Critical Phenomena and Percolation Theory: III

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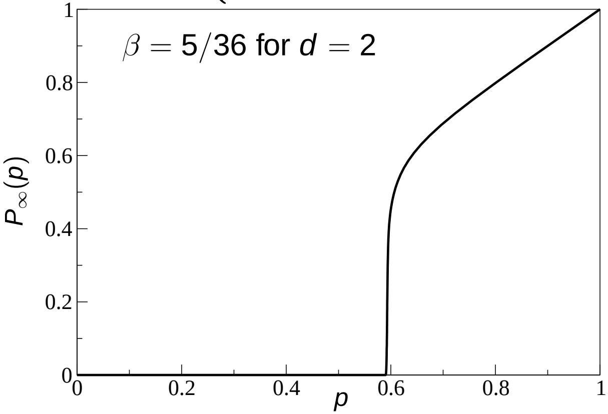
Outline

1. Critical exponents
   - Order parameter: β
   - Average cluster size: γ
   - Characteristic cluster size & cluster no. density: σ, τ & $\mathcal{G}$
2. Scaling relations
   - Average cluster size
   - Correlation length ξ
   - Mass of percolating cluster for $p > p_c$
3. Finite-size scaling
   - Average cluster size
   - Cluster number density
   - Finite-size scaling: Warning
4. Self-similarity, fixed points & the correlation length

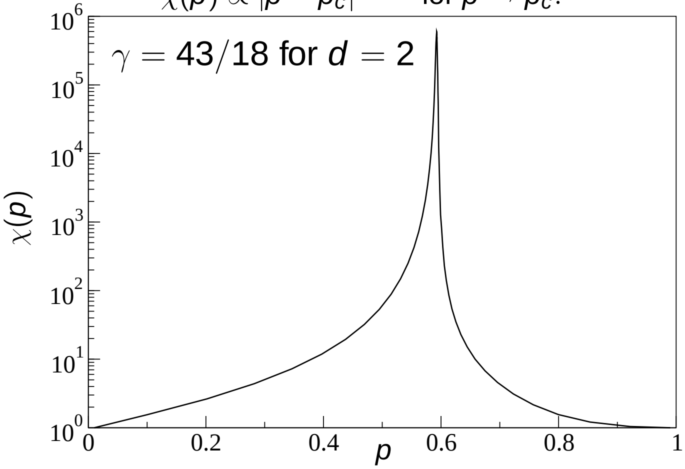
The critical exponent β characterises the abrupt pick-up of the order parameter

$$P_\infty(p) = \begin{cases} 0 & \text{for } p \leq p_c \\ (p - p_c)^\beta & \text{for } p \to p_c^+. \end{cases}$$



The critical exponent γ characterises the divergence of the average cluster size:

$$\chi(p) \propto |p - p_c|^{-\gamma} \quad \text{for } p \to p_c.$$



The critical exponent σ characterises the divergence of the characteristic cluster size:

$$s_\xi(p) \propto |p - p_c|^{-1/\sigma} \quad \text{for } p \to p_c.$$

The critical exponent τ and the scaling function $\mathcal{G}$ enter into the scaling ansatz of the cluster number density

$$n(s,p) \propto s^{-\tau} \mathcal{G}\left(s/s_\xi\right) \quad \text{for } p \to p_c, s \gg 1.$$

Generally, in dimensions greater that one:

$$n(s,p) \propto \begin{cases} s^{-\tau} & \text{for } 1 \ll s \ll s_\xi \\ \text{decays rapidly} & \text{for } s \gg s_\xi. \end{cases}$$

$$
\begin{aligned}
\chi(p) &= \frac{\sum_{s=1}^{\infty} s^2 n(s,p)}{\sum_{s=1}^{\infty} s n(s,p)} \propto \sum_{s=1}^{\infty} s^2 n(s,p) \\
&\approx \sum_{s=1}^{\infty} s^{2-\tau} \mathcal{G}(s/s_\xi) \\
&\approx \int_1^{\infty} s^{2-\tau} \mathcal{G}(s/s_\xi)\, ds \\
&= \int_{1/s_\xi}^{\infty} (u s_\xi)^{2-\tau} \mathcal{G}(u) s_\xi\, du && \text{with } u = s/s_\xi ; ds = s_\xi du \\
&= s_\xi^{3-\tau} \int_{1/s_\xi}^{\infty} u^{2-\tau} \mathcal{G}(u)\, du \\
&\to s_\xi^{3-\tau} \int_0^{\infty} u^{2-\tau} \mathcal{G}(u)\, du && \text{for } p \to p_c \\
&\propto |p - p_c|^{-\frac{3-\tau}{\sigma}} = |p - p_c|^{-\gamma} && \text{for } p \to p_c .
\end{aligned}
$$

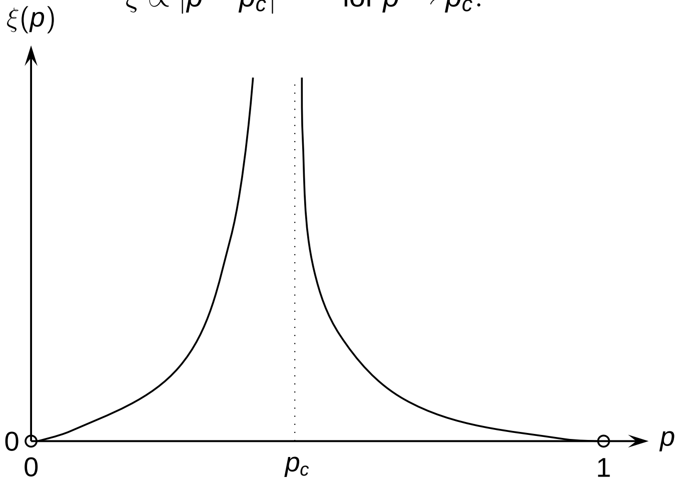
Scaling relation: $\gamma = \frac{3-\tau}{\sigma}$.
Bethe lattice: $\gamma = 1 \ ; \sigma = \frac{1}{2} \ ; \tau = \frac{5}{2}$.
$d = 1$: $\gamma = 1 \ ; \sigma = 1 \ ; \tau = 2$.
$d = 2$: $\gamma = \frac{43}{18}; \sigma = \frac{36}{91}; \tau = \frac{187}{91}$.

Using $P_\infty(p) + \sum_{s=1}^{\infty} sn(s,p) = p$, one can derive another
Scaling relation: $\beta = \frac{\tau-2}{\sigma}$.
Bethe lattice: $\beta = 1 \ ; \sigma = \frac{1}{2} \ ; \tau = \frac{5}{2}$.
$d = 1$: $\beta = 0 \ ; \sigma = 1 \ ; \tau = 2$.
$d = 2$: $\beta = \frac{5}{36}; \sigma = \frac{36}{91}; \tau = \frac{187}{91}$.

Hence, there are **only two independent critical exponents** among β, γ, σ and τ.

The linear scale ξ of the characteristic cluster size $s_\xi \propto \xi^D$.
Because $s_\xi \to \infty$ for $p \to p_c$, so does ξ:

$$\xi \propto |p - p_c|^{-\nu} \quad \text{for } p \to p_c.$$



$s_\xi \propto \xi^D$, where D is fractal dimension of percolating cluster

$\xi =$ characteristic length scale

$=$ typical radius of largest finite cluster (definition for all p)

For $p > p_c$, the percolating infinite cluster is excluded.

- Finite clusters reside inside holes of percolating cluster
- $\xi =$ typical radius of the largest holes in percolating cluster

At $p = p_c$ where $\xi = \infty$:

- Finite clusters of all sizes
- Holes of all sizes in the percolating cluster

Introduced two new critical exponents: D and ν.
However, we can derive further two scaling relations:

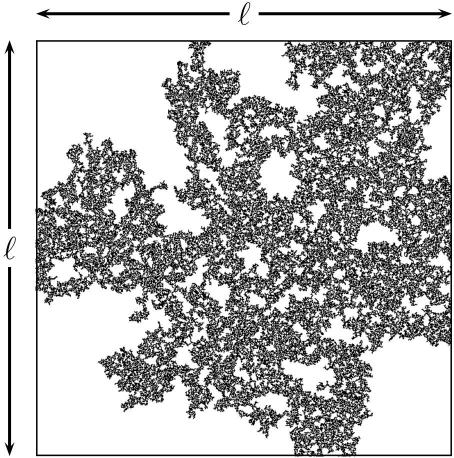
$$\begin{aligned} s_\xi &\propto \xi^D \\ &\propto |p - p_c|^{-\nu D} \quad \text{for } p \to p_c \\ &\propto |p - p_c|^{-1/\sigma} \quad \text{for } p \to p_c \end{aligned}$$

Scaling relation: $D = \frac{1}{\sigma\nu}$
Bethe lattice: $D = 4$; $\sigma = \frac{1}{2}$; $\nu = \frac{1}{2}$.
$d = 1$: $D = 1$; $\sigma = 1$; $\nu = 1$.
$d = 2$: $D = \frac{91}{48}$; $\sigma = \frac{36}{91}$; $\nu = \frac{4}{3}$.

Is the percolating cluster fractal?

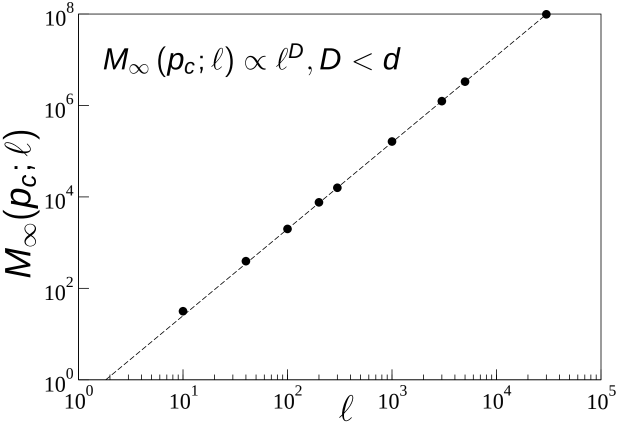
- When $p = p_c$, the correlation length $\xi = \infty$.
  Percolating cluster is fractal on all length scales ℓ.
- When $p \neq p_c$, the correlation length $\xi < \infty$.
  Percolating cluster is fractal on length scales $\ell \ll \xi$.
  Percolating cluster is uniform on length scales $\ell \gg \xi$.

Consider a window of size ℓ in an infinite lattice.
Let $M_\infty(p;\ell)$ denote the mass of percolating infinite cluster in window of size ℓ at occupation probability p.



$$P_\infty(p;\ell) = \frac{M_\infty(p;\ell)}{\ell^d}$$

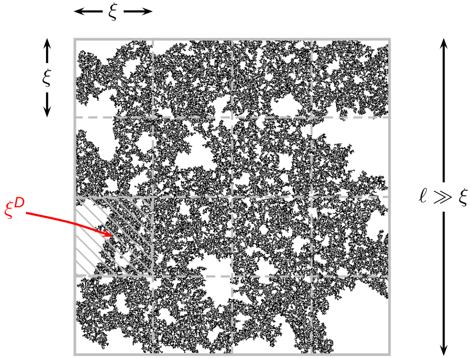
$$P_\infty(p) = \lim_{\ell\to\infty} P_\infty(p;\ell) = 0 \quad \text{for } p \leq p_c.$$



Mass of perc. cluster in window of size ℓ when $p > p_c \Rightarrow \xi < \infty$:

$$M_\infty(\xi, \ell) = \begin{cases} \ell^D & \text{for } \ell \ll \xi \text{ - looks fractal} \\ (\ell/\xi)^d \xi^D = \underbrace{\xi^{D-d}}_{\text{density}} \underbrace{\ell^d}_{\text{volume}} & \text{for } \ell \gg \xi \text{ - looks homogenous} \end{cases}$$

No. boxes of size ξ; Mass within a box of size ξ.



$$\begin{aligned} M_\infty(\xi,\ell) &= P_\infty(p;\ell)\ell^d \\ &= P_\infty(p)\ell^d \\ &= (p-p_c)^\beta \ell^d \\ &\propto \xi^{-\beta/\nu}\ell^d \end{aligned}$$

mass = density · volume

when $\ell \gg \xi$

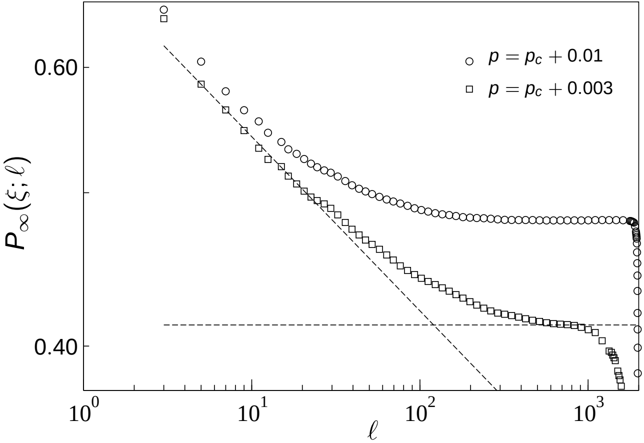
Scaling relation: $D - d = -\beta/\nu$.

$d = 1$: $D = 1$; $\beta = 0$; $\nu = 1$.

$d = 2$: $D = \frac{91}{48}$; $\beta = \frac{5}{36}$; $\nu = \frac{4}{3}$.

Bethe lattice: $D = 4$; $\beta = 1$; $\nu = \frac{1}{2}$.

$$P_\infty(\xi,\ell) = \frac{M_\infty(\xi,\ell)}{\ell^d} = \begin{cases} \ell^{D-d} & \text{for } \ell \ll \xi \\ \xi^{D-d} & \text{for } \ell \gg \xi \end{cases}$$



Percolation is defined on an infinite lattice $L = \infty$.
However, we cannot simulate $L = \infty$.

- $L \gg \xi$ ☺
  To all intents & purposes such systems appear to be ∞.
  We have clusters of all sizes up to ξ in linear size.
  ξ is an (inherent) upper cut-off scale set by p.
- $L \ll \xi$ ☹
  Such systems are finite.
  We have clusters of all sizes up to L in linear size.
  L is an (external) upper cut-off scale set by the system.
  This is a **finite-size effect**.

At $p = p_c$, $\xi = \infty$ so $L \ll \xi$ FOR ANY L.
Divergences of quantities such as χ are capped.
However, we can exploit the finiteness of the lattice at $p = p_c$ where necessarily $L \ll \xi$ to extract critical exponents.

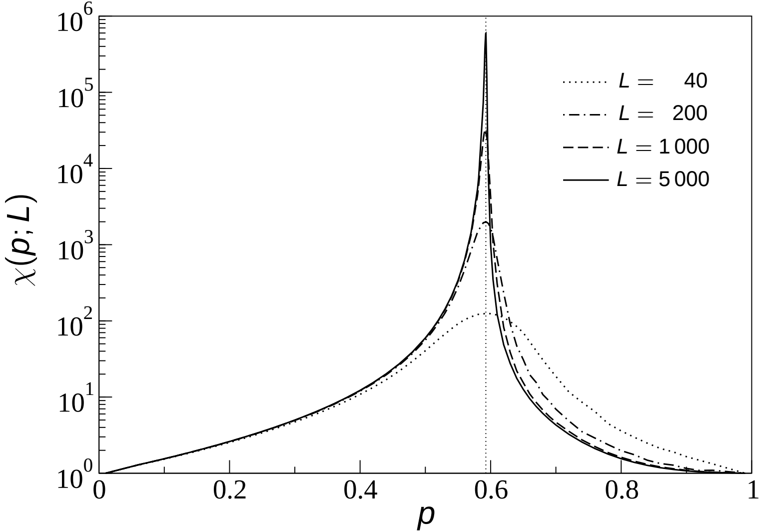
Consider an infinite lattice $L = \infty$.
For p close to p_c, $\xi \propto |p - p_c|^{-\nu} \Rightarrow |p - p_c| \propto \xi^{-1/\nu}$.

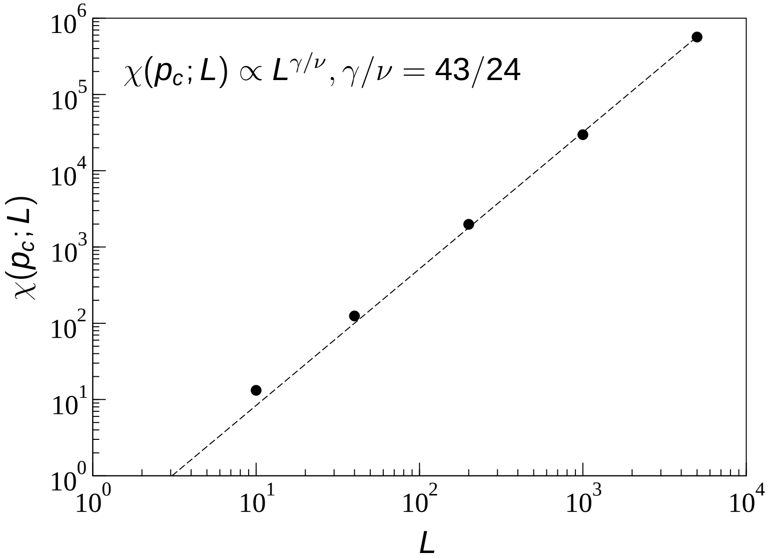
$$\chi(p) \propto |p - p_c|^{-\gamma} \propto \xi^{\gamma/\nu} \quad \text{for } p \to p_c$$

Now consider finite lattice $L < \infty$:

$$\chi(p; L) = \begin{cases} \xi^{\gamma/\nu} & \text{for } L \gg \xi \\ L^{\gamma/\nu} & \text{for } L \ll \xi \end{cases}$$

At $p = p_c$, $\xi = \infty$, so $L \ll \xi$ for ALL system sizes L.
Hence, we expect finite-size scaling $\chi(p_c; L) \propto L^{\gamma/\nu}$.
Extract γ/ν by measuring scaling of $\chi(p_c; L)$ with system size L.


10^6
10^5
10^4
10^3
10^2
10^1
10^0
$\chi(p;L)$
0
0.2
0.4
0.6
0.8
1
p
$L = 40$
$L = 200$
$L = 1\,000$
$L = 5\,000$


$\chi(p_c; L) \propto L^{\gamma/\nu}, \gamma/\nu = 43/24$
10^0
10^1
10^2
10^3
10^4
10^5
10^6
$\chi(p_c; L)$
L

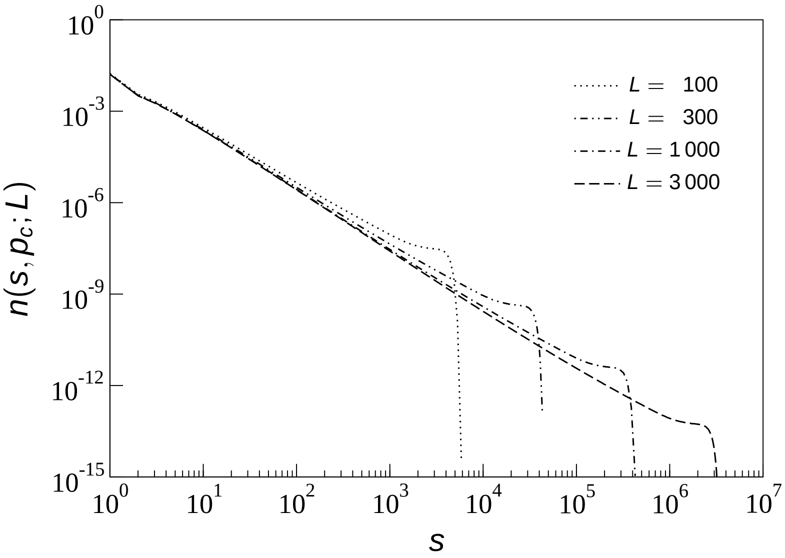
Consider an infinite lattice $L = \infty$.

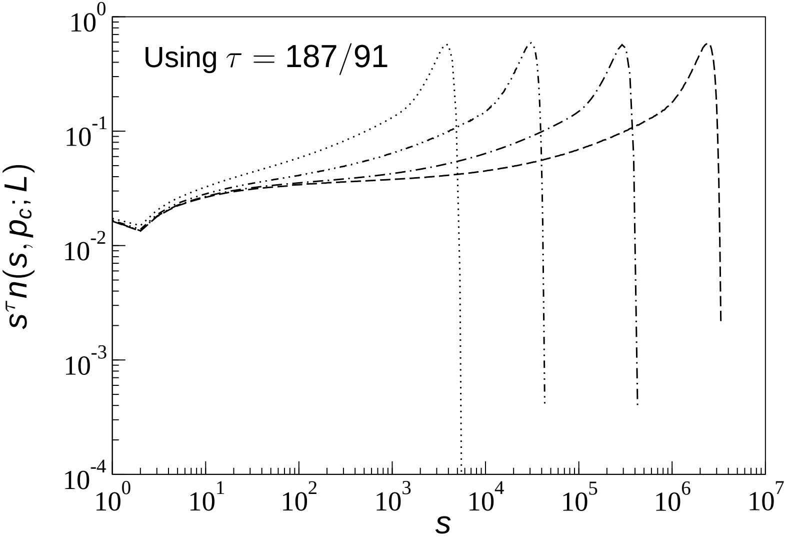
$$
\begin{aligned}
n(s,p) &\propto s^{-\tau}\mathcal{G}\left(s/s_\xi\right) & &\text{for } p \to p_c, s \gg 1 \\
s_\xi(p) &\propto |p - p_c|^{-1/\sigma} \propto \xi^{1/(\sigma\nu)} \propto \xi^D & &\text{for } p \to p_c
\end{aligned}
$$

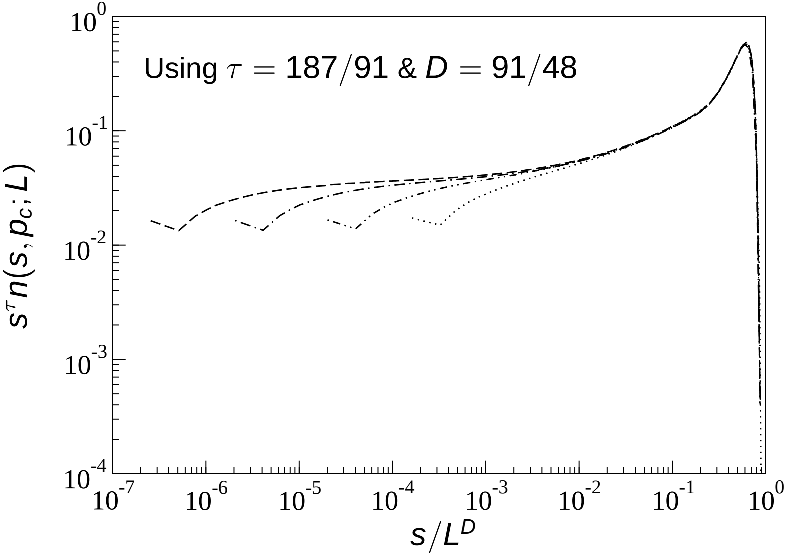
Now consider finite lattice $L < \infty$:

$$
n(s,p;L) = \begin{cases} s^{-\tau}\mathcal{G}\left(s/\xi^D\right) & \text{for } L \gg \xi, p \to p_c, s \gg 1 \\ s^{-\tau}\tilde{\mathcal{G}}\left(s/L^D\right) & \text{for } L \ll \xi, p \to p_c, s \gg 1 \end{cases}
$$

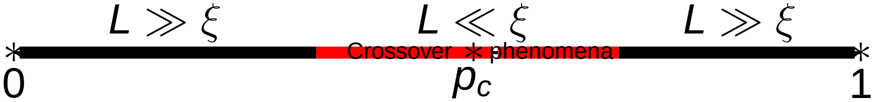
At $p = p_c$, $\xi = \infty$, so $L \ll \xi$ for ALL system sizes L.
Hence, we expect finite-size scaling $n(s,p;L) \propto s^{-\tau}\tilde{\mathcal{G}}\left(s/L^D\right)$.
Extract τ and D by data collapse, plotting $s^\tau n(s,p;L)$ vs. s/L^D.


$L = 100$
$L = 300$
$L = 1\,000$
$L = 3\,000$
$n(s, p_c; L)$
s
10^{0}
10^{-3}
10^{-6}
10^{-9}
10^{-12}
10^{-15}
10^{1}
10^{2}
10^{3}
10^{4}
10^{5}
10^{6}
10^{7}


Using $\tau = 187/91$
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{1}
10^{2}
10^{3}
10^{4}
10^{5}
10^{6}
10^{7}
$s^{\tau} n(s, p_c; L)$
s


Using $\tau = 187/91$ & $D = 91/48$
$s^\tau n(s, p_c; L)$
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-7}
10^{-6}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
s/L^D

At $p = p_c$, the correlation length $\xi = \infty$, i.e., ALWAYS $L \ll \xi$. Measure critical exponents by investigating how the quantities scale with system size at $p = p_c$.



ξ

0 at $p = 0$. Empty lattice. Trivially self-similar.

∞ at $p = p_c$.Density at critical value p_c. Non-trivially self-similar.

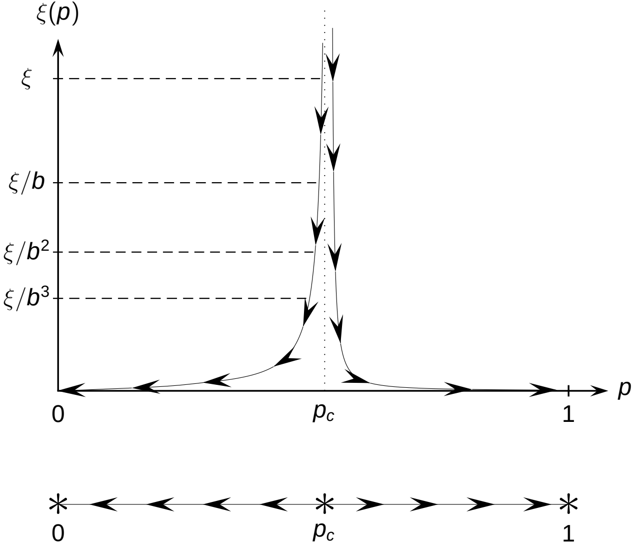
0 at $p = 1$. Fully occupied lattice. Trivially self-similar.

ξ :Fluctuations away from the trivially self-similar configurations. Self-similarity can be identified with fixed points of a rescaling transformation which reduces length scales by a factor $b > 1$. Assume $\xi < \infty$:

$$\xi \mapsto \xi/b \mapsto \xi/b^2 \mapsto \xi/b^3 \mapsto \cdots\cdots \quad \lim_{n\to\infty} \xi/b^n = 0$$

$$p_1^- > p_2^- > p_3^- > p_4^- > \cdots\cdots \quad \lim_{n\to\infty} p_n^- = 0$$

$$p_1^+ < p_2^+ < p_3^+ < p_4^+ < \cdots\cdots \quad \lim_{n\to\infty} p_n^+ = 1$$


$\xi(p)$
ξ
ξ/b
ξ/b^2
ξ/b^3
p
0
p_c
1
0
p_c
1

The solutions to the fixed points equation

$$\xi = \xi/b \Leftrightarrow \xi = \begin{cases} 0 & \text{associated with } p = 0 \text{ or } p = 1 \\ \infty & \text{associated with } p = p_c. \end{cases}$$

Fixed points $p^\star = 0$ and $p^\star = 1$ are stable fixed points.
Fixed point $p^\star = p_c$ is an unstable fixed point.

At $p = p_c$, the system is delicately poised in a non-trivial self-similar state between two trivially self-similar states.

Thank you for listening!

For a comprehensive introduction to percolation, please see K. Christensen and N.R. Moloney, *Complexity and Criticality*, Imperial College Press (2005), Chapter 1.

Access to animations, please visit **www.complexityandcriticality.com**.