Deriving Closed Form Solutions for the Material Wait time Problem

Strategic Planning and Modeling*

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Problem Statement

In many cases we want to calculate the expected wait time to build, or to ship, a product when not all components, or order line items, are immediately available. This problem has been originally solved in the context of System Response Time [1]. The problem studied by Kruger [1] is formulated as follows: What is the appropriate Safety Stock $\boldsymbol{SS}$ to achieve type I service level $\boldsymbol{SL}$, when the demand information is known $\boldsymbol{x}$ periods ahead of the actual demand realization (actual shipment). This study builds on this logic to answer the question: *'What is the distribution of the wait time $\boldsymbol{X}$ given an immediate availability (Service Level) of $\boldsymbol{SL}$'*

Time series Formulation

Initially we consider a demand process simply modeled as a time series $\boldsymbol{D(t)}$. The series is assumed to be stationary and IID from a distribution with mean and standard deviation $\boldsymbol{\mu}, \boldsymbol{\sigma}$ respectively. Because of stationarity we can drop the functional $\boldsymbol{t}$. Later in this analysis we will change the demand model to be a random process. We assume that demand is fulfilled by pulling from an inventory of parts or FGI. Care needs to be taken in properly offsetting the build time from the material wait time in case of parts pull. The inventory is managed through a *Base Stock Policy* (or commonly known as *Periodic Review Order Up To Point* policy with review period $\boldsymbol{R}$). The replenishment action has a stochastic Lead Time $\boldsymbol{L(t)}$ which is stationary (again because of

*The content of this document is drawn from the *SKU reduction* and *Order prioritization* projects 2003 with Claude and SteveK stationarity we drop the functional t)and IID from a distribution with mean and standard deviation $\boldsymbol{l, s}$. We want to calculate the probability $\boldsymbol{P(t \leq T)}$ where $\boldsymbol{T}$ is the random variable describing the time an order has to wait until material becomes available if the required material (components or line items) are not immediately available. From now on when referring to material we are interchangeably referring to either components or line items.

From Kruger [1] we are able to solve for the required safety stock in inventory

$$SS(t, SL) = z_{SL} \sqrt{\sigma^2 (l + R - t) + s^2 \mu^2} - t \ \mu \tag{1}$$

In this formulation we see that the solution is given assuming normally distributed demand over lead time plus review period. If the normality assumption is relaxed and the formulation is changed to reflect the total inventory level (or in other words the Order Up to Point) then we get:

$$I(t, SL) = \mathbb{F}^{-1}_{(l+R-t)\mu, \sqrt{\sigma^2 \left((l+R-t) \wedge 0\right) + s^2 \mu^2}} (SL) \tag{2}$$

and the solution can be numerically calculated, depending on the distribution $\mathbb{F}$ of demand over lead time plus review period. Of course we may need to transform the mean

$$(l + R - t)\mu \tag{3}$$

and standard deviation

$$\sqrt{\sigma^2 \left((l + R - t) \wedge 0\right) + s^2 \mu^2}, \tag{4}$$

given in the above formulation, to the appropriate parameters of the distribution. For example, if we assume that $\mathbb{F}$ is Gamma distributed with parameters a and b then formula (2) becomes:

$$\begin{aligned} I(t, SL) =& \mathbb{F}^{-1}_{a,b}(SL) && \text{with} \\ a =& \left(\frac{(l + R - t)\mu}{\sqrt{\sigma^2 \left((l + R - t) \wedge 0\right) + s^2 \mu^2}} \right)^2 && \text{and} \\ b =& \frac{\left(\sqrt{\sigma^2 \left((l + R - t) \wedge 0\right) + s^2 \mu^2} \right)^2}{(l + R - t)\mu} \end{aligned}$$

Another way to justify formula (2) is to derive it directly from the formulas for the mean and variance of the marginal distribution of a random sum of random variables: $Var(I) = E(L+R)Var(D) + Var(L+R)[E(D)]^2$ and $E(I) = E(L+R)E(D)$ with R the review period, D the demand random variable with mean μ and standard deviation of σ, L the lead time with mean l and standard deviation of s and I the distribution of the demand over the exposure period $L+R$. We can see that taking away t period of the exposure $L+R$ reduces the marginal distribution of the random sum to the quantities: $Var\big(I(t)\big) = E(L+R-t)Var(D) + Var(L+R-t)[E(D)]^2$ and $E\big(I(t)\big) = E(L+R-t)E(D)$ which leads to the pdf $\boldsymbol{f}_{(l+R-t)\mu,\sqrt{\sigma^2\big((l+R-t)\wedge 0\big)+s^2\mu^2}}$. The symbol $a \wedge b$ means maximum of a and b. Having this, we can estimate the inventory position (or Order Up To Point), such as to ensure with probability SL that we will fulfill the demand in t periods from when the demand is realized by taking the inverse as described in formula (2).

In many cases we are interested not to calculate the required inventory as a function of the immediate service level and wait time, but to calculate the distribution of wait times as function of the immediate service level derived from a given inventory level. This situation is more often appearing in situations were the inventory investment is probably less negotiable than the wait time.

In this context from equation (1) we can derive the following formula for the probability $\boldsymbol{P(t \leq T)}$ where $\boldsymbol{T}$ is the random variable describing the time an order has to wait until material becomes available if the required material (components or line items) are not immediately available.

$$P(t \leq T|SL) = 1 - \mathbb{F}_{(l+R)\mu,\sqrt{\sigma^2(l+R)+s^2\mu^2}}\big(I(T, 1-SL)\big) \tag{5}$$

To see how this formula can be derived, first lets discuss what is the meaning of the expression (2). Every time period, past the realization of the demand, introduces a new distribution, conditioned to the event that one more period has elapsed (since the demand occurred), by revealing the new arrivals of the pipeline demand. In other words, every period that occurs, after the moment the demand occurred, changes the *posterior* distribution of the demand over the exposure period by reducing the *posterior* lead time exposure by one unit. Let's name this distribution *Posterior Demand Distribution*(PDD). The probability density function of PDD is given by the marginal distribution of the random sum of random variable $\boldsymbol{f}_{(l+R-t)\mu,\sqrt{\sigma^2\big((l+R-t)\wedge 0\big)+s^2\mu^2}}$. Given that time t has elapsed since the demand occurred, the immediate Service Level SL is a posterior probability expressed

by the PDD distribution. A point D on the PDD distribution where $P(d > D) = 1 - SL$ is such that the probability of a demand d occurring as a RV from the PDD distribution to be greater than D is $1-SL$. This is the level of PDD that was not fulfilled immediately with probability $1-SL$ and is covered by the expression $D = I(t, 1-SL) = \mathbb{F}^{-1}_{(l+R-t)\mu, \sqrt{\sigma^2\left((l+R-t)\wedge 0\right)+s^2\mu^2}}(1-SL)$
The probability of satisfying the PDD unfulfilled demand D up to time t is equal to the probability that the demand distribution over the lead time is greater than the demand D on the PDD distribution, which leads to formula (5). As before, the formulation presented here assumes normally distributed $\mathbb{F}$ but one can use the same approach for other distributions, as demonstrated for equation (2). In general formula (5) can be expressed as:

$$P(t \leq T|SL) = 1 - \mathbb{F}_{\alpha,\beta,\ldots}\left(I_{\alpha',\beta',\ldots}(T, 1-SL)\right)$$

One inherent assumption here is that we order to replenish the demand therefore the distribution of the pipeline demand is equal to the distribution of the demand during the exposure period. Of course deviation from this assumption resulting from forecast inaccuracies and operational deviations from the Order Up to Point planning logic may result in reality discrepancies.

By combining equations (2) and (5) we can calculate the distribution of the material wait time as a function of the inventory investment driven by the desired immediate availability. In figure (1) we plot the material wait times distributions for five different service levels.

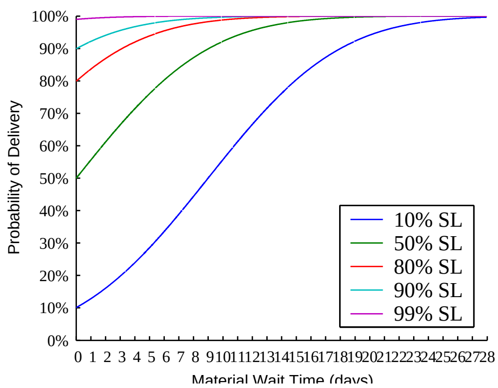

Figure 1: Material wait time from a normal distribution of $\mu = 10, \sigma = 2, l = 14, s = 7$

Another sensitivity, that it is good to know, is of the variability of lead time. Figure 2 plots the distributions of the material wait time under 50%

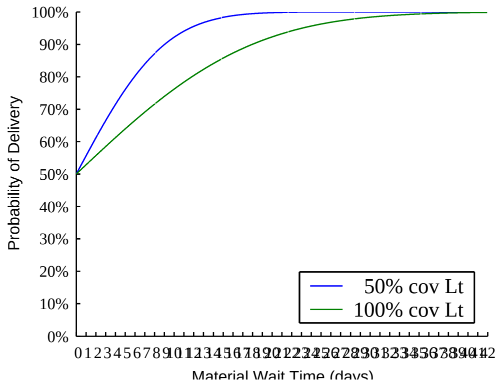

Figure 2: $\mu = 10, \sigma = 2, l = 14, s = [7, 14]$

and 100% COV of lead time. We can observe that the material wait time distribution is sensitive to lead time variability. *Extra care should be taken when the leat time is not known and/or variable*

Demand Process Formulation

The formulation described in the previous section dealt with answering the questions of *"how much inventory given a SRT"* and *"what material wait time distribution given an inventory investment"*. These questions were answered having in mind the following assumptions:

1. ASAP orders. No prior knowledge
2. FIFO order processing
3. Immediate material allocation to the order
4. Order up to point replenishment

The above assumptions are rather restrictive and in many cases do not apply, especially assumption 3. If one is to investigate the distribution of material wait time by relaxing the above assumptions, it is very difficult to get a solution in a closed form. In this case simulation is the recommended path. Even in that case there are issues. If one is to build a simulation of the queuing system describing the order handling environment, the demand must be modeled as random process $\boldsymbol{D}_t$ and not as time series $\boldsymbol{D}(t)$. Modeling the demand as a process transforms the way we measure it. Instead of measuring the total unit demand in a given time interval we measure arrival rate of orders **and** we also measure the distribution of the size of the order. This reformulation makes the traditional calculation of the inventory level

$$\mathbb{F}^{-1}_{(l+R)\mu,\sqrt{\sigma^2(l+R)+s^2\mu^2}}(SL) \tag{6}$$

inadequate. In order to adjust the formulation we need to introduce some new notation. First the order rate $\boldsymbol{\lambda}$ and then the order size modeled as a random variable with mean and standard deviation $\boldsymbol{\mu}_o, \sigma_o$ respectively. all other parameters remail the same as before.

From equation 6 we can calculate the required inventory expressed in terms of the new formulation

$$S = \mathbb{F}^{-1}_{(l+R)\mu_o\lambda,\sqrt{\lambda(\sigma_o^2+\mu_o^2)(l+R)+s^2(\mu_o\lambda)^2}}(SL) \tag{7}$$

References and Notes

References

[1] Gregory A. Kruger *The Supply Chain Approach to Planning and Procurement*, February 1997 Hewlett Packard Journal.