The effect of set-theoretic assumptions on spectra of $\mathcal{L}_{\omega_1,\omega}$-sentences

Workshop on Set-Theoretical Aspects of the Model Theory of Strong Logic

CRM CENTRE DE RECERCA MATEMÀTICA

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Preliminaries

Definition

- $\mathcal{L}_{\omega_1,\omega}$ is the closure of first-order formulas under countable disjunctions and conjunctions.
- $\mathcal{L}_{\omega_1,\omega}$ satisfies the Downward Lowenheim-Skolem Theorem.

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Definition

Let P be a property and ϕ an $\mathcal{L}_{\omega_1,\omega}$-sentence.
The P-spectrum of ϕ is the set

$$\{\kappa \mid \text{the models of } \phi \text{ of size } \kappa \text{ satisfy } P\}$$

Examples for P:

- "there exists a model"- ME-Spec(ϕ)
- "amalgamation"- AP-Spec(ϕ)
- "joint embedding"- JEP-Spec(ϕ)
- "has maximal models"
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- *By Downward Lowenheim-Skolem, ME-Spec(ϕ) is downward closed.*
- *This is not the case for AP-Spec(ϕ) or JEP-Spec(ϕ).*

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If ME-Spec(ϕ)=$[\aleph_0, \aleph_\alpha]$, then ϕ *characterizes* $\aleph_\alpha$.

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Let *Char* be the set of cardinals characterized by $\mathcal{L}_{\omega_1,\omega}$-sentences.

Questions

1. *What cardinals are in Char?*
2. *What are the closure properties of Char?*
3. *What is absolute?*
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Theorem (Morley, López-Escobar)

Let Γ be a countable set of sentences of $\mathcal{L}_{\omega_1,\omega}$. If Γ has models of cardinality $\beth_\alpha$ for all $\alpha < \omega_1$, then it has models of all infinite cardinalities.

Corollary

If $\aleph_\alpha$ is characterized by an $\mathcal{L}_{\omega_1,\omega}$-sentence, then $\aleph_\alpha < \beth_{\omega_1}$.

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Theorem (Malitz, Baumgartner)

For every $\alpha < \omega_1$, there exists a complete $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ_α that characterizes $\beth_\alpha$.

Thus, $\beth_{\omega_1}$ is optimal and is called the *Hanf number* for $\mathcal{L}_{\omega_1,\omega}$.

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Alephs

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Hjorth's theorem generalizes to

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Assume $\aleph_\beta$ is characterized by ϕ^β. Then for every $\alpha < \omega_1$, there exists an $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ^β_α that characterizes $\aleph_{\beta+\alpha}$.

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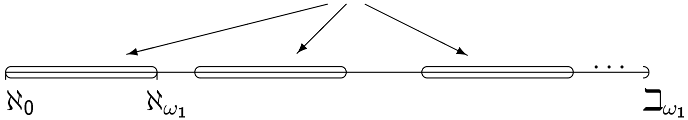
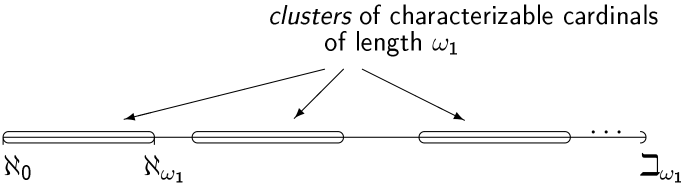
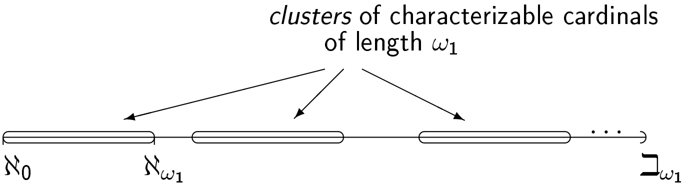
clusters of characterizable cardinals of length ω_1





Under GCH there is only one cluster.



Under GCH there is only one cluster.
Under $\neg$GCH, it is consistent that there exist non-characterizable cardinals below $\beth_{\omega_1}$.

Conjecture (Shelah)

If $2^{\aleph_0} > \aleph_{\omega_1}$, then every $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ that has a model in $\aleph_{\omega_1}$, also has a model in $2^{\aleph_0}$.

In particular, no cardinal between $\aleph_{\omega_1}$ and $2^{\aleph_0}$ is characterized by an $\mathcal{L}_{\omega_1,\omega}$-sentence.

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The set of cardinals characterized by $\mathcal{L}_{\omega_1,\omega}$-sentences is closed under

1. *successor;*
2. *countable unions;*
3. *countable products;*
4. *powerset.*

Many of the results are true even for the set of cardinals characterized by **complete** $\mathcal{L}_{\omega_1,\omega}$-sentences.But it takes considerable more work.

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Question (S.)

Let C be the least set of cardinals that contains $\aleph_0$ and is closed under (1)-(4). Does C contain all characterizable cardinals?

Consistently yes, e.g. under GCH.

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Consistently no.

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- There exists an $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ that characterizes the maximum of $2^{\aleph_0}$ and $\mathcal{B}$.

$$\mathcal{B} = \max\{\kappa | \text{there exists a Kurepa tree with} \kappa \text{ many branches}\}.$$

- **Theorem:** If ZFC is consistent, then so is

$$\text{ZFC} + (2^{\aleph_0} < \aleph_{\omega_1} = \mathcal{B} < 2^{\aleph_1}) + \text{"}\mathcal{B} \text{ is not in the set } C \text{ described above"}.$$

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The set of cardinals characterized by $\mathcal{L}_{\omega_1,\omega}$-sentences is also closed under

5. *powers;*
6. $\kappa \mapsto Ded(\kappa)$;
7. $\kappa \mapsto \max\{\lambda | \text{there exists a } \kappa\text{-Kurepa tree with } \lambda \text{ many branches}\}$;
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We have only partial results for similar closures properties when *complete* $\mathcal{L}_{\omega_1,\omega}$-sentences are concerned.

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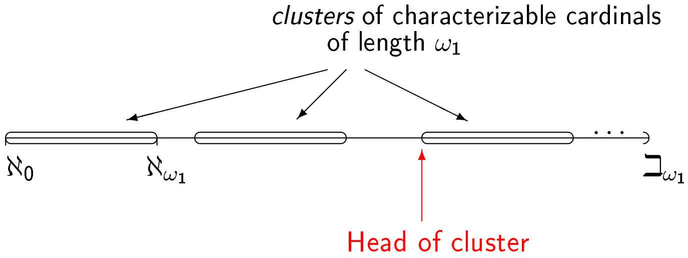

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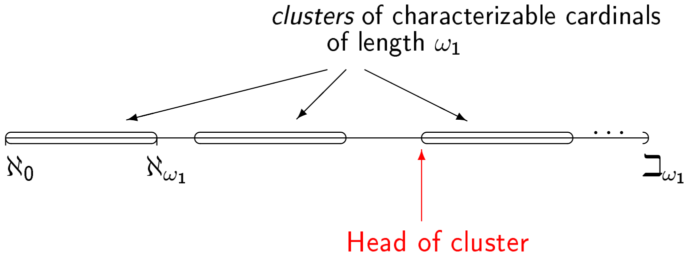

Idea: The head of a cluster of characterizable cardinals determines the properties of the whole cluster.

Theorem (S.)

If $(\aleph_\alpha)^\kappa$ is characterized by a complete sentence, then the same is true for $(\aleph_{\alpha+\beta})^\kappa$, for all $\beta < \omega_1$.

Note: We do not assume that either $\aleph_\alpha$ or κ is characterizable.

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If $\aleph_\alpha$ and $\kappa^{\aleph_\alpha}$ are characterized by a complete sentence, then the same is true for $\kappa^{\aleph_{\alpha+\beta}}$, for all $\beta < \omega_1$.

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The question for $\beta = 0$ remains open. It is known under extra assumptions.

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Observation

All these theorems and closure properties indicate a definability issue.

Theorem (Morley, Barwise)

Let A be a countable admissible set with $o(A) = \gamma$, and let ϕ be a sentence of $\mathcal{L}_{\omega_1,\omega} \cap A$. Then either

- *ϕ characterizes some $\aleph_\beta < \beth_\gamma$, or*
- *ϕ has arbitrarily large models.*

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Is membership in Char a model-theoretic or set-theoretic question?

View #1: There is a nice model-theoretic description of *Char* that is absolute. E.g. it is the smallest set closed under ...
Set theory enter the picture only because things like the size of $2^{\aleph_0}$ are not absolute.

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- Is the following property absolute? "ME-Spec(ϕ) is right-closed".
  It relates to the following set-theoretic question: Is it consistent that for every $\alpha < \omega_1$ there exists a Kurepa tree with exactly $\aleph_\alpha$ branches, but there is no Kurepa tree with $\aleph_{\omega_1}$ branches?
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Theorem (Boney, Baldwin)

Let κ be strongly compact and $\boldsymbol{K}$ be an AEC with Lowenheim-Skolem number less than κ. If $\boldsymbol{K}$ satisfies DAP$([\mu, <\kappa))$, then $\boldsymbol{K}$ satisfies DAP$([\mu, \infty))$.

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