeq \Gamma \to \mathsf{U}$$

Diagrams over graphs

Definition

A diagram D over Γ is a pair $\langle D_0, D_1\rangle$ where

$$\begin{aligned} D_0 &: \Gamma_0 \to \mathsf{Type} \\ D_1 &: \prod_{(i,j:\Gamma_0)} \prod_{(q:\Gamma_1(i,j))} D_0(i) \to D_0(j). \end{aligned}$$

As with families of graphs, we have the notion of a total space of a diagram: it is the graph of its elements.

Definition

When $D$is a diagram over Γ, define $\int_\Gamma D$ by

$$\begin{aligned} (\textstyle\int_\Gamma D)_0 &:\equiv \textstyle\sum_{(i:\Gamma_0)} D_0(i) \\ (\textstyle\int_\Gamma D)_1(\langle i,u\rangle,\langle j,v\rangle) &:\equiv \textstyle\sum_{(q:\Gamma_1(i,j))} D_1(q,u) = v. \end{aligned}$$

Families of diagrams and their sums

- If D is a diagram over Γ, a family E of diagrams over D is a diagram over $\int_\Gamma D$.
- Thus a family E of diagrams over D consists of

$$E_0 \colon \prod_{\{i:\Gamma_0\}} D_0(i) \to \mathsf{Type}$$
$$E_1(\langle q, p\rangle) \colon E_0(u) \to E_0(v)$$

  for every $\langle q, p\rangle : (\int_\Gamma D)_1(\langle i, u\rangle, \langle j, v\rangle)$.
- When E is a diagram over D, we define the diagram $\Sigma(D, E)$ over Γ by

$$(\Sigma(D, E))_0(i) : \sum_{(u:D_0(i))} E_0(u)$$
$$(\Sigma(D, E))_1(q, \langle u, z\rangle) : \langle D_1(q, u), E_1(q, u, D_1(q, u), \mathsf{refl}_{D_1(q,u)}, z)\rangle$$

A classifier for diagrams

Definition

We define the graph $\mathsf{U}^{\to}$ by

$$\mathsf{U}^{\to}_0 :\equiv \mathsf{Type}$$
$$\mathsf{U}^{\to}_1(X, Y) :\equiv X \to Y.$$

- So diagrams over Γ are precisely morphisms $\Gamma \to \mathsf{U}^{\to}$.
- A family of diagrams over D is a morphism $\int_\Gamma D \to \mathsf{U}^{\to}$.

Likewise we can define $\mathsf{U}^{\leftarrow}$ and $\mathsf{U}^{\simeq}$.

- U classifies all the graph morphisms.
- morphisms into $\mathsf{U}^{\to}$ correspond to left fibrations of graphs.
- morphisms into $\mathsf{U}^{\leftarrow}$ are the *contravariant* diagrams; $\mathsf{U}^{\leftarrow}$ classifies the right fibrations of graphs.
- $\mathsf{U}^{\simeq}$ classifies the cartesian transformations.

Fibrations

Definition

A left (edge) fibration of graphs is a morphism $f : \Delta \to \Gamma$ if for every $u : \Delta_0$, $j : \Gamma_0$ and $q : \Gamma_1(f_0(u), j)$, the type

$$\mathsf{leftLiftings}(u, j, q) :\equiv \sum_{(v:\Delta_0)} \sum_{(p:\Delta_1(u,v))} \sum_{(\beta:f_0(v)=j)} \beta_*(f_1(p)) = q$$

is contractible.

Theorem

Write $\mathsf{leftFib}(\Gamma) :\equiv \sum_{(\Delta:\mathsf{Graph})} \sum_{(f:\Delta\to\Gamma)} \mathsf{isLeftFib}(f)$. *Then there is an equivalence*

$$\mathsf{leftFib}(\Gamma) \simeq \Gamma \to \mathsf{U}^{\to}.$$

Equifibered families of diagrams

- For a diagram D over Γ, define the type of equifibered families over D to be

$$\mathsf{equiFib}(D) :\equiv \textstyle\int_\Gamma D \to \mathsf{U}^{\simeq}.$$

Theorem

A diagram $E : \mathsf{leftFib}(\int_\Gamma D)$ over $\int_\Gamma D$ is an equifibered precisely when the square

$$\begin{array}{ccc}
\sum_{(u:D_0(i))} E_0(u) & \longrightarrow & \sum_{(v:D_0(j))} E_0(v) \\
{\scriptstyle \mathsf{pr}_1}\downarrow & & \downarrow{\scriptstyle \mathsf{pr}_1} \\
D_0(i) & \xrightarrow[D_1(q)]{} & D_0(j)
\end{array}$$

is always a pullback.

Colimits as higher inductive types

Let D be a diagram over Γ. Define $\mathsf{colim}(D)$ to be the higher inductive type with basic constructors

$$\alpha_0 : \prod_{\{i:\Gamma_0\}} D_0(i) \to \mathsf{colim}(D)$$
$$\alpha_1 : \prod_{\{i,j:\Gamma_0\}} \prod_{(q:\Gamma_1(i,j))} \prod_{(u:D_0(i))} \alpha_0(D_1(q,u)) = \alpha_0(u)$$

The induction principle for $\mathsf{colim}(D)$ is that for any family $P : \mathsf{colim}(D) \to \mathsf{Type}$, if there are

$$A_0 : \prod_{\{i:\Gamma_0\}} \prod_{(u:D_0(i))} P(\alpha_0(x))$$
$$A_1 : \prod_{\{i,j:\Gamma_0\}} \prod_{(q:\Gamma_1(i,j))} \prod_{(u:D_0(i))} \alpha_1(q,u)_*(A_0(D_1(q,u))) = A_0(u)$$

then there is a section $f : \prod_{(x:\mathsf{colim}(D))} P(x)$ satisfying

$$f(\alpha_0(u)) \equiv A_0(u) \qquad \text{for each } u : D_0(i)$$
$$f(\alpha_1(q,u)) = A_1(q,u) \qquad \text{for each } q : \Gamma_1(i,j) \text{ and } u : D_0(i).$$

The universal property of colimits

Theorem

For any family $P : \mathrm{colim}(D) \to \mathrm{Type}$, *the type* $\prod_{(x:\mathrm{colim}(D))} P(x)$ *is equivalent to the type of pairs* $\langle A_0, A_1 \rangle$ *where*

$$A_0 : \prod_{\{i:\Gamma_0\}} \prod_{(u:D_0(i))} P(\alpha_0(x))$$
$$A_1 : \prod_{\{i,j:\Gamma_0\}} \prod_{(q:\Gamma_1(i,j))} \prod_{(u:D_0(i))} \alpha_1(q,u)_*(A_0(D_1(q,u))) = A_0(u).$$

This equivalence is given by $\lambda f.\, \langle f \circ \alpha_0, \mathrm{ap}_f \circ \alpha_1 \rangle$.

Corollary

For any type X, *the type* $\mathrm{colim}(D) \to X$ *is equivalent to the type of pairs* $\langle A_0, A_1 \rangle$ *where*

$$A_0 : \prod_{\{i:\Gamma_0\}} D_0(i) \to X$$
$$A_1 : \prod_{\{i,j:\Gamma_0\}} \prod_{(q:\Gamma_1(i,j))} \prod_{(u:D_0(i))} A_0(D_1(q,u)) = A_0(u).$$

This equivalence is given by $\lambda f.\, \langle f \circ \alpha_0, \mathrm{ap}_f \circ \alpha_1 \rangle$.

The descent theorem

Theorem

For any diagram D there is an equivalence

$$\mathsf{equiFib}(D) \simeq \mathsf{colim}(D) \to \mathsf{Type}.$$

Theorem

There is an equivalence

$$\mathsf{colim}(\Sigma(D,E)) \simeq \sum_{(w:\mathsf{colim}(D))} \mathsf{colim}(E)(w)$$