Some phenomenological homotopy types

§∅ Introduction and apology

• This is a fascinating conference and I want to thank the organizers for the chance to be here.

• I am a complete newbie and the learning curve here has been steep. My understanding of this material comes mostly from the ∞-category viewpoint, and I hope that will not work on your last nerve.

• In fact, this talk is probably best regarded as comic relief. There is a more detailed version at `arXiv:1202.0694`

§I: Homotopy types and moduli problems

John bf von Neumann modelled **ordinal** numbers as ordered sets built from $\emptyset$s and brackets $\{,\}$'s.

Gottlob **Frege** had previously defined the *cardinality* of a set S to be the class of all sets in bijective correspondence with S. Thus three apples and three oranges have the same cardinal **number**. Unfortunately Frege's cardinality is not a set.

Graeme **Segal** (circa 1966) observed that it is however a **category**, and that if eg

$$S = \{1, \ldots, n\} := \{\mathbf{n}\}$$

is **finite**, we expect that its classifying space

$$[\text{Frege}(\mathbf{n})] \cong B\Sigma_n$$

should be the classifying space of the symmetric group on n letters. This is a much richer and more complex object than v Neumann's number n. For example, its homology groups

$$H^*(B\Sigma_n, \mathbb{Z})$$

are torsion, nontrivial in arbitrarily high dimension, localized at primes $\leq n$, . . .

It is a (prototypical) homotopy type, classifying n-fold covering spaces. I would like to call it the topos of n-tuples but I'm not sure it is really a topos in the technical sense.

1.1 Thinking along these lines suggests regarding a (Euclidean, plane) **triangle** as defined by three unordered distinct points modulo plane congruence. The category of such things is thus the (topological) groupoid

$$[\Sigma_3 \backslash \mathrm{Config}^3(\mathbb{R}^2)/\mathrm{O}_+(2) \rtimes \mathbb{R}^2]$$

(where $\mathrm{Config}^n(X) = X^{\times n} - \mathbf{D}_X$); but I should exclude the locus

$$[\Sigma_3 \backslash \mathrm{Config}^3(\mathbb{R})/\mathbb{R}^2]$$

of degenerate cases (defined by three colinear points).

This is again an interesting homotopy type, with a rich structure, closely related to the braid group on three strands . . .

1.2 Classical **conic sections** provide another such example. This puts us clearly in the world of moduli stacks in algebraic geometry: the topological groupoid

$$\mathcal{E} := [\text{Upper Half Plane}/\mathrm{PSl}_2(\mathbb{Z})]$$

of **elliptic curves** defines yet another phenomenological homotopy type.

[This is usually described as a 2-diml orbifold, with one point having finite isotropy $\mathbb{Z}_2$ and one having isotropy $\mathbb{Z}_3$; but the cusp has isotropy $\mathbb{Z}$. In spite of its long history, it is not well-understood: I don't think a calculation of its (co)homology is anywhere in the literature.]

Time for a **Pop Quiz**: What is the classifying space for necklaces with n beads? [Alternately, cyclically ordered n-tuples.]

(Tentative answer: $B(\mathbb{Z}_n \rtimes \mathbb{Z}_n^\times)$? Note that this is much smaller than $B\Sigma_n$ . . .)

1.3 A terminological digression

I propose to call $\mathcal{E}$ the homotopy type **of**, or the classifying space **for**, elliptic curves.

In general, to describe some class $\mathcal{C}$ of phenomena (eg chrysanthemums), we can imagine a moduli stack $[\mathcal{C}]$ of such things, with a universal family of chrysanthemums over it, and try to interpret families parametrized by X in terms of maps $X \to [\mathcal{C}]$. [Early work by Plato *et al* in this area stalled for about 2500 years till Grothendieck introduced some new ideas.]

Alternately: I'm suggesting that the homotopy type $[\mathcal{C}]$ represents a functor of the form

$$X \mapsto \pi_0(\mathcal{C}/X)? \cong ?\pi_0 \mathrm{Maps}(X, [\mathcal{C}]) .$$

This suggests questions about representability theorems (eg of EH Brown, or more recently J Lurie) in HoTT. I suppose this must be tied up with issues surrounding the axiom of choice?

§II Stratifications and databases

2.1 Moduli problems from algebraic geometry (eg **conic sections**) show clearly that singularities are ubiquitous in interesting classifying problems: eg a conic section can be

• *open* or *closed*, eg a hyperbola or an ellipse; but

• hyperbolas can degenerate into parabolas, while ellipses can degenerate into circles,

• parabolas can degenerate into a pair of crossed lines, and then into just one line, and

• a circle can degenerate into a point, as can a line . . .

In general transformation groupoids $[X/G]$ can be stratified systematically in terms of **orbit types**, ie conjugacy classes of closed subgroups of G.

The orbit category $\mathcal{O}(G)$ has sets S_0 $(= G/H_0)$, S_1 $(= G/H_1)$ with transitive G-actions as objects, and

$$\text{Mor}_{\mathcal{O}}(S_0, S_1) := \pi_0 \text{Maps}_G(S_0, S_1)$$

as morphism sets. [If $G = A$ is abelian this is essentially just its lattice of subgroups.]

A G-space X defines a presheaf

$$\mathcal{O}(G) \ni G/H \mapsto \pi_0(X_H) \in (\text{Sets}) ,$$

(where $X_H := \{x \in X \text{ , iso(x)} \subset H\}$).

The main technical **point of this talk** is to propose the pullback

$$\begin{array}{ccc} \pi_0[X/G] & \dashrightarrow & (\mathrm{Sets})_* \\ \big\downarrow & & \big\downarrow \\ \mathcal{O}(G) & \longrightarrow & (\mathrm{Sets}) \end{array}$$

(a kind of Grothendieck integral) as a functor

$$[X/G] \to \pi_0[X/G]$$

from topological groupoids to (finitely presented, if we're lucky) small categories: more precisely, to **databases** in the sense of D Spivak, `arXiv 1009.1166`.

[I'm thinking of the resulting small categories as directed graphs, graded by dimension: with generic objects in codimension zero, and more singular (or exceptional, or specialized) objects as lying on strata of positive codimension.

For example, the circle group $\mathbb{T}$ acts on S^2 by rotation, with the north and south poles as fixed points. Its π_0, according to the prescription above, is the little category

$$\circ \longleftarrow \bullet \longrightarrow \circ$$

representing the degeneration of the generic orbit type (with trivial isotropy) at the poles (with isotropy $\mathbb{T}$.

The stratification of quotients is somehnow reminiscent of modal logic with values in a probability space, but filtration by codimension is much finer than that: every set of positive codimension in Euclidean space has zero measure. I think it is a familiar feeling, say in trying to prove some theorem, that the result is generically true, but an obstruction to a general proof lies in positive codimension. These incidence diagrams encode some of that kind of information.

The construction above is **not** homotopy invariant as presented here. A definition in terms of topological groupoids rather than global quotients would be more natural. Work of Gepner and Henriques `arxiv:0701916` (among others) gives me hope . . .

§**III** A few more examples:

- (the type of) Riemannian metrics on a compact manifold M

- Knots, following Vassiliev

- isolated singularities of smooth functions, following Arnol'd

- structurally stable orbits (with **slices**), following Thom

Some of these (eg knot tables) are manifestly databases, though interpreting them in the framework of transformation groupoids requires some translation. Arnol'd's tables of incidence relations among singularity types also look like a database in Spivak's sense.

• Physicists are interested in critical points of the functional

$$\text{Riemannian metrics on } M \ni g \mapsto \int R(g) d\text{vol}_g \in \mathbb{R} ;$$

its domain is contractible, but it is also invariant under an action of the group $\text{Diff}(M)$ of diffeomorphisms of M. The homotopy-to-geometric quotient map

$$B\text{Diff}(M) = E\text{Diff}(M) \times_{\text{Diff}} \text{Metrics} \to * \times_{\text{Diff}} \text{Metrics}$$

allows us to think of this as something like a Morse function on a smooth blowup of the physicists' configuration space for gravitation.

In fact their configuration space is not so bad: its isotropy groups are compact, so the group is close to the generalized Kac-Moody groups of Broto and Kitchloo . . .

• **Knot theorists** want to classify the components of the quotient space defined by the topological groupoid

$$[\text{Diff}(S^1)\backslash\text{Emb}(S^1,S^3)/\text{Diff}(S^3)]$$

of embeddings of the circle in the three-sphere.

Vassiliev observes that the corresponding space

$$\text{Imm}(S^1,S^3) = \bigcup_{k\geq 0} \text{VI}_k(S^1,S^3) \cup \infty\text{−codim stuff}$$

of immersions is filtered by subspaces of immersions with at most $k \geq 0$ double points: where for example VI_0 is the space of embeddings.

Incidence relations among (closures of) the orbits of the group action on the VI_* lead to the skein relations for things like the Jones and Alexander knot invariants. These incidence relations are not recorded in standard knot tables, but they are part of their implicit structure.

• Arnol'd's school studies isolated singularities of smooth functions

$$f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$$

(eg $n = 1$, $f(x) = x^{k+1}$, $k \geq 2$) under the action of the group $\mathrm{Diff}(\mathbb{C}^k, 0) \times \mathrm{Diff}(\mathbb{C}, 0)$:

More precisely (following Thom) they consider a certain family of Morse unfoldings (ie nice deformations of f) which define an equivariant neighborhood. In nice cases eg $f(x) = x^{k+1}$) the resulting groupoid

$$[\mathrm{Diff}(\mathbb{C}^n, 0)\backslash \mathrm{Morsifications(f)}/\mathrm{Diff}(\mathbb{C}, 0)]$$

has the homotopy type of an Eilenberg-MacLane $K(\pi, 1)$ space, with π the braid group on k strands: very roughly, a normalized deformation

$$x^{k+1} + \sum t_i x^{k-i} = \prod (x - \rho_i)$$

with distinct roots can be interpreted as a point $\{\rho_i\}$ in a configuration space. Morse deformations of more general singularities define analogous subspace arrangements associated with interesting Coxeter/Dynkin diagrams. . .

• Ideas of Rene Thom lie behind most of the preceding constructions: ie, that phenomena in Nature are candidates for classification only if they are **structurally stable** in a suitable sense: thus hurricanes have names, but clouds do not.

One possible formulation of this condition is the existence of **slice neighborhoods** for the orbits of a topological groupoid (cf B Noohi, tt arXiv:0710.2615); for the stack of Riemannian metrics this is an old (and deep) result of D Ebin. A database theory accounting for the orbit structure of some large class of topological stacks, as part of a theory of stratified homotopy types would unify a great range of interesting mathematics.

A personal note

Maybe I should end by thanking my wife, an anthropological linguist interested in the syntax and semantics of classification, for many conversations about these matters.

She tells me that Swahili has twenty-eight words for "it". The linguist E Sapir asserted that every language is a classification system for the world; this has been echoed by JL Borges, who reminds us that we unfortunately do not know what the world **is**. I have tried here to argue that homotopy theory, in its theoretical and practical forms, may be useful in sorting this all out.

THANKS for your attention, and patience!