Statistical physics of tailored random graphs: entropies, processes, and generation

Lecture III. Ising spin models on graphs

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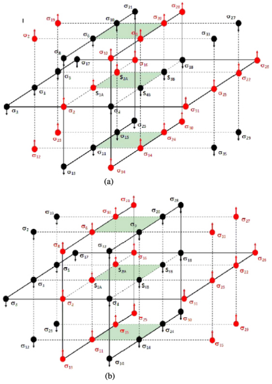


Ising spins on a tailored random graph

Definitions

- N Ising spins $\sigma_i \in \{-1, 1\}$, $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N)$

$$H(\boldsymbol{\sigma}) = -\sum_{i<j} c_{ij} J_{ij} \sigma_i \sigma_j$$

$$\textit{energies}: \quad J_{ij} \in \mathbb{R}, \quad \textit{drawn randomly from} \quad P(J)$$

$$\textit{topology}: \quad p(\mathbf{c}) = Z^{-1} \delta_{\mathbf{k},\mathbf{k}(\mathbf{c})}, \quad k_i(\mathbf{c}) = \sum_j c_{ij}$$

- disorder-averaged free energy density, use $\overline{\log \Sigma} = \lim_{n\to 0} n^{-1} \log \overline{\Sigma^n}$:

$$\begin{aligned}
\bar{f} &= -\lim_{N\to\infty} \frac{1}{\beta N} \overline{\left(\log \sum_{\boldsymbol{\sigma} \in \{-1,1\}^N} \mathrm{e}^{-\beta H(\boldsymbol{\sigma})} \right)} \\
&= -\lim_{N\to\infty} \lim_{n\to 0} \frac{1}{\beta n N} \log \sum_{\boldsymbol{\sigma}^1 \ldots \boldsymbol{\sigma}^n \in \{-1,1\}^N} \overline{\mathrm{e}^{-\beta \sum_{\alpha=1}^n H(\boldsymbol{\sigma}^\alpha)}} \\
&= -\lim_{N\to\infty} \lim_{n\to 0} \frac{1}{\beta n N} \log \sum_{\boldsymbol{\sigma}^1 \ldots \boldsymbol{\sigma}^n \in \{-1,1\}^N} \overline{\mathrm{e}^{\beta \sum_{i<j} c_{ij} J_{ij} \sum_{\alpha=1}^n \sigma_i^\alpha \sigma_j^\alpha}}
\end{aligned}$$

Average over graphs

change to Erdös-Rènyi measure,
to enable expansion for large N

$$p_{\rm ER}(\mathbf{c})=\prod_{i<j}\Big[\frac{\langle k\rangle}{N}\delta_{c_{ij},1}+(1-\frac{\langle k\rangle}{N})\delta_{c_{ij},0}\Big],\qquad \langle k\rangle=\frac{1}{N}\sum_i k_i \tag{1}$$

Now:

$$\begin{aligned}
p(\mathbf{c}) &= \frac{\delta_{\mathbf{k},\mathbf{k}(\mathbf{c})}}{Z}=\frac{\delta_{\mathbf{k},\mathbf{k}(\mathbf{c})}}{Z}\frac{p_{\rm ER}(\mathbf{c})}{p_{\rm ER}(\mathbf{c})}=\frac{\delta_{\mathbf{k},\mathbf{k}(\mathbf{c})}}{Z}\frac{p_{\rm ER}(\mathbf{c})}{\left(\frac{\langle k\rangle}{N}\right)^{\sum_{i<j}c_{ij}}\left(1-\frac{\langle k\rangle}{N}\right)^{\frac{1}{2}N(N-1)-\sum_{i<j}c_{ij}}}\\
&= \frac{\delta_{\mathbf{k},\mathbf{k}(\mathbf{c})}}{Z}\frac{p_{\rm ER}(\mathbf{c})}{\left(\frac{\langle k\rangle}{N}\right)^{\frac{1}{2}\sum_{ij}c_{ij}}\left(1-\frac{\langle k\rangle}{N}\right)^{\frac{1}{2}N(N-1)-\frac{1}{2}\sum_{ij}c_{ij}}}\\
&= \frac{\delta_{\mathbf{k},\mathbf{k}(\mathbf{c})}}{Z}\frac{p_{\rm ER}(\mathbf{c})}{\left(\frac{\langle k\rangle}{N}\right)^{\frac{1}{2}N\langle k\rangle}\left(1-\frac{\langle k\rangle}{N}\right)^{\frac{1}{2}N(N-1)-\frac{1}{2}N\langle k\rangle}}
\end{aligned}$$

so

$$\sum_{\mathbf{c}}p(\mathbf{c})\Phi(\mathbf{c})=\frac{1}{\mathcal{Z}}\sum_{\mathbf{c}}p_{\rm ER}(\mathbf{c})\delta_{\mathbf{k},\mathbf{k}(\mathbf{c})}\Phi(\mathbf{c})\qquad \mathcal{Z}=\sum_{\mathbf{c}}p_{\rm ER}(\mathbf{c})\delta_{\mathbf{k},\mathbf{k}(\mathbf{c})}$$

let $\boldsymbol{\sigma}_i = (\sigma_i^1, \ldots, \sigma_i^n)$, use $\delta_{k\ell} = (2\pi)^{-1} \int_{-\pi}^{\pi} \mathrm{d}\omega \; \mathrm{e}^{\mathrm{i}\omega(k-\ell)}$

$$\overline{\mathrm{e}^{\beta \sum_{i<j} c_{ij} J_{ij} \sum_{\alpha=1}^{n} \sigma_i^\alpha \sigma_j^\alpha}}$$

$$= \sum_{\mathbf{c}} p(\mathbf{c}) \prod_{i<j} \int \mathrm{d}J \; P(J) \mathrm{e}^{\beta J c_{ij} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j}$$

$$= \frac{1}{\mathcal{Z}} \sum_{\mathbf{c}} p_{\mathrm{ER}}(\mathbf{c}) \delta_{\mathbf{k},\mathbf{k}(\mathbf{c})} \prod_{i<j} \int \mathrm{d}J \; P(J) \mathrm{e}^{\beta J c_{ij} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j}$$

$$= \frac{1}{\mathcal{Z}} \sum_{\mathbf{c}} p_{\mathrm{ER}}(\mathbf{c}) \Big[\prod_i \int_{-\pi}^{\pi} \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{\mathrm{i}\omega(k_i - \sum_j c_{ij})} \Big] \prod_{i<j} \int \mathrm{d}J \; P(J) \mathrm{e}^{\beta J c_{ij} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j}$$

$$= \frac{1}{\mathcal{Z}} \int_{-\pi}^{\pi} \frac{\mathrm{d}\boldsymbol{\omega}}{(2\pi)^N} \mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\cdot\mathbf{k}} \sum_{\mathbf{c}} p_{\mathrm{ER}}(\mathbf{c}) \mathrm{e}^{-\mathrm{i}\sum_{ij} \omega_i c_{ij}} \prod_{i<j} \int \mathrm{d}J \; P(J) \mathrm{e}^{\beta J c_{ij} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j}$$

$$= \frac{1}{\mathcal{Z}} \int_{-\pi}^{\pi} \frac{\mathrm{d}\boldsymbol{\omega}}{(2\pi)^N} \mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\cdot\mathbf{k}} \sum_{\mathbf{c}} p_{\mathrm{ER}}(\mathbf{c}) \prod_{i<j} \Big[\mathrm{e}^{-\mathrm{i}(\omega_i+\omega_j) c_{ij}} \int \mathrm{d}J \; P(J) \mathrm{e}^{\beta J c_{ij} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j} \Big]$$

$$= \frac{1}{\mathcal{Z}} \int_{-\pi}^{\pi} \frac{\mathrm{d}\boldsymbol{\omega}}{(2\pi)^N} \mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\cdot\mathbf{k}} \prod_{i<j} \Big[(1 - \frac{\langle k \rangle}{N}).1 + \frac{\langle k \rangle}{N} \mathrm{e}^{-\mathrm{i}(\omega_i+\omega_j)} \int \mathrm{d}J \; P(J) \mathrm{e}^{\beta J \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j} \Big]$$

$$= \frac{1}{\mathcal{Z}} \int_{-\pi}^{\pi} \frac{\mathrm{d}\boldsymbol{\omega}}{(2\pi)^N} \mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\cdot\mathbf{k}} \prod_{i<j} \Big[1 + \frac{\langle k \rangle}{N} \Big(\mathrm{e}^{-\mathrm{i}(\omega_i+\omega_j)} \int \mathrm{d}J \; P(J) \mathrm{e}^{\beta J \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j} - 1 \Big) \Big]$$

$L_{ij} = \mathcal{O}(1)$,
expand for large N:

$$\prod_{i<j}\left[1+\frac{1}{N}L_{ij}\right] = \prod_{i<j} \mathrm{e}^{\log[1+\frac{1}{N}L_{ij}]} = \mathrm{e}^{\sum_{i<j}[\frac{1}{N}L_{ij}+\mathcal{O}(N^{-2})]} = \mathrm{e}^{N^{-1}\sum_{i<j}L_{ij}+\mathcal{O}(1)}$$

Apply:

$$\overline{\mathrm{e}^{\beta\sum_{i<j}c_{ij}J_{ij}\sum_{\alpha=1}^{n}\sigma_i^\alpha\sigma_j^\alpha}}$$
$$= \frac{1}{\mathcal{Z}}\int_{-\pi}^{\pi}\frac{\mathrm{d}\boldsymbol{\omega}}{(2\pi)^N}\mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\cdot\mathbf{k}+\frac{\langle k\rangle}{N}\sum_{i<j}\left[\mathrm{e}^{-\mathrm{i}(\omega_i+\omega_j)}\int\mathrm{d}J\, P(J)\mathrm{e}^{\beta J\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j}-1\right]+\mathcal{O}(1)}$$
$$= \frac{1}{\mathcal{Z}}\int_{-\pi}^{\pi}\frac{\mathrm{d}\boldsymbol{\omega}}{(2\pi)^N}\mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\cdot\mathbf{k}+\frac{\langle k\rangle}{2N}\sum_{ij}\left[\mathrm{e}^{-\mathrm{i}(\omega_i+\omega_j)}\int\mathrm{d}J\, P(J)\mathrm{e}^{\beta J\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j}-1\right]+\mathcal{O}(1)}$$

$$\beta=0: \quad \mathcal{Z} = \int_{-\pi}^{\pi}\frac{\mathrm{d}\boldsymbol{\omega}}{(2\pi)^N}\mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\cdot\mathbf{k}+\frac{\langle k\rangle}{2N}\sum_{ij}\left[\mathrm{e}^{-\mathrm{i}(\omega_i+\omega_j)}-1\right]+\mathcal{O}(1)}$$

Hence

$$\overline{\mathrm{e}^{\beta\sum_{i<j}c_{ij}J_{ij}\sum_{\alpha=1}^{n}\sigma_i^\alpha\sigma_j^\alpha}}$$
$$= \frac{\int_{-\pi}^{\pi}\mathrm{d}\boldsymbol{\omega}\;\mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\cdot\mathbf{k}+\frac{\langle k\rangle}{2N}\sum_{ij}\mathrm{e}^{-\mathrm{i}(\omega_i+\omega_j)}\int\mathrm{d}J\, P(J)\mathrm{e}^{\beta J\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j}+\mathcal{O}(1)}}{\int_{-\pi}^{\pi}\mathrm{d}\boldsymbol{\omega}\;\mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\cdot\mathbf{k}+\frac{\langle k\rangle}{2N}\sum_{ij}\mathrm{e}^{-\mathrm{i}(\omega_i+\omega_j)}+\mathcal{O}(1)}}$$

The free energy

use previous result:

$$
\begin{aligned}
-\beta\bar{f} &= \lim_{N\to\infty}\lim_{n\to 0}\frac{1}{nN}\log\sum_{\boldsymbol{\sigma}_1\ldots\boldsymbol{\sigma}_N}\frac{\int_{-\pi}^{\pi}\mathrm{d}\boldsymbol{\omega}\;\mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\cdot\mathbf{k}+\frac{\langle k\rangle}{2N}\sum_{ij}\mathrm{e}^{-\mathrm{i}(\omega_i+\omega_j)}\int\mathrm{d}J\;P(J)\mathrm{e}^{\beta J\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j}+\mathcal{O}(1)}}{\int_{-\pi}^{\pi}\mathrm{d}\boldsymbol{\omega}\;\mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\cdot\mathbf{k}+\frac{\langle k\rangle}{2N}\sum_{ij}\mathrm{e}^{-\mathrm{i}(\omega_i+\omega_j)}+\mathcal{O}(1)}} \\
&= \lim_{N\to\infty}\lim_{n\to 0}\frac{1}{nN}\log\left\{2^{nN}\left\langle\frac{\int_{-\pi}^{\pi}\mathrm{d}\boldsymbol{\omega}\;\mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\cdot\mathbf{k}+\frac{\langle k\rangle}{2N}\sum_{ij}\mathrm{e}^{-\mathrm{i}(\omega_i+\omega_j)}\int\mathrm{d}J\;P(J)\mathrm{e}^{\beta J\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j}+\mathcal{O}(1)}}{\int_{-\pi}^{\pi}\mathrm{d}\boldsymbol{\omega}\;\mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\cdot\mathbf{k}+\frac{\langle k\rangle}{2N}\sum_{ij}\mathrm{e}^{-\mathrm{i}(\omega_i+\omega_j)}+\mathcal{O}(1)}}\right\rangle_{\{\boldsymbol{\sigma}_i\}}\right. \\
&= \log 2+\lim_{N\to\infty}\lim_{n\to 0}\frac{1}{nN}\log\left\langle\int_{-\pi}^{\pi}\mathrm{d}\boldsymbol{\omega}\;\mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\cdot\mathbf{k}+\frac{\langle k\rangle}{2N}\sum_{ij}\mathrm{e}^{-\mathrm{i}(\omega_i+\omega_j)}\int\mathrm{d}J\;P(J)\mathrm{e}^{\beta J\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j}+\mathcal{O}(1)}\right\rangle_{\{\boldsymbol{\sigma}_i\}} \\
&\qquad -\lim_{N\to\infty}\lim_{n\to 0}\frac{1}{nN}\log\int_{-\pi}^{\pi}\mathrm{d}\boldsymbol{\omega}\;\mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\cdot\mathbf{k}+\frac{\langle k\rangle}{2N}\sum_{ij}\mathrm{e}^{-\mathrm{i}(\omega_i+\omega_j)}+\mathcal{O}(1)} \\
&= \log 2+\lim_{N\to\infty}\lim_{n\to 0}\frac{1}{n}\Big[\Phi(\beta)-\Phi(0)\Big]
\end{aligned}
$$

Left to do:

$$\Phi(\beta) = \frac{1}{N}\log\left\langle \int_{-\pi}^{\pi} \mathrm{d}\boldsymbol{\omega}\ \mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\cdot\mathbf{k}+\frac{\langle k\rangle}{2N}\sum_{ij}\mathrm{e}^{-\mathrm{i}(\omega_i+\omega_j)}\int \mathrm{d}J\ P(J)\mathrm{e}^{\beta J\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j}} \right\rangle_{\{\boldsymbol{\sigma}_i\}}$$

notes:

- link with graphs ensemble

$$\Phi(0) = \frac{1}{N}\log\int_{-\pi}^{\pi}\mathrm{d}\boldsymbol{\omega}\ \mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\cdot\mathbf{k}+\frac{\langle k\rangle}{2N}\sum_{ij}\mathrm{e}^{-\mathrm{i}(\omega_i+\omega_j)}} = \log(2\pi)+\frac{1}{2}\langle k\rangle+\frac{1}{N}\log\mathcal{Z}$$

- all site-dependent variables appear in quantity of the form

$$\frac{1}{N}\sum_{ij} G(\omega_i.\boldsymbol{\sigma}_i;\omega_j,\boldsymbol{\sigma}_j)$$

Order parameters

- define
  $$\mathcal{P}(\boldsymbol{\sigma},\omega|\{\boldsymbol{\sigma}_i,\omega_i\}) = \frac{1}{N}\sum_i \delta_{\boldsymbol{\sigma},\boldsymbol{\sigma}_i}\delta(\omega-\omega_i)$$
  so that
  $$\Phi(\beta) = \frac{1}{N}\log\Big\langle \int_{-\pi}^{\pi} \mathrm{d}\boldsymbol{\omega}\ \mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\cdot\mathbf{k}}$$
  $$\times\ \mathrm{e}^{\frac{1}{2}\langle k\rangle N\int \mathrm{d}\omega\mathrm{d}\omega'\sum_{\boldsymbol{\sigma}\boldsymbol{\sigma}'}\mathcal{P}(\boldsymbol{\sigma},\omega|\ldots)\mathcal{P}(\boldsymbol{\sigma}',\omega'|\ldots)\mathrm{e}^{-\mathrm{i}(\omega+\omega')}\int \mathrm{d}J\ P(J)\mathrm{e}^{\beta J\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'}}\Big\rangle_{\{\boldsymbol{\sigma}_i\}}$$

- introduce for *each* $(\boldsymbol{\sigma},\omega)$:
  $$1 = \int \mathrm{d}\mathcal{P}(\boldsymbol{\sigma},\omega)\ \delta\Big[\mathcal{P}(\boldsymbol{\sigma},\omega) - \frac{1}{N}\sum_i \delta_{\boldsymbol{\sigma},\boldsymbol{\sigma}_i}\delta(\omega-\omega_i)\Big]$$
  $$= \int \frac{\mathrm{d}\mathcal{P}(\boldsymbol{\sigma},\omega)\mathrm{d}\hat{\mathcal{P}}(\boldsymbol{\sigma},\omega)}{2\pi}\mathrm{e}^{\mathrm{i}\hat{\mathcal{P}}(\boldsymbol{\sigma},\omega)\big[\mathcal{P}(\boldsymbol{\sigma},\omega)-\frac{1}{N}\sum_i \delta_{\boldsymbol{\sigma},\boldsymbol{\sigma}_i}\delta(\omega-\omega_i)\big]}$$

  discretise ω,
  $\hat{\mathcal{P}}(\ldots) \to \hat{\mathcal{P}}(\ldots)N\Delta\omega$
  $$1 = \int \frac{\mathrm{d}\mathcal{P}(\boldsymbol{\sigma},\omega)\mathrm{d}\hat{\mathcal{P}}(\boldsymbol{\sigma},\omega)}{2\pi/N\Delta\omega}\mathrm{e}^{\mathrm{i}N\Delta\omega\hat{\mathcal{P}}(\boldsymbol{\sigma},\omega)\big[\mathcal{P}(\boldsymbol{\sigma},\omega)-\frac{1}{N}\sum_i \delta_{\boldsymbol{\sigma},\boldsymbol{\sigma}_i}\delta(\omega-\omega_i)\big]}$$

use $\Delta\omega\sum_\omega \to \int \mathrm{d}\omega$:

$$
\begin{aligned}
1 &= \lim_{\Delta\omega\to 0} \prod_{\omega,\boldsymbol{\sigma}} \int \frac{\mathrm{d}\mathcal{P}(\boldsymbol{\sigma},\omega)\mathrm{d}\hat{\mathcal{P}}(\boldsymbol{\sigma},\omega)}{2\pi/N\Delta\omega} \mathrm{e}^{\mathrm{i}N\Delta\omega\hat{\mathcal{P}}(\boldsymbol{\sigma},\omega)\left[\mathcal{P}(\boldsymbol{\sigma},\omega)-\frac{1}{N}\sum_i \delta_{\boldsymbol{\sigma},\boldsymbol{\sigma}_i}\delta(\omega-\omega_i)\right]} \\
&= \lim_{\Delta\omega\to 0} \int \Big[\prod_{\omega,\boldsymbol{\sigma}} \frac{\mathrm{d}\mathcal{P}(\boldsymbol{\sigma},\omega)\mathrm{d}\hat{\mathcal{P}}(\boldsymbol{\sigma},\omega)}{2\pi/N\Delta\omega}\Big] \mathrm{e}^{\mathrm{i}N\Delta\omega\sum_{\omega,\boldsymbol{\sigma}}\hat{\mathcal{P}}(\boldsymbol{\sigma},\omega)\left[\mathcal{P}(\boldsymbol{\sigma},\omega)-\frac{1}{N}\sum_i \delta_{\boldsymbol{\sigma},\boldsymbol{\sigma}_i}\delta(\omega-\omega_i)\right]} \\
&= \int \{\mathrm{d}\mathcal{P}\mathrm{d}\hat{\mathcal{P}}\} \mathrm{e}^{\mathrm{i}N\sum_{\boldsymbol{\sigma}}\int \mathrm{d}\omega\, \hat{\mathcal{P}}(\boldsymbol{\sigma},\omega)\mathcal{P}(\boldsymbol{\sigma},\omega)-\mathrm{i}\sum_i \hat{\mathcal{P}}(\boldsymbol{\sigma}_i,\omega_i)}
\end{aligned}
$$

with short hand (path integral measure):
$\{\mathrm{d}\mathcal{P}\mathrm{d}\hat{\mathcal{P}}\} = \lim_{\Delta\omega\to 0}\prod_{\omega,\boldsymbol{\sigma}}[\mathcal{P}(\boldsymbol{\sigma},\omega)\mathrm{d}\hat{\mathcal{P}}(\boldsymbol{\sigma},\omega)N\Delta\omega/2\pi]$

result:
factorisation over sites!

$$
\begin{aligned}
\Phi(\beta) &= \frac{1}{N}\log \int \{\mathrm{d}\mathcal{P}\mathrm{d}\hat{\mathcal{P}}\} \mathrm{e}^{\mathrm{i}N\sum_{\boldsymbol{\sigma}}\int \mathrm{d}\omega\, \hat{\mathcal{P}}(\boldsymbol{\sigma},\omega)\mathcal{P}(\boldsymbol{\sigma},\omega)} \\
&\quad \times \mathrm{e}^{\frac{1}{2}\langle k\rangle N\sum_{\boldsymbol{\sigma},\boldsymbol{\sigma}'}\int \mathrm{d}\omega\mathrm{d}\omega'\, \mathcal{P}(\boldsymbol{\sigma},\omega)\mathcal{P}(\boldsymbol{\sigma}',\omega')\mathrm{e}^{-\mathrm{i}(\omega+\omega')}\int \mathrm{d}J\, P(J)\mathrm{e}^{\beta J\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'}} \\
&\quad \times \prod_i \Big\langle \int_{-\pi}^{\pi} \mathrm{d}\omega_i\, \mathrm{e}^{\mathrm{i}\omega_i k_i - \mathrm{i}\hat{\mathcal{P}}(\boldsymbol{\sigma}_i,\omega_i)} \Big\rangle_{\boldsymbol{\sigma}_i}
\end{aligned}
$$

detour ...
ensemble entropy

use

$$p(\mathbf{c}) = \frac{\delta_{\mathbf{k},\mathbf{k}(\mathbf{c})}}{Z} \frac{p_{\rm ER}(\mathbf{c})}{\left(\frac{\langle k\rangle}{N}\right)^{\frac{1}{2}N\langle k\rangle}\left(1-\frac{\langle k\rangle}{N}\right)^{\frac{1}{2}N(N-1)-\frac{1}{2}N\langle k\rangle}}$$

$$\Phi(0) = \log(2\pi) + \frac{1}{2}\langle k\rangle + \frac{1}{N}\log\mathcal{Z}$$

$$0\log 0 = \lim_{\epsilon\downarrow 0}\epsilon\log\epsilon = 0$$

$$\begin{aligned}
S &= -\frac{1}{N}\sum_{\mathbf{c}} p(\mathbf{c})\log p(\mathbf{c}) = -\frac{1}{N}\sum_{\mathbf{c}}\left(\frac{\delta_{\mathbf{k},\mathbf{k}(\mathbf{c})}}{Z}\right)\log\left(\frac{\delta_{\mathbf{k},\mathbf{k}(\mathbf{c})}}{Z}\right) = \frac{1}{N}\log Z \\
&= \frac{1}{N}\log\sum_{\mathbf{c}}\delta_{\mathbf{k},\mathbf{k}(\mathbf{c})}\frac{p_{\rm ER}(\mathbf{c})}{\left(\frac{\langle k\rangle}{N}\right)^{\frac{1}{2}N\langle k\rangle}\left(1-\frac{\langle k\rangle}{N}\right)^{\frac{1}{2}N(N-1)-\frac{1}{2}N\langle k\rangle}} \\
&= \frac{1}{N}\log\mathcal{Z} - \frac{1}{2}\langle k\rangle\log\left(\frac{\langle k\rangle}{N}\right) - \frac{1}{2}[(N-1)-\langle k\rangle]\log\left(1-\frac{\langle k\rangle}{N}\right) \\
&= \frac{1}{N}\log\mathcal{Z} + \frac{1}{2}\langle k\rangle\Big[\log(N/\langle k\rangle)+1\Big] + \mathcal{O}(N^{-1}) \\
&= \Phi(0) - \log(2\pi) + \frac{1}{2}\langle k\rangle\log(N/\langle k\rangle) + \mathcal{O}(N^{-1})
\end{aligned}$$

Saddle-point equations

$$\prod_i \Big\langle \int_{-\pi}^{\pi} \mathrm{d}\omega_i \; \mathrm{e}^{\mathrm{i}\omega_i k_i - \mathrm{i}\hat{\mathcal{P}}(\boldsymbol{\sigma}_i,\omega_i)} \Big\rangle_{\boldsymbol{\sigma}_i} = \mathrm{e}^{\sum_i \log\langle \int_{-\pi}^{\pi} \mathrm{d}\omega \; \mathrm{e}^{\mathrm{i}\omega k_i - \mathrm{i}\hat{\mathcal{P}}(\boldsymbol{\sigma},\omega)}\rangle_{\boldsymbol{\sigma}}}$$

$$= \mathrm{e}^{N\sum_k p(k)\log\langle \int_{-\pi}^{\pi} \mathrm{d}\omega \; \mathrm{e}^{\mathrm{i}\omega k - \mathrm{i}\hat{\mathcal{P}}(\boldsymbol{\sigma},\omega)}\rangle_{\boldsymbol{\sigma}} + \mathcal{O}(\sqrt{N})}$$

now

$$\lim_{N\to\infty} \Phi(\beta) = \lim_{N\to\infty} \frac{1}{N} \log \int \{\mathrm{d}\mathcal{P}\mathrm{d}\hat{\mathcal{P}}\} \; \mathrm{e}^{N\Psi[\mathcal{P}.\hat{\mathcal{P}}]+\mathcal{O}(\sqrt{N})} = \mathrm{extr}_{\{\mathcal{P},\hat{\mathcal{P}}\}} \Psi_\beta[\mathcal{P}.\hat{\mathcal{P}}]$$

$$\Psi_\beta[\mathcal{P},\hat{\mathcal{P}}] = \mathrm{i}\sum_{\boldsymbol{\sigma}} \int \mathrm{d}\omega \; \hat{\mathcal{P}}(\boldsymbol{\sigma},\omega)\mathcal{P}(\boldsymbol{\sigma},\omega) + \sum_k p(k) \log \Big\langle \int_{-\pi}^{\pi} \mathrm{d}\omega \; \mathrm{e}^{\mathrm{i}\omega k - \mathrm{i}\hat{\mathcal{P}}(\boldsymbol{\sigma},\omega)} \Big\rangle_{\boldsymbol{\sigma}}$$
$$+ \frac{1}{2}\langle k\rangle \sum_{\boldsymbol{\sigma},\boldsymbol{\sigma}'} \int \mathrm{d}\omega\mathrm{d}\omega' \; \mathcal{P}(\boldsymbol{\sigma},\omega)\mathcal{P}(\boldsymbol{\sigma}',\omega')\mathrm{e}^{-\mathrm{i}(\omega+\omega')} \int \mathrm{d}J \; P(J)\mathrm{e}^{\beta J \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'}$$

extrema:

$$\frac{\delta\Psi}{\delta\mathcal{P}} = 0: \qquad \mathrm{i}\hat{\mathcal{P}}(\boldsymbol{\sigma},\omega) = -\langle k\rangle \mathrm{e}^{-\mathrm{i}\omega} \sum_{\boldsymbol{\sigma}'} \int \mathrm{d}\omega' \; \mathcal{P}(\boldsymbol{\sigma}',\omega')\mathrm{e}^{-\mathrm{i}\omega'} \int \mathrm{d}J \; P(J)\mathrm{e}^{\beta J\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'}$$

$$\frac{\delta\Psi}{\delta\hat{\mathcal{P}}} = 0: \qquad \mathcal{P}(\boldsymbol{\sigma},\omega) = \sum_k p(k) \frac{\mathrm{e}^{\mathrm{i}\omega k - \mathrm{i}\hat{\mathcal{P}}(\boldsymbol{\sigma},\omega)}}{\sum_{\boldsymbol{\sigma}'} \int_{-\pi}^{\pi} \mathrm{d}\omega' \; \mathrm{e}^{\mathrm{i}\omega' k - \mathrm{i}\hat{\mathcal{P}}(\boldsymbol{\sigma}',\omega')}}$$

- new quantities and short-hands:

$$\mathcal{D}(\boldsymbol{\sigma}) = \int_{-\pi}^{\pi} d\omega\; \mathcal{P}(\boldsymbol{\sigma},\omega)e^{-i\omega}, \qquad \hat{\mathcal{P}}(\boldsymbol{\sigma},\phi) = i\langle k\rangle e^{-i\omega}\gamma(\boldsymbol{\sigma}), \qquad \int dJ\, P(J)\ldots = \langle \ldots \rangle_J$$

new eqns:

$$\gamma(\boldsymbol{\sigma}) = \sum_{\boldsymbol{\sigma}'} \mathcal{D}(\boldsymbol{\sigma}')\langle e^{\beta J \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'}\rangle_J$$

$$\mathcal{D}(\boldsymbol{\sigma}) = \sum_k p(k) \frac{\int_{-\pi}^{\pi} d\omega\; e^{i\omega(k-1)+\langle k\rangle e^{-i\omega}\gamma(\boldsymbol{\sigma})}}{\sum_{\boldsymbol{\sigma}'}\int_{-\pi}^{\pi} d\omega\; e^{i\omega k+\langle k\rangle e^{-i\omega}\gamma(\boldsymbol{\sigma}')}}$$

- do ω-integrals:

$$\begin{aligned}\int_{-\pi}^{\pi} d\omega\; e^{i\omega k+\langle k\rangle e^{-i\omega}\gamma(\boldsymbol{\sigma})} &= \sum_{\ell\geq 0}\frac{\langle k\rangle^\ell \gamma^\ell(\boldsymbol{\sigma})}{\ell!}\int_{-\pi}^{\pi} d\omega\; e^{i\omega k - e\omega\ell} \\ &= 2\pi\sum_{\ell\geq 0}\frac{\langle k\rangle^\ell \gamma^\ell(\boldsymbol{\sigma})}{\ell!}\delta_{k\ell} = \begin{cases} 2\pi\langle k\rangle^k\gamma^k(\boldsymbol{\sigma})/k! & \text{if } k\geq 0 \\ 0 & \text{if } k<0\end{cases}\end{aligned}$$

$$\mathcal{D}(\boldsymbol{\sigma}) = \sum_{k>0} p(k)\frac{k}{\langle k\rangle}\frac{\gamma^{k-1}(\boldsymbol{\sigma})}{\sum_{\boldsymbol{\sigma}'}\gamma^k(\boldsymbol{\sigma}')}, \qquad \gamma(\boldsymbol{\sigma}) = \sum_{\boldsymbol{\sigma}'}\mathcal{D}(\boldsymbol{\sigma}')\langle e^{\beta J\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'}\rangle_J$$

use saddle point eqns:

$$\lim_{N\to\infty}\Phi(\beta) = \text{extr}_{\{\mathcal{P},\hat{\mathcal{P}}\}}\Psi_\beta[\mathcal{P}.\hat{\mathcal{P}}]$$
$$= \log(2\pi) - \frac{1}{2}\langle k\rangle + \sum_k p(k)\log\langle\gamma^k(\boldsymbol{\sigma})\rangle_{\boldsymbol{\sigma}} + \sum_k p(k)\log[\langle k\rangle^k/k!]$$

$\beta = 0$: $\gamma(\boldsymbol{\sigma}) = 1$, $\mathcal{D}(\boldsymbol{\sigma}) = 2^{-n}$

$$\lim_{N\to\infty}\Phi(0) = \log(2\pi) - \frac{1}{2}\langle k\rangle + \sum_k p(k)\log[\langle k\rangle^k/k!]$$

- free energy:

$$-\beta\bar{f} = \log 2 + \lim_{N\to\infty}\lim_{n\to 0} n^{-1}\Big[\Phi(\beta) - \Phi(0)\Big]$$
$$= \lim_{n\to 0}\frac{1}{n}\sum_k p(k)\log\langle\gamma^k(\boldsymbol{\sigma})\rangle_{\boldsymbol{\sigma}}$$

- ensemble entropy:

$$
\begin{aligned}
S &= \Phi(0) - \log(2\pi) + \frac{1}{2}\langle k\rangle \log(\frac{N}{\langle k\rangle}) + \mathcal{O}(N^{-1}) \\
&= \frac{1}{2}\langle k\rangle \log(\frac{N}{\langle k\rangle}) + \frac{1}{2}\langle k\rangle + \sum_k p(k)\log\tilde{p}(k) + \epsilon_N \qquad \tilde{p}(k) = \mathrm{e}^{-\langle k\rangle}\langle k\rangle^k/k!
\end{aligned}
$$

$$
\begin{aligned}
S &= \overbrace{\frac{1}{2}\langle k\rangle \log(N/\langle k\rangle) + \frac{1}{2}\langle k\rangle}^{\text{entropy of ER ensemble}} - \overbrace{\sum_k p(k)\log[p(k)/\tilde{p}(k)]}^{\text{dissimilarity } p(k) \text{ vs } \tilde{p}(k)} \\
&\quad - \overbrace{-\sum_k p(k)\log p(k)}^{\text{entropy of } p(k)} + \epsilon_N
\end{aligned}
$$

Replica symmetric solutions

$\mathcal{D}(\boldsymbol{\sigma})$ invariant under all permutations of $\{1,\ldots,n\}$

$$\mathcal{D}(\boldsymbol{\sigma})=\int\!\mathrm{d}h\,\mathcal{D}(h)\,\frac{\mathrm{e}^{\beta h\sum_\alpha \sigma_\alpha}}{[2\cosh(\beta h)]^n}$$

- work out $\gamma(\boldsymbol{\sigma})$:

$$\begin{aligned}\gamma(\boldsymbol{\sigma}) &= \int\!\frac{\mathrm{d}h\mathrm{d}J\,\mathcal{D}(h)P(J)}{[2\cosh(\beta h)]^n}\sum_{\boldsymbol{\sigma}'}\mathrm{e}^{\beta h\sum_\alpha \sigma'_\alpha+\beta J\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'}\\ &= \int\!\frac{\mathrm{d}h\mathrm{d}J\,\mathcal{D}(h)P(J)}{\cosh^n(\beta h)}\prod_{\alpha=1}^{n}\cosh[\beta(h+J\sigma_\alpha)]\end{aligned}$$

- any $F(\sigma=\pm 1)$:

$$F(\sigma)=a\mathrm{e}^{b\sigma}:\qquad a=\sqrt{F(1)F(-1)},\qquad b=\log\sqrt{F(1)/F(-1)}$$

$$\cosh[\beta(h+J\sigma)]=\sqrt{\cosh[\beta(h+J)]\cosh[\beta(h-J)]}\,\mathrm{e}^{\frac{1}{2}\sigma\log[\cosh[\beta(h+J)]/\cosh[\beta(h-J)]]}$$

- any A,B:

$$\frac{1}{2}\log[\cosh(A+B)/\cosh(A-B)]=\mathrm{atanh}[\tanh(A)\tanh(B)]$$

$$\gamma(\boldsymbol{\sigma}) = \int \mathrm{d}h\, \mathcal{D}(h) \Big\langle \Big[\frac{\sqrt{\cosh[\beta(h+J)]\cosh[\beta(h-J)]}}{\cosh(\beta h)} \Big]^n \mathrm{e}^{\mathrm{atanh}[\tanh(\beta h)\tanh(\beta J)]\sum_\alpha \sigma_\alpha} \Big\rangle_J$$

- let $C_{n,k} = \sum_{\boldsymbol{\sigma}} \gamma^k(\boldsymbol{\sigma})$, claim: $\lim_{n\to 0} C_{n,k} = 1$

$$C_{n,k} = \int \mathrm{d}h\, \mathcal{D}(h) \Big\langle \Big[\frac{\sqrt{\cosh[\beta(h+J)]\cosh[\beta(h-J)]}}{\cosh(\beta h)} \Big]^n \sum_{\boldsymbol{\sigma}} \mathrm{e}^{\mathrm{atanh}[\tanh(\beta h)\tanh(\beta J)]\sum_\alpha \sigma_\alpha} \Big\rangle_J$$

$$= \int \mathrm{d}h\, \mathcal{D}(h) \Big\langle \Big[\frac{\sqrt{\cosh[\beta(h+J)]\cosh[\beta(h-J)]}}{\cosh(\beta h)} \cosh\Big(\mathrm{atanh}[\tanh(\beta h)\tanh(\beta J)]\Big) \Big]^n \Big\rangle_J$$

- remaining eqn:

$$\int \mathrm{d}h\, \mathcal{D}(h) \frac{\mathrm{e}^{\beta h \sum_\alpha \sigma_\alpha}}{[2\cosh(\beta h)]^n} = \sum_{k>0} p(k) \frac{k}{\langle k \rangle C_{n,k}} \gamma^{k-1}(\boldsymbol{\sigma})$$

$$= \sum_{k>0} p(k) \frac{k}{\langle k \rangle C_{n,k}} \int \Big[\prod_{\ell<k} \mathrm{d}h_\ell \mathcal{D}(h_\ell) \Big] \Big\langle \Big[\prod_{\ell<k} \frac{\sqrt{\cosh[\beta(h_\ell+J_\ell)]\cosh[\beta(h_\ell-J_\ell)]}}{\cosh(\beta h_\ell)} \Big]^n$$

$$\times\, \mathrm{e}^{\sum_\alpha \sigma_\alpha \sum_{\ell<k} \mathrm{atanh}[\tanh(\beta h_\ell)\tanh(\beta J_\ell)]} \Big\rangle_{J_1\ldots J_{k-1}}$$

$$\int \mathrm{d}h \, \frac{\mathrm{e}^{\beta h \sum_\alpha \sigma_\alpha}}{[2\cosh(\beta h)]^n} \, \mathcal{D}(h) \; =$$

$$\int \mathrm{d}h \frac{\mathrm{e}^{\beta h \sum_\alpha \sigma_\alpha}}{[2\cosh(\beta h)]^n} \sum_{k>0} p(k) \frac{k}{\langle k \rangle} \frac{[2\cosh(\beta h)]^n}{C_{n,k}} \int \Big[\prod_{\ell<k} \mathrm{d}h_\ell \mathcal{D}(h_\ell) \Big]$$

$$\Big\langle \Big[\prod_{\ell<k} \frac{\sqrt{\cosh[\beta(h_\ell + J_\ell)] \cosh[\beta(h_\ell - J_\ell)]}}{\cosh(\beta h_\ell)} \Big]^n \delta\Big[h - \frac{1}{\beta} \sum_{\ell<k} \mathrm{atanh}[\tanh(\beta h_\ell) \tanh(\beta J_\ell)] \Big] \Big\rangle_{J_1 \ldots J_{k-1}}$$

after $n \to 0$:

$$\mathcal{D}(h) = \sum_{k>0} p(k) \frac{k}{\langle k \rangle} \int \Big[\prod_{\ell<k} \mathrm{d}h_\ell \mathcal{D}(h_\ell) \Big] \Big\langle \delta\Big[h - \frac{1}{\beta} \sum_{\ell<k} \mathrm{atanh}[\tanh(\beta h_\ell) \tanh(\beta J_\ell)] \Big] \Big\rangle_{J_1 \ldots J_{k-1}}$$

interpretation?

- return to $\mathcal{P}(\boldsymbol{\sigma},\omega)$:

$$\mathcal{P}(\boldsymbol{\sigma})=\int_{-\pi}^{\pi}\mathrm{d}\omega\;\mathcal{P}(\boldsymbol{\sigma},\omega)=\lim_{n\to 0}\lim_{N\to\infty}\frac{1}{N}\sum_i\lim_{n\to 0}\overline{\langle\prod_{\alpha\leq n}\delta_{\sigma_\alpha,\sigma_i^\alpha}\rangle}$$

hence

$$m=\lim_{N\to\infty}\frac{1}{N}\sum_i\overline{\langle\sigma_i\rangle}=\lim_{n\to 0}\sum_{\boldsymbol{\sigma}}\mathcal{P}(\boldsymbol{\sigma})\frac{1}{n}\sum_{\alpha=1}^{n}\sigma_\alpha$$

$$q=\lim_{N\to\infty}\frac{1}{N}\sum_i\overline{\langle\sigma_i\rangle^2}=\lim_{n\to 0}\sum_{\boldsymbol{\sigma}}\mathcal{P}(\boldsymbol{\sigma})\frac{1}{n(n-1)}\sum_{\alpha\neq\beta=1}^{n}\sigma_\alpha\sigma_\beta$$

- $\mathcal{P}(\boldsymbol{\sigma})$ in terms of $\mathcal{D}(h)$:

$$\mathcal{P}(\boldsymbol{\sigma})=\sum_k p(k)\frac{\int_{-\pi}^{\pi}\mathrm{d}\omega\;\mathrm{e}^{\mathrm{i}\omega k+\langle k\rangle\mathrm{e}^{-\mathrm{i}\omega}}\gamma(\boldsymbol{\sigma})}{\sum_{\boldsymbol{\sigma}'}\int_{-\pi}^{\pi}\mathrm{d}\omega\;\mathrm{e}^{\mathrm{i}\omega k+\langle k\rangle\mathrm{e}^{-\mathrm{i}\omega}}\gamma(\boldsymbol{\sigma}')}=\sum_k\frac{p(k)}{C_{n,k}}\gamma^k(\boldsymbol{\sigma})$$

$$=\sum_k p(k)\int\Big[\prod_{\ell\leq k}\mathrm{d}h_\ell\mathcal{D}(h_\ell)\Big]\Big\langle\mathrm{e}^{\sum_\alpha\sigma_\alpha\sum_{\ell\leq k}\mathrm{atanh}[\tanh(\beta h_\ell)\tanh(\beta J_\ell)]}\Big[1+\mathcal{O}(n)\Big]\Big\rangle_{J_1\ldots J_k}$$

result:

$$m=\int \mathrm{d}h\, D(h) \tanh(\beta h), \qquad q=\int \mathrm{d}h\, D(h) \tanh^2(\beta h)$$

$$D(h)=\sum_k p(k)\int \Big[\prod_{\ell\leq k} \mathrm{d}h_\ell\, \mathcal{D}(h_\ell)\Big]\Big\langle \delta\Big[h-\frac{1}{\beta}\sum_{\ell\leq k} \mathrm{atanh}[\tanh(\beta h_\ell)\tanh(\beta J_\ell)]\Big)\Big\rangle_{J_1\ldots J_k}$$

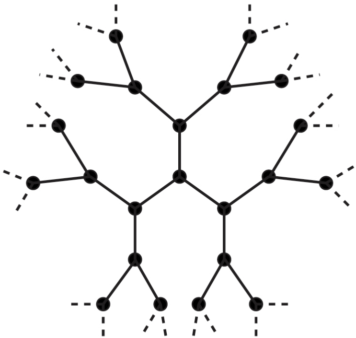
$$\mathcal{D}(h)=\sum_{k>0} p(k)\frac{k}{\langle k\rangle}\int \Big[\prod_{\ell< k} \mathrm{d}h_\ell \mathcal{D}(h_\ell)\Big]\Big\langle \delta\Big[h-\frac{1}{\beta}\sum_{\ell< k} \mathrm{atanh}[\tanh(\beta h_\ell)\tanh(\beta J_\ell)\Big]\Big\rangle_{J_1\ldots J_{k-1}}$$

effective field distr:

$D(h)$: for *true* graph
$\mathcal{D}(h)$: for *cavity* graph

typical form of order parameter eqns
for **locally tree-like** graphs

alternative derivation via
belief propagation or cavity methods



Continuous phase transitions

$$\mathcal{D}(h)=\sum_{k>0}p(k)\frac{k}{\langle k\rangle}\int\Big[\prod_{\ell<k}\mathrm{d}h_\ell\mathcal{D}(h_\ell)\Big]\Big\langle\delta\Big[h-\frac{1}{\beta}\sum_{\ell<k}\mathrm{atanh}[\tanh(\beta h_\ell)\tanh(\beta J_\ell)]\Big]\Big\rangle_{J_1\ldots J_{k-1}}$$

- at $T=\beta^{-1}=\infty$: $\mathcal{D}(h)=\delta(h)$

  gives: $D(h)=\delta(h),\quad m=q=0$

  paramagnetic state,
  is in fact a soln at any β

- bifurcations away from $\delta(h)$:
  expand eqns in width of $\mathcal{D}(h)$

  $\mathcal{D}(h)=\epsilon^{-1}W(h/\epsilon),\quad 0<\epsilon\leq 1$

  bifurcating state always has
  $\int\mathrm{d}h\,P(h)h^2>0$

$$1st\ order\ bifurcation: \quad \delta(h)\ \to\ P(h)\quad with\quad \int\mathrm{d}h\,P(h)h\neq 0$$

$$2nd\ order\ bifurcation: \quad \delta(h)\ \to\ P(h)\quad with\quad \int\mathrm{d}h\,P(h)h=0$$

- first order:

$$\int \mathrm{d}h\, \mathcal{D}(h)h = \sum_{k>0} p(k)\frac{k}{\langle k\rangle}\int \Big[\prod_{\ell<k} \mathrm{d}h_\ell\, \mathcal{D}(h_\ell)\Big]\Big\langle \frac{1}{\beta}\sum_{\ell<k} \mathrm{atanh}[\tanh(\beta h_\ell)\tanh(\beta J_\ell)\Big\rangle_{J_1\ldots J_{k-1}}$$

$$\epsilon \int \mathrm{d}x\; W(x)x = \frac{1}{\beta}\sum_{k>0} p(k)\frac{k(k-1)}{\langle k\rangle}\int \mathrm{d}x\; W(x)\Big\langle \mathrm{atanh}[\tanh(\beta\epsilon x)\tanh(\beta J)\Big\rangle_J$$

$$= \epsilon \sum_{k>0} p(k)\frac{k(k-1)}{\langle k\rangle}\int \mathrm{d}x\; W(x)x\langle \tanh(\beta J)\rangle_J + \mathcal{O}(\epsilon^2)$$

bifurcation of ferromagn state
with $m, q \neq 0$ at:

$$P \to F: \qquad (\langle k^2\rangle/\langle k\rangle - 1)\int \mathrm{d}J\; P(J)\tanh(\beta J) = 1$$

- second order:

$$\int \mathrm{d}h\, \mathcal{D}(h) h^2 = \sum_{k>0} p(k) \frac{k}{\langle k \rangle} \int \Big[\prod_{\ell<k} \mathrm{d}h_\ell\, \mathcal{D}(h_\ell) \Big] \Big\langle \Big(\frac{1}{\beta} \sum_{\ell<k} \mathrm{atanh}[\tanh(\beta h_\ell) \tanh(\beta J_\ell) \Big)^2 \Big\rangle_{J_1 \ldots J_{k-1}}$$

$$\epsilon^2 \int \mathrm{d}x\; W(x) x^2 = \frac{1}{\beta^2} \sum_{k>0} p(k) \frac{k}{\langle k \rangle} \Big\{ (k-1) \int \mathrm{d}x\; W(x) \Big\langle \mathrm{atanh}^2[\tanh(\beta \epsilon x) \tanh(\beta J) \Big\rangle_J$$

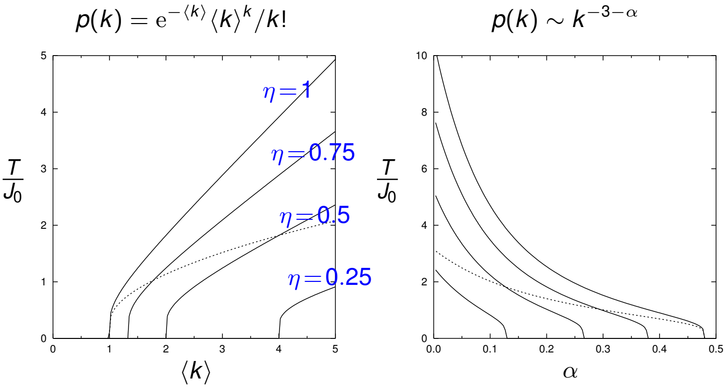
$$+ (k-1)(k-2) \Big[\int \mathrm{d}x\; W(x) \Big\langle \mathrm{atanh}[\tanh(\beta \epsilon x) \tanh(\beta J) \Big\rangle_J \Big]^2 \Big\}$$

$$= \epsilon^2 \sum_{k>0} p(k) \frac{k(k-1)}{\langle k \rangle} \int \mathrm{d}x\; W(x) x^2 \langle \tanh^2(\beta J) \rangle_J + \mathcal{O}(\epsilon^3)$$

bifurcation of spin-glass state
with $m = 0$, $q \neq 0$ at:

$$P \to SG: \qquad (\langle k^2 \rangle / \langle k \rangle - 1) \int \mathrm{d}J\; P(J) \tanh^2(\beta J) = 1$$

Phase diagrams



$P(J) = \frac{1}{2}(1+\eta)\delta(J-J_0) + \frac{1}{2}(1-\eta)\delta(J+J_0)$
with $J_0 \geq 0$

$$\textit{solid lines}: \quad P \to F, \quad \eta\,(\langle k^2\rangle - \langle k\rangle)\tanh(\beta J_0) = 1$$
$$\textit{dashed line}: \quad P \to SG, \quad (\langle k^2\rangle - \langle k\rangle)\tanh^2(\beta J_0) = 1$$

Further steps

- Graphs with controlled $p(k)$ and controlled $W(k, k')$
- Replica symmetry breaking
- Dynamics of spins on tailored random graphs
- Spins on 'small world' graphs
- Fast spins and slowly evolving graphs

some references

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