A Categorical Description of Homotopy Type Theory

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Goals

Short term:

- To describe Hott conceptually.

Mid term:

- To explore new directions in Hott.

Long term:

- To contribute to the development of a large scale formalization of mathematics

Axiomatic Homotopy Theory

Henry Whitehead (1950):

The ultimate aim of algebraic homotopy is to construct a purely algebraic theory, which is equivalent to homotopy theory in the same sort of way that analytic is equivalent to pure projective geometry.

An axiomatization should be useful for thinking and computing.

The main axiomatisations of homotopy theory.:

- **Quillen**: *Homotopical algebra* (1967)
- **Lurie**: *Higher topos theory* (2008)

Other axiomatizations

- **Gabriel, Zisman**: *Calculus of fractions and homotopy theory* (1967)
- **Ken Brown**: *Abstract Homotopy Theory an Generalised Sheaf Cohomology* (1973)
- **Heller**: *Homotopy theories* (1988)
- **Baues**: *Algebraic homotopy* (1989).

Criteria

Baues (1989) suggests two criteria for an axiom system:

- *The axioms should be sufficiently strong to permit the basic constructions of homotopy theory*;
- *The axioms should be as weak (and as simple) as possible, so that the constructions of homotopy theory are available in as many contexts as possible*.

In the case of homotopy type theory, we may add the desiderata:

- *It should give a notion of elementary higher topos*.

The emergence of Homotopy Type Theory

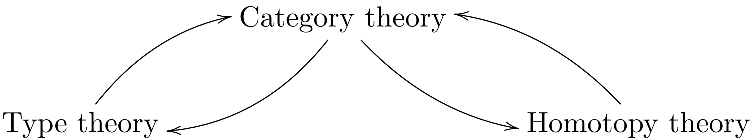
Type theory:

- **Martin-Löf**: *Intuitionistic Theory of Types* (1971, 1975, 1984)
- **Coqand, Huet**: *The Calculus of Constructions* (1988)

Homotopy Type Theory:

- **Hofmann, Streicher**: *The groupoid interpretation of type theory* (1995)
- **Awodey, Warren**: *Homotopy theoretic models of identity types* (2006∼2007)
- **Voevodsky**: *Notes on type systems* (2006∼2009)

Category theory as a bridge



There is no category theory in the book:

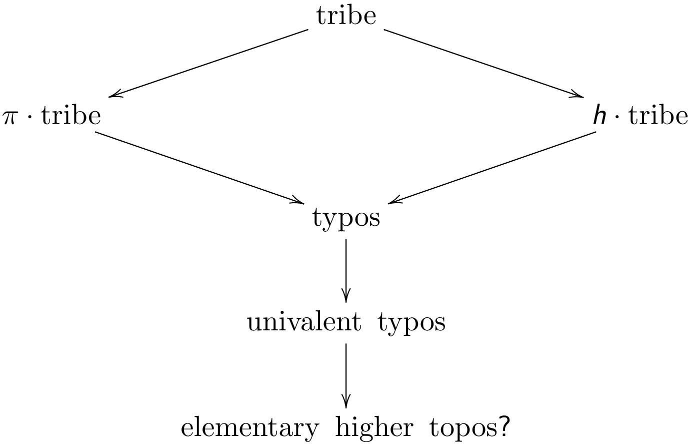
HOMOTOPY TYPE THEORY

Univalent Foundations of Mathematics

Category Theory and Type Theory

- **Lawvere**: *Equality in hyperdoctrines and comprehension schema as an adjoint functor* (1968)
- **Benabou**: *Des catégories fibrés* (1980)
- **Hyland, Johnstone and Pitts**: *Tripos theory* (1980)
- **Seely**: *Lccc and type theory* (1984)
- **Curien**: *Substitution up to isomorphism* (1990)
- **Streicher**: *Semantics of Type Theory* (1991)
- **Jacobs**: *Categorical Logic and Type Theory* (1999)
- **Curien, Garner, Hofmann**: *Revisiting the categorical interpretation of dependant type theory* (2013)

Overview of the talk



Squarable maps

We say that an object X in a category $\mathcal{C}$ is **squarable** if the cartesian product $A \times X$ exists for every object $A \in \mathcal{C}$.

We say that a map $p : X \to B$ is **squarable** if the object (X, p) of the category $\mathcal{C}/B$ is squarable. This means that the pullback square

$$\begin{array}{ccc} A \times_B X & \xrightarrow{p_2} & X \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{f} & B \end{array}$$

exists for every map $f : A \to B$. The projection $p_1 : A \times_B X \to B$ is the **base change** of the map $p : X \to B$ *along* $f : A \to B$.

Tribe

Definition

A **tribe** is a category $\mathcal{C}$ equipped with a class of maps $\mathcal{F} \subseteq \mathcal{C}$ satisfying the following conditions:

- $\mathcal{F}$ is closed under composition and contains the isomorphisms;
- every map in $\mathcal{F}$ is squarable and $\mathcal{F}$ is closed under base changes;
- $\mathcal{C}$ has a terminal object $\star$ and every map $X \to \star$ belongs to $\mathcal{F}$.

A map in $\mathcal{F}$ is a **fibration** or a **family** of the tribe.

Remarks on the notion of tribes

- Martin Hyland is using the notion in his work.
- Paige North is using the notion to construct models of type theory.
- For the relation between tribes and comprehension categories see a forthcoming paper of Lumsdaine and Warren: *An overlooked coherence construction for dependant type theory*.

Tribes abound

- A category with finite products is a tribe, if the fibrations are the projections;
- A category with finite limits is a tribe, if every map is a fibration;
- The category of small groupoids **Grpd** is a tribe if the fibrations are the Grothendieck fibrations;
- The category of Kan complexes **Kan** is a tribe, if the fibrations are the Kan fibrations.
- The syntactic category of type theory is a tribe, if the fibrations are the maps isomorphic to the display maps.

A **type** E is an object of a tribe $\mathcal{C}$. Notation:

$$\vdash E : type$$

A **term** of type E is a map $a : \star \to E$. Notation:

$$\vdash a : E$$

The **fiber** $E(a)$ of a fibration $p : E \to A$ at a point $a : A$ is defined by the pullback square

$$\begin{array}{ccc} E(a) & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ \star & \xrightarrow{\;a\;} & A. \end{array}$$

Dependant types and contexts

A fibration $p : E \to A$ should be regarded as an **internal family**

$$(E(x) : x \in A)$$

of objects parametrized by the codomain of p.

The family is called a **dependant type** $E(x)$ **in context** $x : A$,

$$x : A \vdash E(x) : type$$

A **term** $t(x)$ of type $E(x)$ in context $x : A$ is a section t of the fibration $p : E \to A$,

$$x : A \vdash t(x) : E(x)$$

The local category of $\mathcal{C}(A)$

The **local category** of $\mathcal{C}$ at $A \in \mathcal{C}$ is the full subcategory $\mathcal{C}(A)$ of $\mathcal{C}/A$ whose objects (X, p) are the fibrations $p : X \to A$ with codomain A.

An object of $\mathcal{C}(A)$ is a dependant types in context A.

The category $\mathcal{C}(A)$ is a tribe, where a map $f : (X, p) \to (Y, q)$ is a fibration if the underlying map $f : X \to Y$ in $\mathcal{C}$ is a fibration.

We have $\mathcal{C}(\star) = \mathcal{C}$.

Morphisms of tribes

Definition

A **morphism of tribes** $F : \mathcal{C} \to \mathcal{D}$ is a functor which

- takes fibrations to fibrations;
- preserves base changes of fibrations;
- preserves terminal objects.

The tribes form a 2-category in which a 2-cell is a natural transformation.

For example, the base change functor

$$f^\star : \mathcal{C}(B) \to \mathcal{C}(A)$$

is a morphism of tribes for any map $f : A \to B$ in a tribe $\mathcal{C}$.

For every fibration $Y \to B$ and every term $a : A$ we have

$$f^\star(Y)(a) = Y(f(a)).$$

Hence the functor $f^\star$ is a **change of parameters**.

Extension of context

In particular, the base change functor $i_A : \mathcal{C} \to \mathcal{C}(A)$ along the map $A \to \star$ is a morphism of tribes. By definition,
$i_A(X) = X_A = (A \times X, p_1)$.

The object $X_A = i_A(X) \in \mathcal{C}(A)$ is the constant family

$$x : A \vdash X : type$$

with value X.

The functor $i_A : \mathcal{C} \to \mathcal{C}(A)$ is expressed by the following deduction rule:

$$\frac{\vdash X : type}{x : A \vdash X : type.}$$

Push-forward and sum

To every fibration $f : A \to B$ in tribe $\mathcal{C}$ is associated a *summation* functor $\Sigma_f : \mathcal{C}(A) \to \mathcal{C}(B)$ defined by putting $\Sigma_f(X, p) = (X, fp)$.

$$\Sigma_f(X)(b) = \sum_{f(a)=b} X(a)$$

The functor Σ_f is left adjoint to the functor $f^\star$.

In particular, the forgetful functor $\mathcal{C}(A) \to \mathcal{C}$ is a summation functor,

$$\Sigma_A : \mathcal{C}(A) \to \mathcal{C}.$$

The domain of a fibration $p : X \to A$ is its *total space*,

$$X = \sum_{a:A} X(a).$$

Display maps

In type theory, the projection

$$\sum_{a:A} X(a) \to A$$

associated to a dependant type $a : A \vdash E(a) : \mathit{type}$ is called a **display map**.

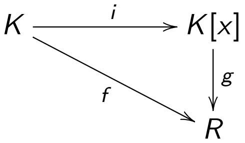
The display map of a dependant type $(X, p) \in \mathcal{C}(A)$ is the fibration $p : X \to A$.

Polynomial rings

Recall that a polynomial ring $K[x]$ is obtained by freely adding to a commutative ring K a new element x.

The freeness of the extension $i : K \to K[x]$ means that

for every homomorphism of commutative rings $f : K \to R$ and every element $r \in R$ there exists a unique homomorphism $g : K[x] \to R$ such that $gi = f$ and $g(x) = r$,

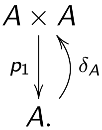


The element $x \in K[x]$ is **generic**.

Generic term

The base change functor $i : \mathcal{C} \to \mathcal{C}(A)$ is a morphism of tribes. Recall that $i(X) = X_A = (A \times X, p_1)$.

The diagonal δ_A is a term of type $i(A)$ in $\mathcal{C}(A)$.



Theorem

The extension $i : \mathcal{C} \to \mathcal{C}(A)$ is obtained by freely adding a term δ_A of type A to the tribe $\mathcal{C}$.

Thus, $\mathcal{C}(A) = \mathcal{C}[\delta_A]$. The term $\delta_A : i(A)$ is **generic**.

Variables=generic terms

For example, the terms x, y and z figuring in the context of a type declaration

$$x : A,\ y : B,\ z : C \vdash E(x, y, z) : \mathit{type}$$

are distinct and generic. This defines a fibration $E \to A \times B \times C$.

Equivalently, this defines an object

$$(E, p) \in \mathcal{C}(A \times B \times C) = \mathcal{C}[x, y, z]$$

Products along a map

Let $f : A \to B$ be a squarable map in a category $\mathcal{C}$ and let $E = (E, p) \in \mathcal{C}/A$.

We say that an object $\Pi_f(E) \in \mathcal{C}/B$ equipped with a map $\epsilon : \Pi_f(E) \times_B A \to E$ is the **product** of E along f,

$$\begin{array}{ccc} E & & \Pi_f(E) \\ \downarrow & & \downarrow \\ A & \xrightarrow{\ f\ } & B \end{array}$$

if the map $g \mapsto \epsilon(g \times_B A)$ is a bijection

$$Hom_B(X, \Pi_f(E)) \simeq Hom_A(f^\star(X), E)$$

for every object $X \in \mathcal{C}/B$.

The map $\epsilon : \Pi_f(E) \times_B A \to E$ is called the **evaluation**.

Π-tribes

Definition

We say that a tribe $\mathcal{C}$ is Π-**closed**, and that it is a Π-**tribe**, if every fibration $E \to A$ has a product along every fibration $f : A \to B$, and if the structure map $\Pi_f(E) \to B$ is a fibration,

$$\begin{array}{ccc} E & & \Pi_f(E) \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B. \end{array}$$

Formally,

$$\Pi_f(E)(b) = \prod_{f(a)=b} E(a)$$

for $b : B$.

Π-tribes abound

- A locally cartesian closed category, if every map is a fibration;
- A cartesian closed category, if the fibrations are the projections;
- (Hofmann-Streicher) The category of small groupoids **Grpd**, if the fibrations are the Grothendieck fibrations;
- (Streicher, Voevodsky) The category of Kan complexes **Kan** if the fibrations are the Kan fibrations;
- (Gambino-Garner) The syntactic category of type theory, if the fibrations are the maps isomorphic to display maps.

If $\mathcal{C}$ is a Π-tribe, then so is the tribe $\mathcal{C}(A)$ for every object $A \in \mathcal{C}$.

The base change functor $f^\star : \mathcal{C}(B) \to \mathcal{C}(A)$ has a right adjoint

$$\Pi_f : \mathcal{C}(A) \to \mathcal{C}(B)$$

for every fibration $f : A \to B$. Moreover, Π_f is a morphisms of tribes (it takes fibrations to fibrations).

In particular, the functor $i_A : \mathcal{C} \to \mathcal{C}(A)$ has a right adjoint

$$\Pi_A : \mathcal{C}(A) \to \mathcal{C}$$

for every objet A,

$$\Pi_A(X, p) = \prod_{a:A} X(a)$$

A Π-tribe $\mathcal{C}$ is cartesian closed:

$$[A, B] = \Pi_A(i_A(B))$$

where $i_A : \mathcal{C} \to \mathcal{C}(A)$ is the base change functor.

Theorem

A tribe $\mathcal{C} = (\mathcal{C}, \mathcal{F})$ is Π-closed if and only if the following two conditions are satisfied:

- *the category $\mathcal{C}(A)$ is cartesian closed for every object $A \in \mathcal{C}$ and the local exponential functor $[X, -]_A : \mathcal{C}(A) \to \mathcal{C}(A)$ preserves fibrations for every object $X \in \mathcal{C}(A)$;*
- *the base-change functor $f^\star : \mathcal{C}(B) \to \mathcal{C}(A)$ is cartesian closed for every map $f : A \to B$.*

Π-functors

Definition

A morphism of tribes $F : \mathcal{C} \to \mathcal{D}$ between Π-tribes is **Π-closed** if it preserves the products $\Pi_f(X)$.

The Π-tribes form a 2-category in which 1-cell is a Π-closed morphism and a 2-cell is a natural isomorphism.

Examples of Π-closed morphisms

The base change functor $f^\star : \mathcal{C}(B) \to \mathcal{C}(A)$ is a Π-closed morphism for any map $f : A \to B$ in a Π-tribe $\mathcal{C}$.

The Yoneda functor $y : \mathcal{C} \to \hat{\mathcal{C}} = [\mathcal{C}^{op}, Set]$ is a Π-closed morphism for any small Π-tribe $\mathcal{C}$.

Generic term

If A is an object of a Π-tribe $\mathcal{C}$, then the base change functor $i : \mathcal{C} \to \mathcal{C}(A)$ is a Π-closed morphism.

Theorem

The extension $i : \mathcal{C} \to \mathcal{C}(A)$ is obtained by freely adding a term x_A of type A to the Π-tribe $\mathcal{C}$.

Thus, $\mathcal{C}(A) = \mathcal{C}[x_A]$ in the 2-category of Π-tribes.

The term $x_A : i(A)$ is **generic**.

Generic map

Let A and B be two objects of Π-tribe $\mathcal{C}$ and let $i : \mathcal{C} \to \mathcal{C}([A, B])$ be the base change functor.

If $\epsilon : [A, B] \times A \to B$ is the evaluation, then the map

$$\langle p_1, \epsilon \rangle : [A, B] \times A \to [A, B] \times B$$

is a map $g : i(A) \to i(B)$.

Theorem

The extension $i : \mathcal{C} \to \mathcal{C}([A, B])$ is obtained by freely adding a map $g : A \to B$ to the Π-tribe $\mathcal{C}$.

Thus, $\mathcal{C}([A, B]) = \mathcal{C}[g]$ in the 2-category of Π-tribes.

The map $g : i(A) \to i(B)$ is **generic**

Polynomial functors

Let $\mathcal{C}$ be a Π-tribe.

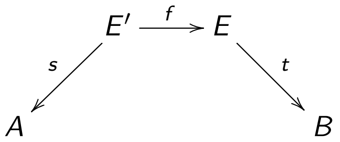
A *polynomial* $P = (s, f, t) : A \to B$ is a triple of maps

$$\begin{array}{ccccc} & E' & \xrightarrow{f} & E & \\ {\scriptstyle s}\swarrow & & & & \searrow{\scriptstyle t} \\ A & & & & B \end{array}$$

where f and t are fibrations.

The *polynomial functor* $P : \mathcal{C}(A) \to \mathcal{C}(B)$ is defined to be the composite

$$\begin{array}{ccccc} & \mathcal{C}(E') & \xrightarrow{\Pi_f} & \mathcal{C}(E) & \\ {\scriptstyle s^\star}\nearrow & & & & \searrow{\scriptstyle \Sigma_t} \\ \mathcal{C}(A) & & & & \mathcal{C}(B) \end{array}$$



We have

$$P(X)(b) = \sum_{t(k)=b} \prod_{f(e')=e} X(s(e'))$$

for every $X \in \mathcal{C}(A)$.

Polynomial functors are closed under composition.

A *polynomial monad* (P, μ, η) is a polynomial $P : A \to A$ equipped with a monad structure $\mu : P \circ P \to P$ and $\eta : I_A \to P$.

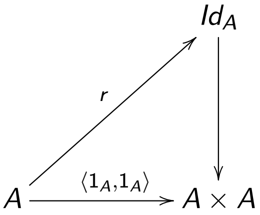
Martin Hyland has a *Dialectica interpretation* in type theory which is using polynomials.

Type theory and homotopical algebra

Awodey and Warren:

Martin-Löf type theory can be interpreted in a model category:

- types are interpreted as fibrant objects;
- display maps are interpreted as fibrations;
- the identity type $Id_A \to A \times A$ is a path object for A;
- the reflexivity term $r : A \to Id_A$ is an acyclic cofibration.



The relation $u \pitchfork f$

Recall that a map $u : A \to B$ in a category $\mathcal{C}$ is said to have the **left lifting property** with respect to a map $f : X \to Y$,

if every commutative square

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{b} & Y \end{array}$$

has a diagonal filler $d : B \to X$, $du = a$ and $fd = b$.

Notation: $u \pitchfork f$

For a class of maps $\mathcal{S} \subseteq \mathcal{C}$, let us put

$$^{\pitchfork}\mathcal{S} = \{u \in \mathcal{C} : \forall f \in \mathcal{S} \quad u \pitchfork f\}$$

Homotopical tribes

We say that a map in a tribe $\mathcal{C} = (\mathcal{C}, \mathcal{F})$ is **anodyne** if it belongs to the class ${}^{\pitchfork}\mathcal{F}$.

Definition

We say that a tribe $\mathcal{C}$ is **homotopical** and that it is a **h-tribe**, if the following two conditions are satisfied

- every map $f : A \to B$ admits a factorization $f = pu$ with u an anodyne map and p a fibration;
- the base change of an anodyne map along a fibration is anodyne.

Remark: The first condition implies the second when $\mathcal{C}$ is a Π-tribe.

If $\mathcal{C}$ is a h-tribe, then so is the tribe $\mathcal{C}(A)$ for every object $A \in \mathcal{C}$.

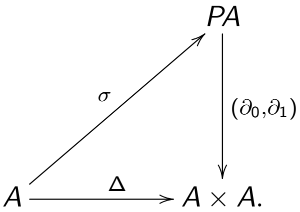
Examples of *h*-tribes

- (Hofmann-Steicher) The category of groupoids **Grpd**, if the fibrations are the Grothendieck fibrations; the anodyne maps are the monic categorical equivalences.
- (Awodey-Warren-Voevodsky) The category of Kan complexes **Kan**, if the fibrations are the Kan fibrations; the anodyne maps are the monic homotopy equivalences.
- (Gambino-Garner) The syntactic category of type theory, if a fibration is a map isomorphic to a display map.

Path object

Let $\mathcal{C}$ be a h-tribe.

A **path object** for an object $A \in \mathcal{C}$ is a factorisation of the diagonal $\Delta : A \to A \times A$ as an anodyne map $\sigma : A \to PA$ followed by a fibration $(\partial_0, \partial_1) : PA \to A \times A$,



Mapping path object and factorization

In a h-tribe, the factorization of a map $f : A \to B$ as an anodyne map $u : A \to E$ followed by a fibration $p : E \to B$ can be constructed from a path object $(PB, \sigma, \partial_0, \partial_1)$ for B.

By construction, the object E is the **mapping path object** $P(f)$ defined by the pullback square

$$\begin{array}{ccc} P(f) & \xrightarrow{p_2} & PB \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \partial_0} \\ A & \xrightarrow{f} & B. \end{array}$$

We have $u = \langle 1_A, \sigma f \rangle$ and $p = \partial_1 p_2$.

Identity type

In Martin-Löf type theory, there is a type constructor which associates to every type A a dependant type

$$x : A, y : A \ \vdash Id_A(x, y) : type$$

called the **identity type** of A together with a term

$$x : A \ \vdash r(x) : Id_A(x, x)$$

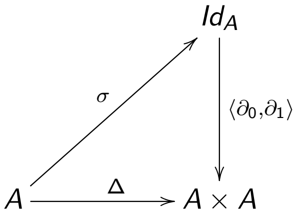
called the **reflexivity term**.

A term $p : Id_A(x, y)$ is regarded as a **proof** that $x = y$.

The reflexivity term $r(x) : Id_A(x, x)$ is the proof that $x = x$.

Identity type as a path object

Equivalently, for every $A \in \mathcal{C}$ there is a diagram



with $\langle \partial_0, \partial_1 \rangle$ the display map

$$\sum_{x:A, y:A} Id_A(x, y) \to A \times A$$

The J-rule

Awodey and Warren:

The reflexivity term $r : A \to Id_A$ is anodyne by the J-rule:

If $p : E \to Id_A$ is a fibration, then every commutative square

$$\begin{array}{ccc} A & \xrightarrow{u} & E \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle p} \\ Id_A & = & Id_A \end{array}$$

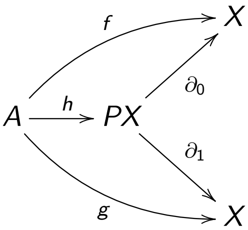
has a diagonal filler $d = J(u)$,

$$\begin{array}{ccc} A & \xrightarrow{u} & E \\ {\scriptstyle r}\downarrow & \overset{d}{\nearrow} & \downarrow{\scriptstyle p} \\ Id_A & = & Id_A \end{array}$$

Homotopy between two maps

Let $\mathcal{C}$ be a h-tribe.

A **homotopy** $h : f \rightsquigarrow g$ between two maps $f, g : A \to X$ in $\mathcal{C}$ is a map $h : A \to PX$ such that $\partial_0 h = f$ and $\partial_1 h = g$,



Type theorists regard h as a **proof** that $f = g$,

$$A \vdash h : Id_X(f, g).$$

Two maps $f, g : A \to X$ are **homotopic**, $f \sim g$, if there exists a homotopy $h : f \rightsquigarrow g$.

The homotopy category

Let $\mathcal{C}$ be a h-tribe.

Theorem

The homotopy relation $f \sim g$ is a congruence on the arrows of $\mathcal{C}$.

The **homotopy category** $Ho(\mathcal{C})$ is the quotient category $\mathcal{C}/\sim$.

A map $f : X \to Y$ in $\mathcal{C}$ is called a **homotopy equivalence** if it is invertible in $Ho(\mathcal{C})$.

An object X is **contractible** if the map $X \to \star$ is a homotopy equivalence.

Every anodyne map is a homotopy equivalence.

h-functors

Definition

We say that a morphism of tribes $F : \mathcal{C} \to \mathcal{D}$ between h-tribes is a h-**morphism** if it preserves the homotopy relation: $f \sim g \Rightarrow F(f) \sim F(g)$.

The h-tribes form a 2-category in which a 1-cell is h-morphism and a 2-cell is a natural transformation.

For example, the base change functor $u^\star : \mathcal{C}(B) \to \mathcal{C}(A)$ is a h-morphism for any map $u : A \to B$ in a h-tribe $\mathcal{C}$.

A h-morphism $F : \mathcal{C} \to \mathcal{D}$ induces a functor

$$Ho(F) : Ho(\mathcal{C}) \to Ho(\mathcal{D}).$$

Generic terms

If A is an object of a h-tribe $\mathcal{C}$, then the base change functor $i_A : \mathcal{C} \to \mathcal{C}(A)$ is a h-functor.

Theorem

The extension $i_A : \mathcal{C} \to \mathcal{C}(A)$ is obtained by freely adding a term x_A of type A to the h-tribe $\mathcal{C}$.

Thus, $\mathcal{C}(A) = \mathcal{C}[x_A]$ in the 2-category of h-tribes.

Weak equivalences

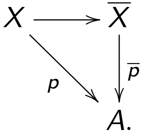
Let $\mathcal{C}$ be a h-tribe.

We say that a map $f : (X, p) \to (Y, q)$ in $\mathcal{C}/A$ is a **weak equivalence** if the underlying map $f : X \to Y$ in $\mathcal{C}$ is a homotopy equivalence.

For every object $(X, p) \in \mathcal{C}/A$ there is a weak equivalence

$$(X, p) \to (\overline{X}, \overline{p})$$

with codomain an object of $\mathcal{C}(A)$,



The object $(\overline{X}, \overline{p})$ is a **fibrant replacement** of (X, p).

Let $\mathcal{C}$ be a *h*-tribe.

Theorem

If $\mathcal{W}_A$ is the class of weak equivalences in $\mathcal{C}/A$, then the inclusion $\mathcal{C}(A) \to \mathcal{C}/A$ induces an equivalence of categories,

$$Ho(\mathcal{C}(A)) \to \mathcal{W}_A^{-1}(\mathcal{C}/A).$$

Theorem

The functor $Ho(f^\star) : Ho(\mathcal{C}(B)) \to Ho(\mathcal{C}(A))$ has a left adjoint

$$\tilde{\Sigma}_f : Ho(\mathcal{C}(A)) \to Ho(\mathcal{C}(B)).$$

for any map $f : A \to B$. Moreover, the functor $\tilde{\Sigma}_f$ is conservative and the adjunction $\tilde{\Sigma}_f \vdash Ho(f^\star)$ is an equivalence of categories if f is a homotopy equivalence.

Homotopy initial objects

Let $\mathcal{C}$ be a h-tribe.

An object $J \in \mathcal{C}$ is **homotopy initial** if every fibration $p : E \to J$ has a section $\sigma : J \to E$,

$$\begin{array}{c} E \\ p \downarrow \; \curvearrowright \sigma \\ J. \end{array}$$

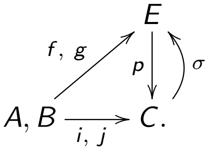
A homotopy initial object J is initial in the homotopy category $Ho(\mathcal{C})$: for every object X, there is a (homotopy unique) map $J \to X$,

$$\begin{array}{ccc} J \times X & \xrightarrow{p_2} & X \\ p_1 \downarrow \; \curvearrowright \sigma & & \\ J & & \end{array}$$

Homotopy coproducts

The **homotopy coproduct** of two objects A and B is an object C equipped with a pair of maps $i, j : A, B \to C$ which is homotopy initial in the category of pairs of maps $f, g : A, B \to E$.

The initially means that for every pair $f, g : A, B \to E$ and every fibration $p : E \to C$ such that $pf = i$ and $pg = j$, there exists a section $\sigma : C \to E$ such that $\sigma i = f$ and $\sigma j = g$,

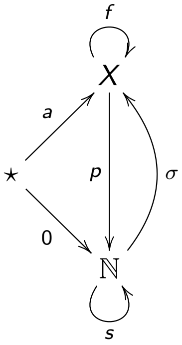


A homotopy coproduct $i, j : A, B \to C$ is a coproduct in the homotopy category $Ho(\mathcal{C})$.

Homotopy natural number object

A **homotopy natural number object** $(\mathbb{N}, s, 0)$ is homotopy initial in the category of triples (X, f, a), where f is an endomorphism of an object X and where $a : X$.

The initially means that if a fibration $p : X \to \mathbb{N}$ is a map of triples $(X, f, a) \to (\mathbb{N}, s, 0)$, then p has a section $\sigma : \mathbb{N} \to X$ which is a map of triples $(\mathbb{N}, s, 0) \to (X, f, a)$,



Inductive types

Recall that a *P-algebra* for an endo-functor $P : \mathcal{C} \to \mathcal{C}$ is a pair (X, α), where $X \in \mathcal{C}$ and $\alpha : P(X) \to X$ is a map in $\mathcal{C}$ (no condition on the action α is required).

Definition

An **inductive type*** is defined to be a homotopy initial object (W, w) in the category of P-algebras, where P is a polynomial endo-functor of $\mathcal{C}$.

The initially means that if a fibration $p : X \to W$ is a map of P-algebras $(X, \alpha) \to (W, w)$, then p has a section $\sigma : W \to X$ which is a map of P-algebras.

(*)(Awodey, Gambino and Sojakova):*Inductive types in homotopy type theory*.

n-types

Let $\mathcal{C}$ be a h-tribe.

Definition

We say that an object $A \in \mathcal{C}$ is a **proposition** if the diagonal $A \to A \times A$ is a homotopy equivalence.

More generally, the fibration $\langle \partial_0, \partial_1 \rangle : PA \to A \times A$ defines an object $P(A)$ of $\mathcal{C}(A \times A)$.

Definition

We say that A is

- a 0-**type** if $P(A)$ is a proposition;
- a $(n+1)$-**type** if $P(A)$ is a n-type.

Typos

Definition

We say that a h-tribe $\mathcal{C}$ is a **typos*** if it is a Π-tribe and the product functor $\Pi_f : \mathcal{C}(A) \to \mathcal{C}(B)$ preserves the homotopy relation for every fibration $f : A \to B$.

If $\mathcal{C}$ is a typos, then so is the tribe $\mathcal{C}(A)$ for any object $A \in \mathcal{C}$.

$(\star)$ Plural: **typoi**

Examples of typoi

Theorem

(**Hofmann and Streicher**) *The category of small groupoids is a typos if the fibrations are the Grothendieck fibrations.*

Theorem

(**Awodey-Warren-Voevodsky**) *The category of Kan complexes is a typos if the fibrations are the Kan fibrations.*

Theorem

(**Gambino-Garner**) *The syntactic category of type theory is a typos if the fibrations are the maps isomorphic to display maps.*

Theorem

*(**Lamarche**) The category of small categories* **Cat** *is a typos, where a fibration is a Grothendieck bifibration.*

Recall that the category of simplicial sheaves in a Grothendieck topos admits a Quillen model structure (the so-called Joyal model structure)* in which the weak equivalences are defined internally.

Theorem

The category of fibrant simplicial sheaves with respect to the Joyal model structure is a typos,

(*)*Letter to Grothendieck* (1984).

From typoi to hyperdoctrines

Let $\mathcal{C}$ be a typos.

If $f : A \to B$ is a map in a typos $\mathcal{C}$, then the functor

$$Ho(f^\star) : Ho(\mathcal{C}(B)) \to Ho(\mathcal{C}(A))$$

has both a left adjoint $\tilde{\Sigma}_f$ and a right adjoint $\tilde{\Pi}_f$.

The functor $A \mapsto Ho(\mathcal{C}(A))$ has the structure of a hyper-doctrine in the sense of Lawvere!

In particular, the homotopy category $Ho(\mathcal{C}(A))$ is cartesian closed for every object $A \in \mathcal{C}$.

Morphisms of typoi

Definition

A **morphism of typoi** $F : \mathcal{C} \to \mathcal{D}$ is a morphism of tribes which preserves

- the internal products $\Pi_f(X)$;
- the homotopy relation.

The typoi form a 2-category in which a 2-cell is a natural isomorphism.

For example, the base change functor $f^\star : \mathcal{C}(B) \to \mathcal{C}(A)$ is a morphism of typoi for any map $f : A \to B$ in a typos $\mathcal{C}$.

Homotopical presheaves

Let $\mathcal{C}$ be a typos.

We call a presheaf $F : \mathcal{C}^{op} \to Set$ **homotopical** if it respects the homotopy relation: $f \sim g \Rightarrow F(f) = F(g)$.

A homotopical presheaf is the same thing as a presheaf $F : Ho(\mathcal{C})^{op} \to Set$.

We say that a homotopical presheaf F is **homotopically representable** if the functor $F : Ho(\mathcal{C})^{op} \to Set$ is representable.

Cribles

Let $\mathcal{C}$ be a typos. Recall that a **crible** $S \subseteq \mathcal{C}$ is a set S of objects of $\mathcal{C}$ such that the implication $A \in S \Rightarrow A' \in S$ is true for every morphism $A' \to A$ in $\mathcal{C}$.

A crible S defines a presheaf $F_S : \mathcal{C}^{op} \to Set$ if we put

$$F_S(A) = \begin{cases} 1 & \text{if } A \in S \\ \emptyset & \text{otherwise} \end{cases}$$

The presheaf F_S is homotopical. We say that S is **homotopically representable** if the resulting presheaf $F_S : Ho(\mathcal{C}) \to Set$ is homotopically representable.

Every object E representing F_S is a proposition (ie the map $E \to \star$ is monic in the homotopy category).

$IsContr(X)$

For every $X \in \mathcal{C}$, the set

$$S = \{A \in \mathcal{C} : X_A \text{ is contractible}\}$$

is a crible. The crible is homotopically representable by the object

$$IsContr(X) = \sum_{x:X} \prod_{y:X} Id_X(x, y)$$

If $X = (X, p) \in \mathcal{C}(B)$, then $IsContr(X) \in \mathcal{C}(B)$.

$IsProp(X)$

For every $X \in \mathcal{C}$, the set

$$S = \{A \in \mathcal{C} : X_A \text{ is } \text{ a proposition } \}$$

is a crible. The crible is homotopically representable by the object

$$IsProp(X) = \prod_{x:X} \prod_{y:X} Id_X(x, y)$$

If $X = (X, p) \in \mathcal{C}(B)$, then $IsProp(X) \in \mathcal{C}(B)$.

$Is(0, X)$

For every $X \in \mathcal{C}$, the set

$$S = \{A \in \mathcal{C} : X_A \text{ is a } 0\text{−object}\}$$

is a crible. The crible is homotopically representable by the object

$$Is(0, X) = \prod_{x:X} \prod_{y:X} IsProp(Id_X(x, y))$$

If $X = (X, p) \in \mathcal{C}(B)$, then $Is(0, X) \in \mathcal{C}(B)$.

$Is(n, X)$

By recursion on $n \geq 0$.

For every $X \in \mathcal{C}$, the set

$$S = \{A \in \mathcal{C} : X_A \text{ is a } (n+1)\text{−object}\}$$

is a crible. The crible is homotopically representable by the object

$$Is(n+1, X) = \prod_{x:X} \prod_{y:X} Is(n, (Id_X(x, y))$$

If $X = (X, p) \in \mathcal{C}(B)$, then $Is(n, X) \in \mathcal{C}(B)$.

$\mathit{IsEq}(f)$

If $f : X \to Y$ is a map in $\mathcal{C}$, then the set

$$S = \{A \in \mathcal{C} : \ f_A : X_A \to Y_A \text{ is an equivalence}\}$$

is a crible. The crible is homotopically representable by the object

$$\mathit{IsEq}(f) = \prod_X \mathit{IsContr}(\overline{X})$$

where $\overline{X} = (\overline{X}, \overline{f})$ is a fibrant replacement of $(X, f) \in \mathcal{C}(Y)$.

If $f : (X, p) \to (Y, q)$ is a map in $\mathcal{C}(B)$, then $\mathit{IsEq}(f) \in \mathcal{C}(B)$.

$Eq(X, Y)$

If X and Y are two objects of $\mathcal{C}$ and $i : \mathcal{C} \to \mathcal{C}([X, Y])$ is the base change functor, then we have a generic map $g : i(X) \to i(Y)$.

The presheaf $F : \mathcal{C}([X, Y])^{op} \to Set$ defined by putting

$$F(A) = \begin{cases} 1, & \text{if } g_A : i(X)_A \to i(Y)_A \text{ is an equivalence} \\ \emptyset & \text{otherwise} \end{cases}$$

is homotopical and representable by the object $Eq(X, Y) = IsEq(g) \in \mathcal{C}([X, Y])$.

Univalent fibration

For every fibration $X \to A$ there is a fibration

$$\langle s, t \rangle : Eq_A(X) \to A \times A,$$

which classifies the homotopy equivalences $X(a) \to X(b)$ between the fibers of $X \in \mathcal{C}(A)$. We have

$$Eq_A(X)(a, b) = Eq(X(a), X(b))$$

for $a : A$ and $b : A$.

By construction, $Eq_A(X) = Eq(p_1^\star(X), p_2^\star(X))$ in $\mathcal{C}(A \times A)$, where $p_i : A \times A \to A$ is a projection.

Univalent fibration

Definition

A fibration $X \to A$ is **univalent** if the unit map $u : A \to Eq_A(X)$ is a homotopy equivalence.

In which case the fibrations

$$\langle s, t \rangle : Eq_A(X) \to A \times A \qquad \langle \partial_0, \partial_1 \rangle : PA \to A \times A$$

are homotopically equivalent:

$$\begin{array}{ccc} A & \xrightarrow{\ u\ } & Eq_A(X) \\ {\scriptstyle \sigma}\downarrow & \nearrow & \downarrow{\scriptstyle \langle s,t \rangle} \\ PA & \xrightarrow[\langle \partial_0, \partial_1 \rangle]{} & A \times A \end{array}$$

Small fibrations and universe

If $\mathcal{C} = (\mathcal{C}, \mathcal{F})$ is a typos, we shall say that a class of maps $\mathcal{F}' \subseteq \mathcal{F}$ is a class of **small fibrations** if the pair $(\mathcal{C}, \mathcal{F}')$ is a typos.

A small fibration $q : U' \to U$ is **universal** if for every small fibration $p : X \to A$ there exists a cartesian square:

$$\begin{array}{ccc} X & \longrightarrow & U' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \longrightarrow & U. \end{array}$$

A **universe** is the codomain of a universal small fibration $U' \to U$.

Martin-Löf axiom: *There is a universe U.*

Univalent typos

Definition

We say that a typos equipped with a universe U is **univalent** if the universal fibration $U' \to U$ is univalent.

Theorem

*(**Voevodsky**) The category of Kan complexes* **Kan** *has the structure of a univalent typos in which the fibrations are the Kan fibrations.*

The theorem was recently extended by Shulman to the category of simplicial objects in the category of presheaves over an elegant Reedy category.

Exercise

Define the notion of higher inductive type in a univalent typos.

Hints: see the n-Lab.

Remark: the notion of higher inductive type should be simpler in a typos having an interval.

Loose ends

- Introduce a notion of interval in a typos (**Warren**)
- Introduce a notion of cofibration in a typos (**Lumsdaine**)
- Define a general notion of higher inductive types
- Define an elementary notion of higher topos (**Shulman**)

I thank you for your attention!