Inner models from Boolean valued higher-order logics and Ω-logic

Daisuke Ikegami
Tokyo Denki University

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From now on...

We work in ZFC.

Gödel's Constructible Hierarchy

Definition

$$
\begin{aligned}
L_0 &= \emptyset, \\
L_{\alpha+1} &= \mathrm{Def}_{\mathrm{FOL}}\big((L_\alpha, \in)\big), \\
L_\gamma &= \bigcup_{\alpha<\gamma} L_\alpha \qquad (\gamma \text{ is limit}), \\
L &= \bigcup_{\alpha\in\mathrm{On}} L_\alpha.
\end{aligned}
$$

Inner models from logics

Definition

Given a logic $\mathcal{L}$ extending FOL with a definability notion,

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\mathrm{L}_0(\mathcal{L}) &= \emptyset, \\
\mathrm{L}_{\alpha+1}(\mathcal{L}) &= \mathsf{Def}_{\mathcal{L}}\big((\mathrm{L}_\alpha(\mathcal{L}), \in)\big), \\
\mathrm{L}_\gamma(\mathcal{L}) &= \bigcup_{\alpha<\gamma} \mathrm{L}_\alpha(\mathcal{L}) \quad (\gamma \text{ is limit}), \\
\mathrm{L}(\mathcal{L}) &= \bigcup_{\alpha\in\mathsf{On}} \mathrm{L}_\alpha(\mathcal{L}).
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Question

What is $\mathrm{L}(\mathcal{L})$ if $\mathcal{L}$ is full 2nd-order logic (SOL)?

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Answer

HOD!

What is HOD?

HOD is the class of all hereditarily ordinal definable sets:

- $x \in \text{HOD}$ if every element of $\text{tr.cl.}(\{x\})$ is 1st-order definable in V with an ordinal parameter α.

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- HOD is the largest transitive proper class s.t. every set in the model is ordinal definable in V.

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- $x \in \mathrm{HOD}$ if every element of tr.cl.$(\{x\})$ is 1st-order definable in V with an ordinal parameter α.
- HOD is a transitive model of ZFC.
- HOD is the largest transitive proper class s.t. every set in the model is ordinal definable in V.
- HOD is very "non-absolute" (e.g., for any real x, there is a poset P such that "$x \in \mathrm{HOD}$" holds in V^P).
- One cannot decide e.g., whether HOD satisfies CH.

Inner models from logics ctd.

Question

What if $\mathcal{L}$ is full n-th order logic for $n \geq 3$?

Inner models from logics ctd.

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Answer

It is the same as HOD.

Inner models from logics ctd.

Question

What about other logics?

Kennedy, Magidor, and Väänänen explored on inner models from first order logic with "generalized quantifiers".

Inner models from logics ctd.

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In this talk, we will discuss inner models from Boolean valued higher order logics and Woodin's Ω-logic. These logics are all dealing with "generic absoluteness".

Inner models from logics: Goal & Motivation

Goal

Construct a model of set theory which is "close to" HOD but easier to analyze.

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Theorem (Woodin)

Let κ be extendible. Then exactly one of the following holds:

1. for every regular $\gamma > \kappa$, γ is measurable in HOD, OR
2. for every singular cardinal $\gamma > \kappa$, γ is singular in HOD and $(\gamma^+)^{\mathrm{HOD}} = \gamma^+$.

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Definition (Woodin)

HOD Conjecture states that the latter case in the above theorem holds.

Inner models from logics: Motivation ctd.

1. HOD Conjecture is connected to the Inner Model Program.
2. HOD Conjecture has an application to the problem on the existence of Reinhardt cardinals in ZF.

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To solve HOD Conjecture, one could expect a fine analysis of HOD. But HOD is very "non-absolute", e.g.,

Proposition (Folklore?)

For any real x, there is a partial order P such that "$x \in \mathrm{HOD}$" in V^P.

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Proposition (Folklore?)

For any real x, there is a partial order P such that “$x \in \mathrm{HOD}$” in V^P.

Question

Can one construct a model of set theory which is “close to” HOD, but invariant under forcing extensions?

Boolean valued 2nd-order logic: background

Two semantics for 2nd-order logic:

1. Full semantics: Highly complex (very powerful), does not enjoy completeness, ω-compactness.
2. Henkin semantics: Very simple (very week), enjoys completeness, ω-compactness.

Boolean valued 2nd-order logic: background

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1. Full semantics: Highly complex (very powerful), does not enjoy completeness, ω-compactness.
2. Henkin semantics: Very simple (very week), enjoys completeness, ω-compactness.

Boolean valued second order logic is a powerful logic sitting between the two semantics.

Boolean valued 2nd-order logic: Boolean valued structures

Definition

Let $\mathcal{L}$ be a relational language. A Boolean valued $\mathcal{L}$-structure is a tuple $M = (A, \mathbb{B}, \{R_i^M\})$ where

1. A is a nonempty set,
2. $\mathbb{B}$ is a complete Boolean algebra, and
3. for each n-ary relational symbol R_i in $\mathcal{L}$, $R_i^M \colon A^n \to \mathbb{B}$.

Boolean valued 2nd-order logic: Boolean valued structures

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2. $\mathbb{B}$ is a complete Boolean algebra, and
3. for each n-ary relational symbol R_i in $\mathcal{L}$, $R_i^M \colon A^n \to \mathbb{B}$.

Example

If $\mathbb{B} = \{0, 1\}$, each R_i^M is a relation in 1st-order logic and M is the same as a 1st-order structure.

Truth of 2nd-order formulas in Boolean valued structures

Basic idea: "subsets" are functions from A to $\mathbb{B}$.

Definition

Let $M = (A, \mathbb{B}, \{R_i\})$ be a Boolean valued $\mathcal{L}$-structure. Then we assign $\|\phi[\vec{a}, \vec{f}]\|^M \in \mathbb{B}$ to each 2nd-order formula ϕ, $\vec{a} \in {}^{<\omega}A$, and $\vec{f} \in {}^{<\omega}({}^{A}\mathbb{B})$ as follows:

1. ϕ is $R_i(\vec{x})$. Then $\|R_i(\vec{x})[\vec{a}]\|^M = R_i^M(\vec{a})$.
2. ϕ is $X(x)$. Then $\|X(x)[a, f]\|^M = f(a)$.
3. Boolean combinations are as usual.
4. ϕ is $\exists x\psi$. Then $\|\exists x\psi[\vec{a}, \vec{f}]\|^M = \bigvee_{b\in A} \|\psi[b, \vec{a}, \vec{f}]\|^M$.
5. ϕ is $\exists X\psi$. Then $\|\exists X\psi[\vec{a}, \vec{f}]\|^M = \bigvee_{g:\, A\to\mathbb{B}} \|\psi[\vec{a}, g, \vec{f}]\|^M$.

Boolean valued 2nd-order logic: Boolean-validity

Definition

Let $\mathcal{L}$ be relational. A 2nd-order $\mathcal{L}$-sentence ϕ is Boolean-valid if $\|\phi\|^M = 1$ for any Boolean valued $\mathcal{L}$-structure M.

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Definition

Let $\mathcal{L}$ be relational. A 2nd-order $\mathcal{L}$-sentence ϕ is Boolean-valid if $\|\phi\|^M = 1$ for any Boolean valued $\mathcal{L}$-structure M.

Lemma

A 2nd-order $\mathcal{L}$-sentence ϕ is Boolean-valid iff for any *1st-order* $\mathcal{L}$-structure M, a partial order $\mathbb{P}$, and a $\mathbb{P}$-generic filter over V, $\left(M, \mathcal{P}(M)^{V[G]}\right) \models \phi$.

Boolean valued 2nd-order logic: Definability

Definition

1. For a set A, $\vec{a} \in A^{<\omega}$, and a second order formula ϕ, $(\phi, \vec{a})$ is suitable to A if for every element x of A, either $\phi[x, \vec{a}]$ or $\neg\phi[x, \vec{a}]$ is Boolean valid with the first order universe A.

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2. Let $(\phi, \vec{a})$ be suitable to A. Then a set $X \subseteq A$ is BVSOL-definable via $(\phi, \vec{a})$ if X is the collection of $x \in A$ such that $\phi[x, \vec{a}]$ is Boolean valid with the first order universe A.

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3. $\mathrm{Def}_{2b}(A)$ is the collection of BVSOL-definable subsets of A via some $(\phi, \vec{a})$ suitable to A.

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One can introduce the constructible hierarchy & universe w.r.t. BVSOL. We write L^{2b}_α and L^{2b} for those.

Inner models from logics: L^{2b}

Remark

L^{2b} is a transitive proper class model of ZF.

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Question

Is L^{2b} a model of AC when there are a proper class of Woodin cardinals?

Inner models from logics: L^{2b} ctd.

Theorem

If there are proper class many Woodin cardinals, then one can show that

$$(\omega, \mathcal{P}(\omega), \in)^{\mathrm{L}^{2b}} \prec (\omega, \mathcal{P}(\omega), \in)^{V}$$

In particular, Projective Determinacy holds in L^{2b}.

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Point: Using large cardinals and projective uniformization, one can prove the following:
For each 2nd-order formula ϕ for the 2nd-order arithmetic, there is a Skolem function f for ϕ such that

1. f is definable in the 2nd-order arithmetic, and
2. f is invariant under set forcings.

Inner models from logics: L^{2b} ctd..

Theorem

Suppose there are a proper class of Woodin cardinals. Then L^{2b} is invariant under set forcings, i.e., for any poset P, $(\mathrm{L}^{2b})^V = (\mathrm{L}^{2b})^{V^P}$.

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Theorem

Suppose there are a proper class of Woodin cardinals. Then L^{2b} is invariant under set forcings, i.e., for any poset P, $(\mathrm{L}^{2b})^V = (\mathrm{L}^{2b})^{V^P}$.

To show: for any poset P and any set A, $Def_{2b}(A) = Def_{2b}^{V^P}(A)$.

$\subseteq$: Easy.

$\supseteq$: Use stationary tower forcings (blackboard?).

Inner models from logics: L^{nb}

One can define L^{nb} for $n \geq 3$ in the same way as L^{2b}.

Remark

L^{nb} is a transitive proper class model of ZF and it is invariant under set forcing extensions.

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One can define L^{nb} for $n \geq 3$ in the same way as L^{2b}.

Remark

L^{nb} is a transitive proper class model of ZF and it is invariant under set forcing extensions.

Question

What are the relationships between L^{mb} and L^{nb} for different m and n?

Main Results: Vague statements

Let L^Ω be the inner model from Woodin's Ω-logic.

Theorem

Under some assumptions on large cardinals and Woodin's Ω-logic,

1. $$\mathrm{L}^{2b} \subsetneq \mathrm{L}^{3b} = \mathrm{L}^{4b} = \cdots = \mathrm{L}^{nb} = \cdots = \mathrm{L}^{\Omega}.$$

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2. The model L^{Ω} is a transitive model of ZFC+GCH.
3. The model L^{Ω} is "very big" w.r.t. inner model theory & descriptive set theory.

Main Results ctd.: Precise statements

Theorem

Suppose that there are proper class many Woodin cardinals. Assume that both the Ω-Conjecture with real parameters and the AD^+-Conjecture hold in any set generic extention. Then

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2. L^{Ω} is a transitive model of ZFC+GCH.
3. The reals in L^{Ω} are exactly those which are Δ^2_1(uB) in a countable ordinal.
4. L^{Ω} is *A*-closed for any universally Baire set *A* which is Δ^2_1(uB).

Background: Ω-logic

Ω-logic: a logic on generic absoluteness

Definition (Ω-validity)

Let ϕ be a Π_2-sentence with a real parameter in set theory.
Then ϕ is *Ω-valid* if ϕ is true in any set forcing extension.

$$\text{Main interest: } 0^\Omega = \{\phi \mid \phi \text{ is } \Omega\text{-valid}\}.$$

Background: Ω-logic ctd.

Example

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3. (Martin-Solovay) Assume $X^{\#}$ exists for any set X. Then every Π^1_3-sentence true in V is Ω-valid.

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3. (Martin-Solovay) Assume $X^{\#}$ exists for any set X. Then every Π^1_3-sentence true in V is Ω-valid.
4. (Woodin) If there are proper class many Woodin cardinals, then for any sentence ϕ true in $\mathrm{L}(\mathbb{R})^V$, $\phi^{\mathrm{L}(\mathbb{R})} \in 0^{\Omega}$.

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Example

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3. (Martin-Solovay) Assume $X^{\#}$ exists for any set X. Then every Π^1_3-sentence true in V is Ω-valid.
4. (Woodin) If there are proper class many Woodin cardinals, then for any sentence ϕ true in $\mathrm{L}(\mathbb{R})^V$, $\phi^{\mathrm{L}(\mathbb{R})} \in 0^\Omega$.
5. (Steel) PFA implies the same as in the item 4.

Strong axioms of infinity give us more statements in 0^Ω.

Background: Universally Baire sets

We will introduce the notion of Ω-provability using universally Baire sets:

Definition

A set of reals A is universally Baire if for any continuous function f from a compact Hausdorff space X to the reals, $f^{-1}(A)$ has the property of Baire in X.

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A set of reals A is universally Baire if for any continuous function f from a compact Hausdorff space X to the reals, $f^{-1}(A)$ has the property of Baire in X.

Remark

A set of reals A is universally Baire if and only if for any partial order P, there are trees T, U on $\omega \times Y$ for some Y such that

$$A = \mathrm{p}[T] \text{ and } \Vdash_P \text{“}\mathrm{p}[\check{T}] = \mathbb{R} \setminus \mathrm{p}[\check{U}]\text{”}.$$

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Using this fact and the trees, one can canonically interpret a uB set A in a set generic extension $V[G]$ (namely $\mathrm{p}[T]$ in $V[G]$). We write A_G for this interpreted set in $V[G]$.

Background: Universally Baire sets ctd.

Example

1. The collection of all uB sets is closed under complements and countable unions, hence every Borel set is universally Baire.

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Background: Universally Baire sets ctd.

Example

1. The collection of all uB sets is closed under complements and countable unions, hence every Borel set is universally Baire.
2. Every $\boldsymbol{\Pi}^1_1$-set of reals is universally Baire.
3. The following are equivalent:
   1. every $\boldsymbol{\Pi}^1_2$-set of reals is universally Baire,
   2. every set has a sharp.

Background: Closure under universally Baire sets

Definition (A-closure)

Let A be universally Baire. An ω-model M of ZFC is A-closed if for any V-generic filter G on a partial order in M,

$$M[G] \cap A_G \in M[G].$$

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Example

1. For an ω-model M of ZFC, the following are equivalent:
   1. M is A-closed for any Π^1_1-set A, and
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2. For an ω-model M of ZFC, the following are equivalent:
   1. M is A-closed for every Π^1_2-set A, and
   2. M is closed under sharps.

Background: Ω-provability

Definition

Let ϕ be a Π_2-sentence with a real parameter in set theory.
Then ϕ is Ω-provable if there is a universally Baire set A such that

$$(\forall M \text{ c.t.m. of ZFC}) \text{ if } M \text{ is } A\text{-closed, then } M \models \phi.$$

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Example

Suppose every set has a sharp. Then any Π^1_3-sentence true in V is Ω-provable.

Background: Ω-Conjecture

Definition

Ω-Conjecture with real parameters states that ϕ is Ω-valid iff ϕ is Ω-provable for all ϕ.

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Remark

Ω-Conjecture with real parameters holds in any set generic extension of a canonical inner model with a proper class of Woodin cardinals.

Background: the effect of Ω-Conjecture

With Ω-Conjecture, one can reduce an Ω-valid Π_2 statement to a Σ^2_1 (uB) statement:

Definition

1. A formula ϕ is Σ^2_1(uB) if it is of the form

$$(\exists A\colon \text{universally Baire})\ (H_{\omega_1}, \in, A) \models \psi,$$

   where ψ is a first order formula.
2. A set $A \subseteq H_{\omega_1}$ is Δ^2_1(uB) if both A and its complement are defined by a Σ^2_1(uB) formula.
3. A set $A \subseteq H_{\omega_1}$ is Δ^2_1(uB) in a countable ordinal if there is a countable ordinal α such that A is Δ^2_1(uB) with parameter α.

Background: $\Delta^2_1(\mathrm{uB})$

Remark

1. All the reals in the mice known to exist so far are $\Delta^2_1(\mathrm{uB})$ in a countable ordinal.

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Remark

1. All the reals in the mice known to exist so far are $\Delta^2_1(\mathrm{uB})$ in a countable ordinal.
2. If M is A-closed for every A which is universally Baire and $\Delta^2_1(\mathrm{uB})$, then M is closed under all the mouse operators known to exist so far.

Background: AD^+-Conjecture

We would like to make Ω-valid statements definable in $H_{\mathfrak{c}^+}$. So we need:

Definition

AD^+-Conjecture states the following:
Suppose A, B are sets of reals such that $\mathrm{L}(A, \mathbb{R})$ and $\mathrm{L}(B, \mathbb{R})$ are models of AD^+.
Assume also that every set of reals in $\mathrm{L}(A, \mathbb{R}) \cup \mathrm{L}(B, \mathbb{R})$ is ω_1-universally Baire.
Then either $\boldsymbol{\Delta}_1^{2\,\mathrm{L}(A,\mathbb{R})} \subseteq \boldsymbol{\Delta}_1^{2\,\mathrm{L}(B,\mathbb{R})}$ or vice versa.

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Theorem (Woodin)

1. Suppose there are proper class many Woodin cardinals and assume that AD^+-Conjecture holds. Then the set of Ω-provable statements is definable in $H_{\mathfrak{c}^+}$.
2. MM implies that AD^+-Conjecture holds.

Inner models from logics: Back to Main Results...

Theorem

Suppose there are a proper class of Woodin cardinals. Assume that the Ω-Conjecture with real parameters and AD^+-Conjecture hold in any set generic extension. Then

$$\mathrm{L}^{3b} = \mathrm{L}^{4b} = \cdots = \mathrm{L}^{nb} = \cdots .$$

For the proof, we introduce L^{Ω} from Ω-logic and show that $Def_{3b} = Def_{\Omega}$.

Inner models from logics: Def_Ω

Definition

Let ϕ be a Σ_2 formula and ψ be a Π_2 formula in the language of set theory. We say (ϕ, ψ) is a Δ_2^{ZFC}-pair if

$$\mathsf{ZFC} \vdash \text{“}(\forall \vec{x})\ \phi(\vec{x}) \leftrightarrow \psi(\vec{x})\text{”}.$$

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Definition

Let A be a first-order structure, $\vec{a} \in A^{<\omega}$, and (ϕ, ψ) be a Δ_2^{ZFC}-pair. Then the triple $(\phi, \psi, \vec{a})$ is suitable to A if for any element x of A, either $\psi[x, \vec{a}, A]$ or $\neg\phi[x, \vec{a}, A]$ is Ω-valid.

Inner models from logics: L^{Ω}

Definition

1. Let $(\phi, \psi, \vec{a})$ be suitable to A. Then a set $X \subseteq A$ is Ω-definable via $(\phi, \psi, \vec{a})$ if $X = \{x \in A \mid (\forall P : \text{ poset})\ V^P \vDash \phi[x, \vec{a}, A]\}$.
2. $\mathrm{Def}_{\Omega}(A)$ is the collection of Ω-definable subset of A via some $(\phi, \psi, \vec{a})$ suitable to A.

One can define $\mathrm{L}^{\Omega}_{\alpha}$ and L^{Ω} in the same way as before.

Inner models from logics: L^Ω

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One can define L^Ω_α and L^Ω in the same way as before.

Remark

L^Ω is a transitive proper class model of ZF and it is invariant under forcing extensions.

Inner models from logics: Proof of Theorem

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To show: $Def_{3b}(A) = Def_{\Omega}(A)$ for any A.

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To show: $Def_{3b}(A) = Def_{\Omega}(A)$ for any A.
$\subseteq$: Easy.
$\supseteq$: If X is in $Def_{\Omega}(A)$, then there are $\vec{a}$, ϕ, and ψ witnessing it.
Using Ω-Conjecture with real parameters & AD^+-Conjecture in $V^{\mathrm{Coll}(\omega,A)}$, one can argue that the Ω-validity of $\phi[x,\vec{a}]$ and $\psi[x,\vec{a}]$ (for $x \in A$) can be expressed as 3rd-order Boolean validity with 1st-order universe A in $V^{\mathrm{Coll}(\omega,A)}$.

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Since Def_{3b} and Def_{Ω} are invariant under forcings, one can argue that there is a 3rd-order formula defining X in BVTOL with 1st-order universe A.

Inner models from logics: L^Ω is "large"

Theorem

Suppose there are proper class many Woodin cardinals and assume that the Ω-Conjecture with real parameters holds in any set generic extension. Then

1. L^Ω is A-closed for every A which is $\Delta^2_1(\mathrm{uB})$ and universally Baire,

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1. L^Ω is A-closed for every A which is $\Delta^2_1(\mathrm{uB})$ and universally Baire,
2. in V, there is a $\Delta^2_1(\mathrm{uB})$ well-ordering on the reals in L^Ω,

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3. the Axiom of Choice holds in L^Ω,
4. the reals in L^Ω are exactly those which are $\Delta^2_1(\mathrm{uB})$ in a countable ordinal, and

Inner models from logics: L^{Ω} is "large"

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5. L^{Ω} satisfies GCH.

Summary

Theorem

Under some assumptions on large cardinals and Ω-logic,

1. $$\mathrm{L}^{2b} \subseteq \mathrm{L}^{3b} = \mathrm{L}^{4b} = \cdots = \mathrm{L}^{nb} = \cdots = \mathrm{L}^{\Omega}.$$
2. L^{Ω} is a transitive proper class model of ZFC + GCH invariant under forcing extensions.
3. L^{Ω} is "large" w.r.t. inner model theory & descriptive set theory.

Questions

What kind of large cardinals could exist in L^{Ω}?

Conjecture

There is NO measurable cardinal in L^{Ω}.

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Does $(\mathrm{L}^{\Omega}_{\alpha} \mid \alpha \in \mathrm{On})$ have some kind of condensation property?