Towards a simple definition of weak ω-categories

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Weak higher dimensional categories and groupoids

- *Modoidal categories* were defined by Mac Lane, and *bicategories* (= *weak 2-categories*) by Benabou.
- *Weak ω-categories & ω-groupoids* were defined by Batanin (see also Grothendieck & Maltsiniotis).
- An explicit description is known up to dimension 3.
- In Martin-Löf ITT, types are weak ω-groupoids (Lumsdaine 2010, van den Berg & Garner 2011).

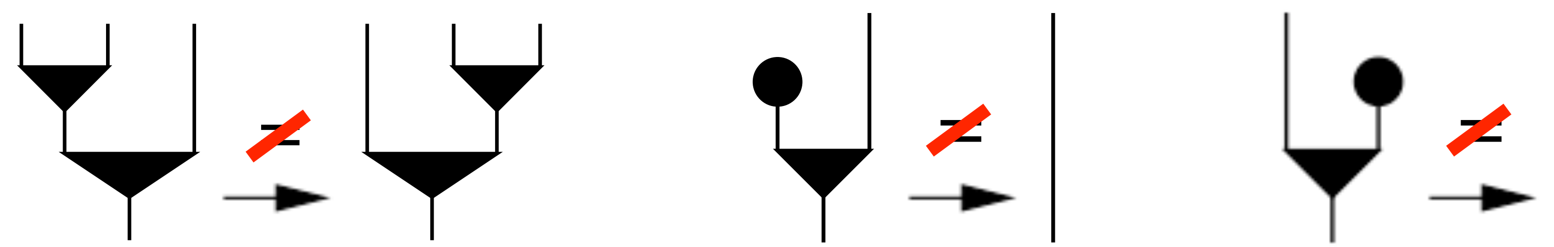
Working programme

- Extract an *explicit* definition of weak ω-groupoids (or weak ω-categories) from Martin-Löf ITT.
- Find connections with other research directions based on *rewriting* (Anick, Squier, Kobayashi, …), using *polygraphs* (Burroni) and *polygraphic resolutions* (Métayer, Lafont, Guiraud & Malbos, …).

Remark: Mac Lane's proof of coherence for monoidal categories is based on rewriting.

The theory of monoids

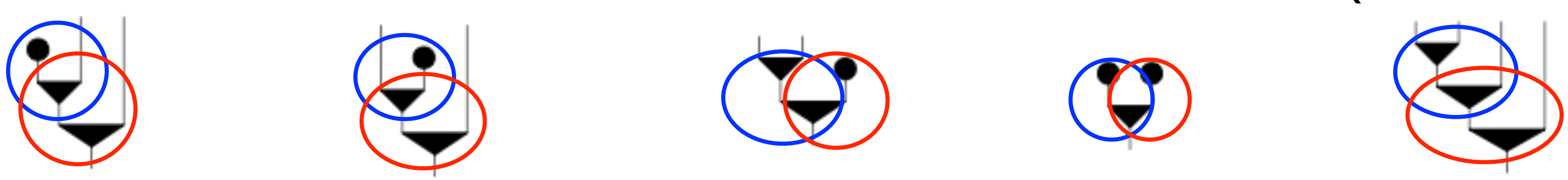
2 generators: 

3 ~~relations~~: rewrite rules: 

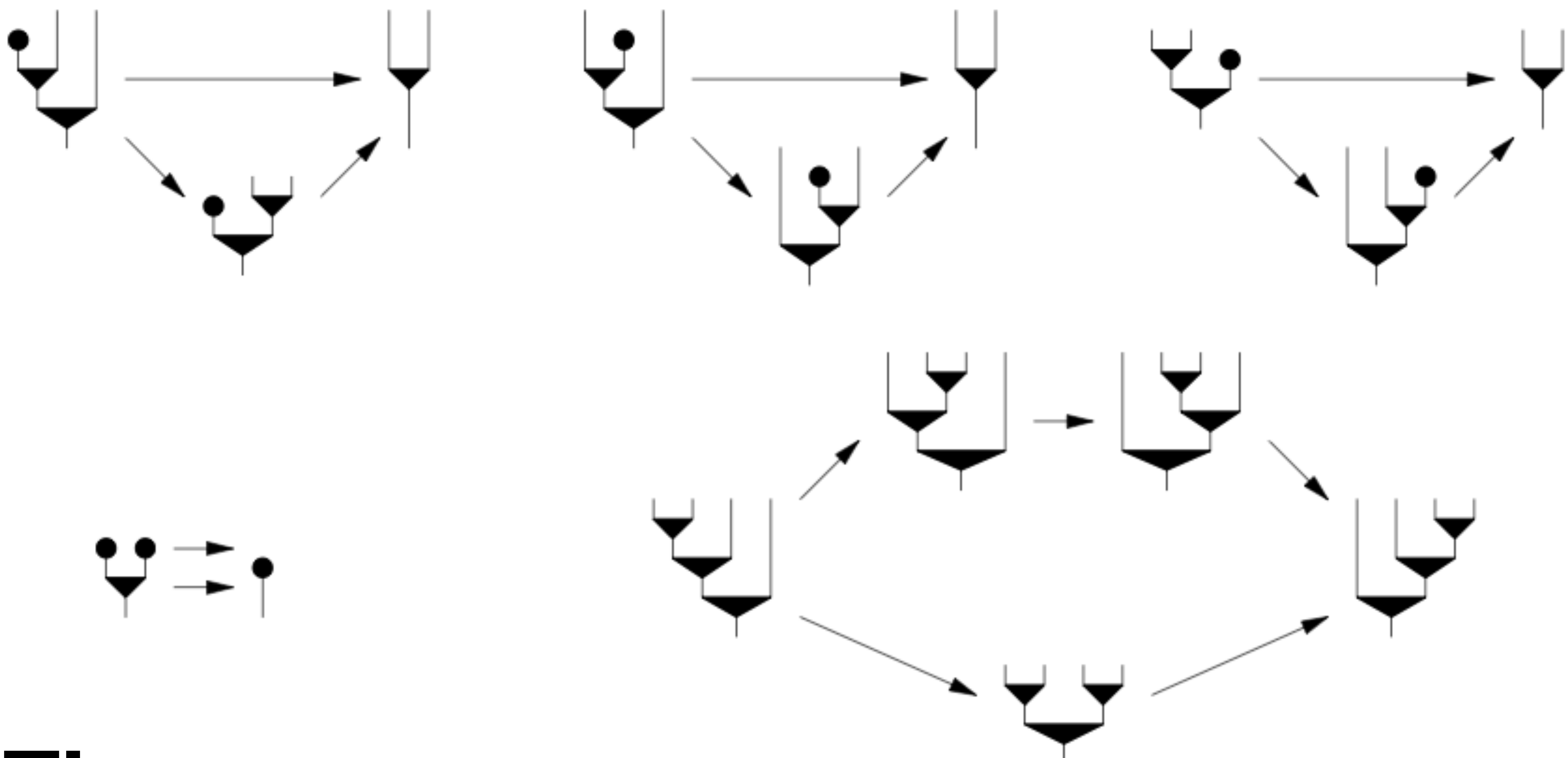
This *presentation* defines the PRO of *finite monotone maps* (= the *augmented simplex category* Δ_a).

This *rewrite system* is *convergent*:

- *Termination* is obvious.
- *Confluence* is checked for the 5 conflicts (*critical peaks*):



Confluence of critical peaks



Theorem:

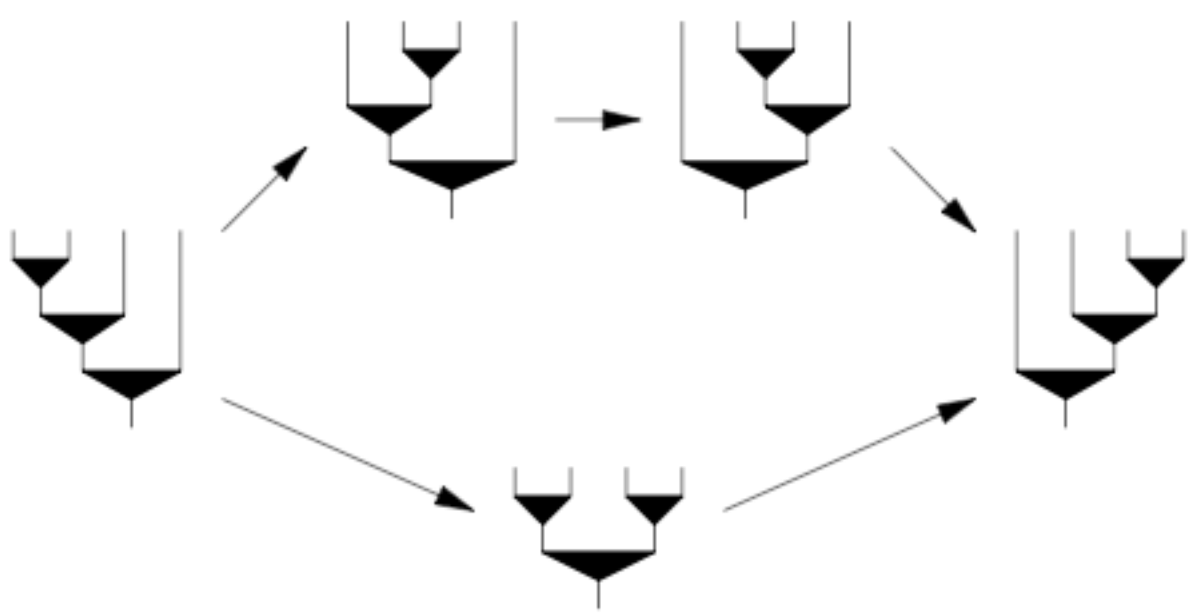
- Any tree *u* reduces to a unique *normal form û*.
- The reduction *u* → *û* is unique modulo the above *confluence diagrams* (+ *interchange*).

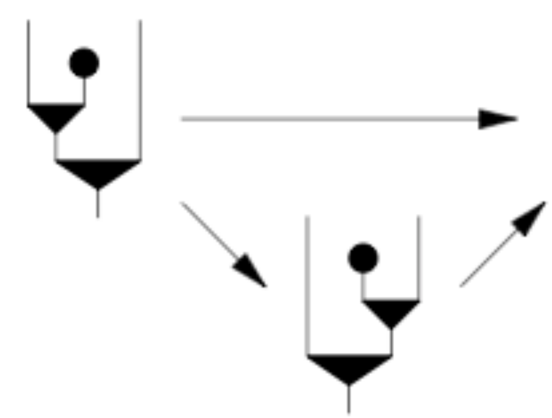
Mac Lane's coherence result

We consider *derivations* instead of *reductions*.
In other words, rewrite rules are formally inverted.

Corollary of the previous theorem:

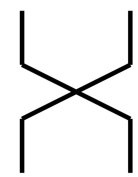
- Any derivation $u \leftrightarrow v$ can be seen as a *fraction* $u \rightarrow \hat{u} \leftarrow v$.
- This derivation is unique modulo the following diagrams:

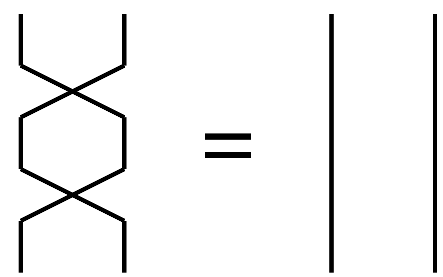
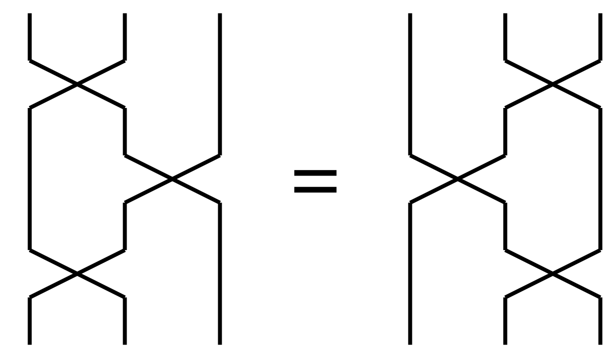




The other 3 diagrams are derivable (Kelly).

The theory of permutations

1 generator:  (permutation of two items)

2 relations:  

This *presentation* defines the PRO of *finite permutations*.

Diagrammatic version of the *group presentation* of $\mathbf{S}_{n+1}$:

$\langle s_1,\ldots,s_n \mid s_i^{\,2} = 1,\ s_i\, s_{i+1} s_i = s_{i+1} s_i\, s_{i+1},\ s_i\, s_j = s_j\, s_i \text{ for } j > i+1 \rangle$

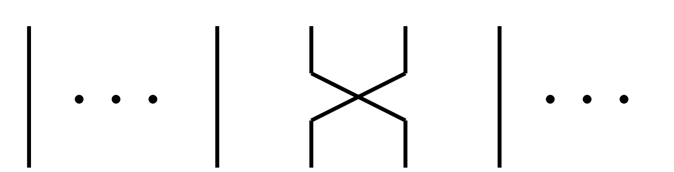
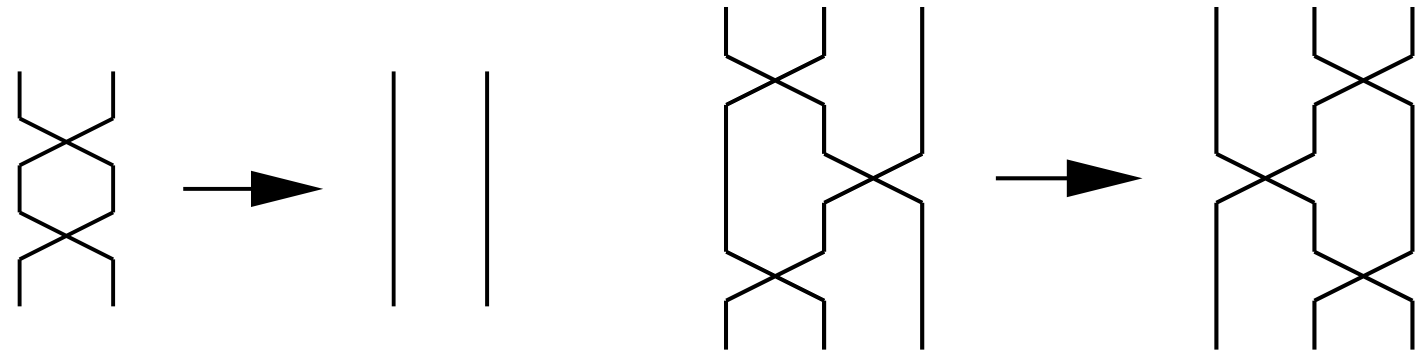
The s_i are the *transpositions*: 

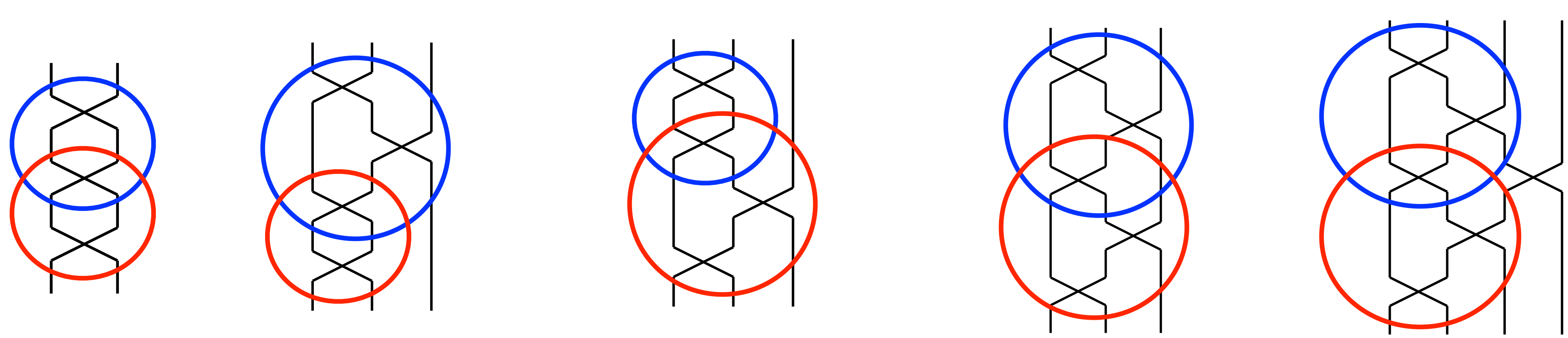
Diagram rewriting

2 rewrite rules: 

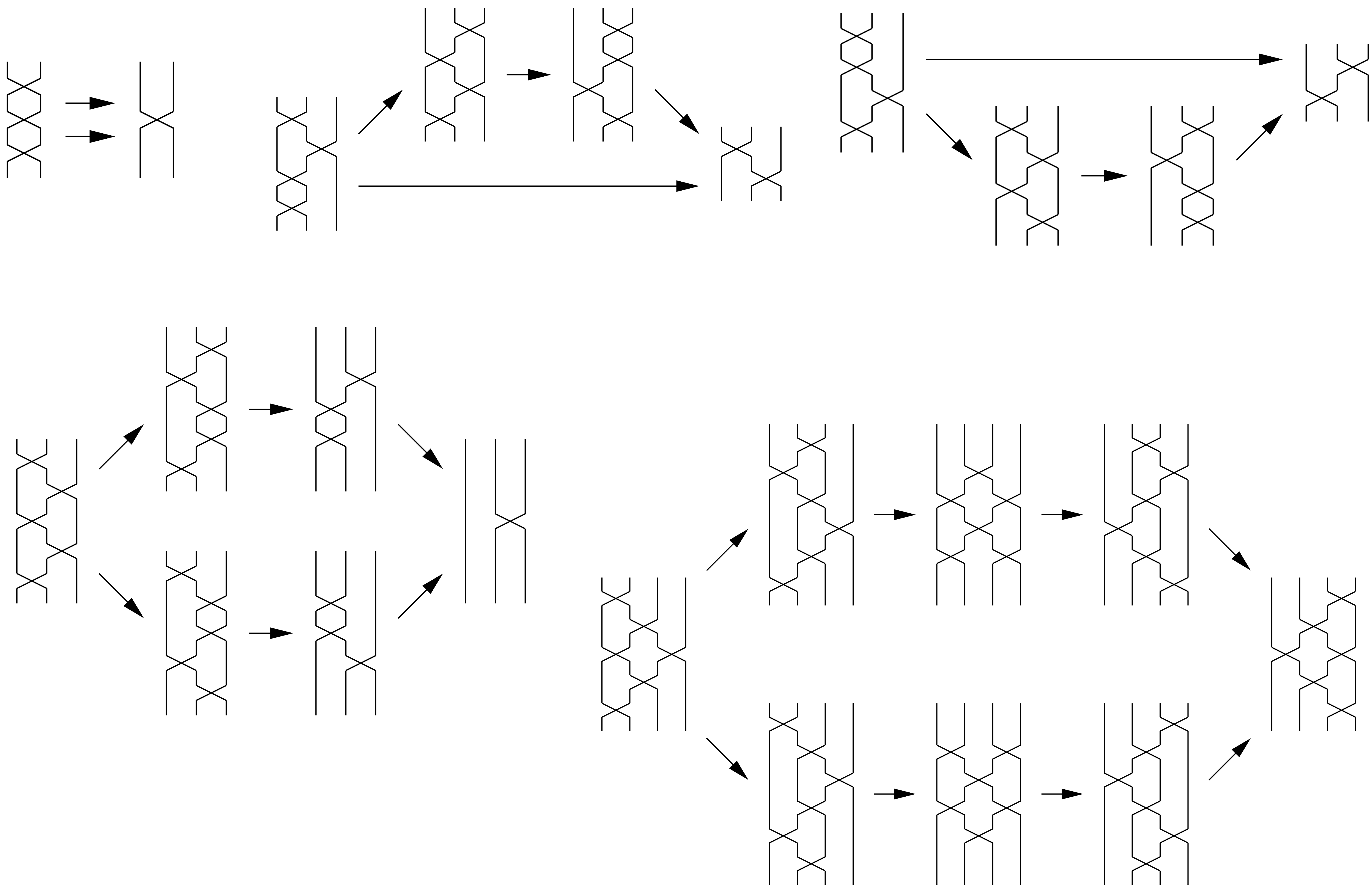
This rewrite system is *convergent*:

- *termination* (existence of a *normal form*)
- *confluence* (uniqueness of this normal form)

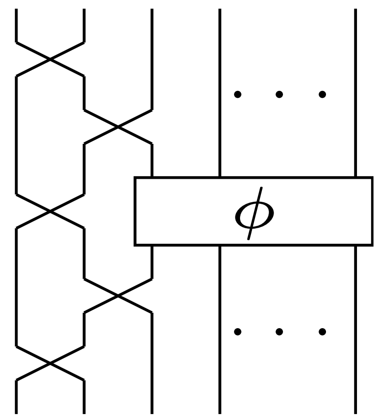
5 conflicts (*critical peaks*):



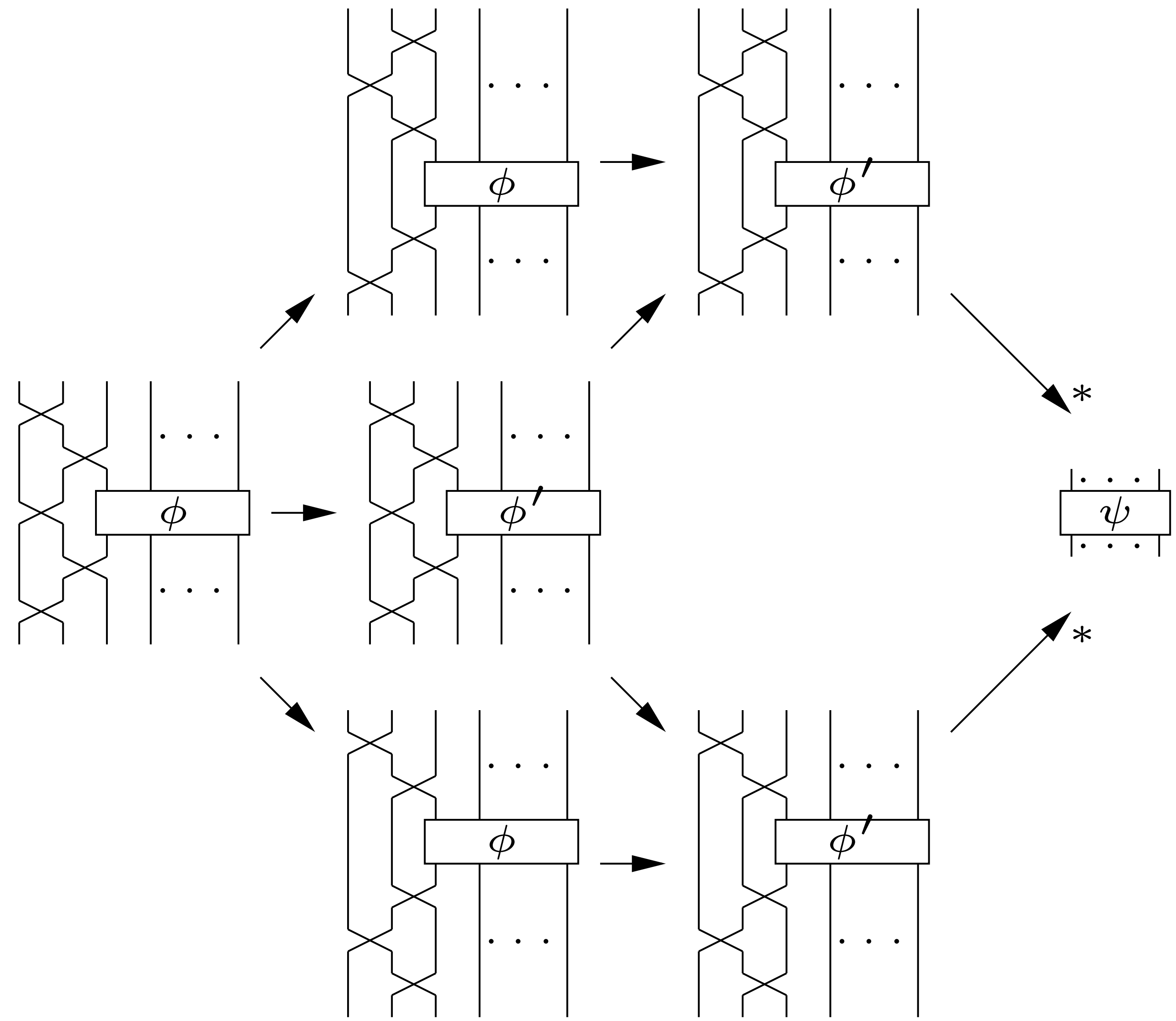
Confluence of critical peaks

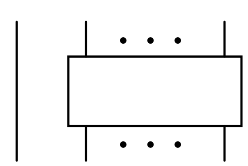
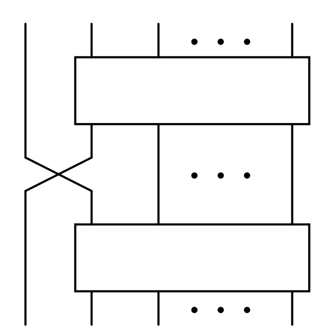


Problem with global conflicts

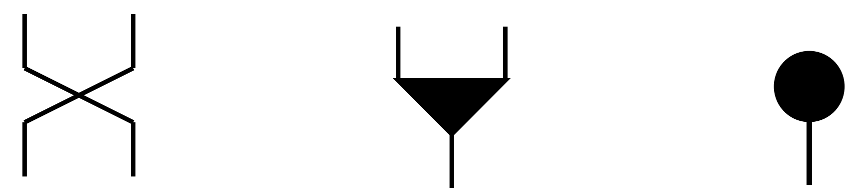
Global conflict: 

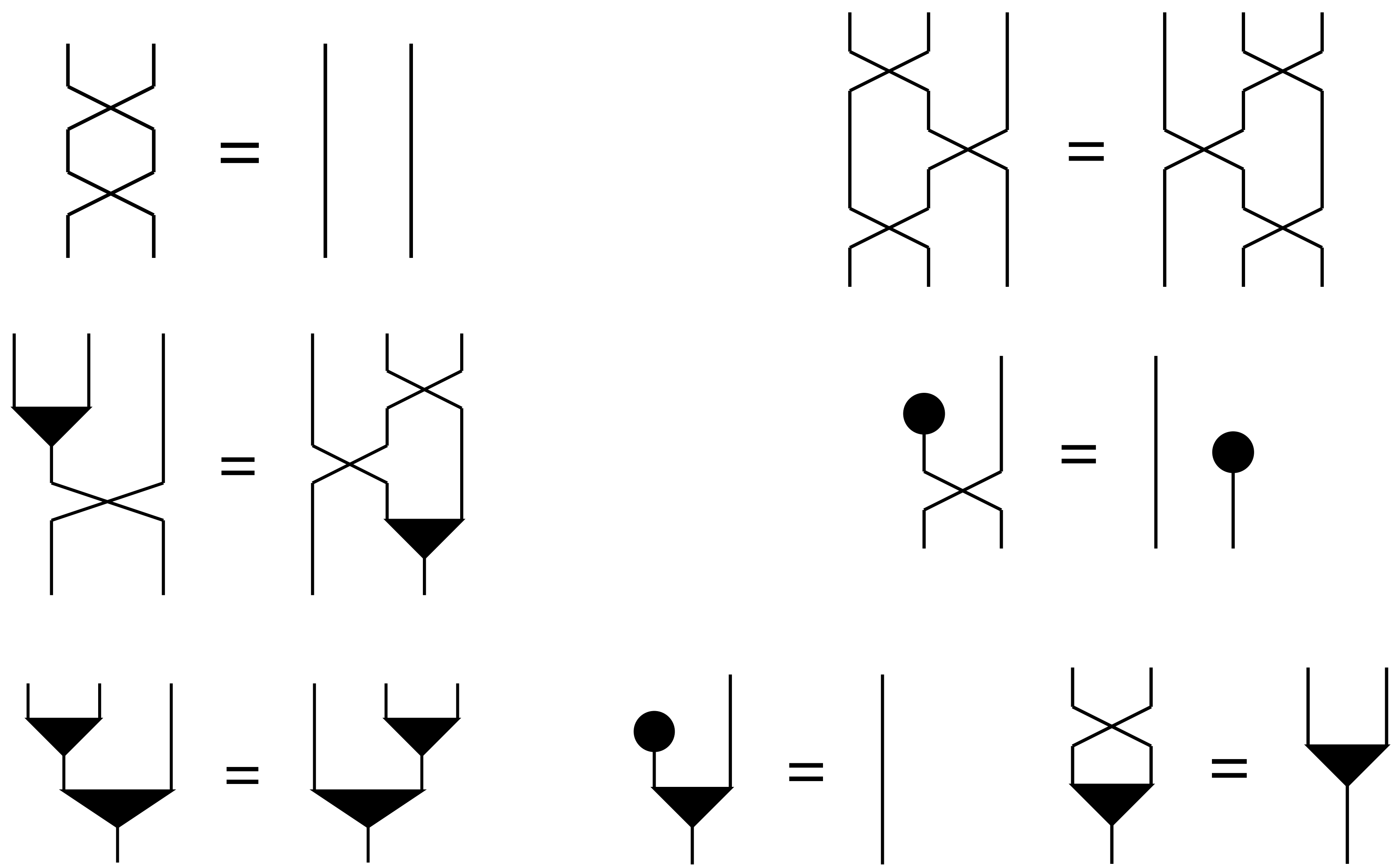
By *noetherian induction*, we can assume that the diagram ϕ is *reduced*:



→ ϕ is of the form  or  (only 2 cases)

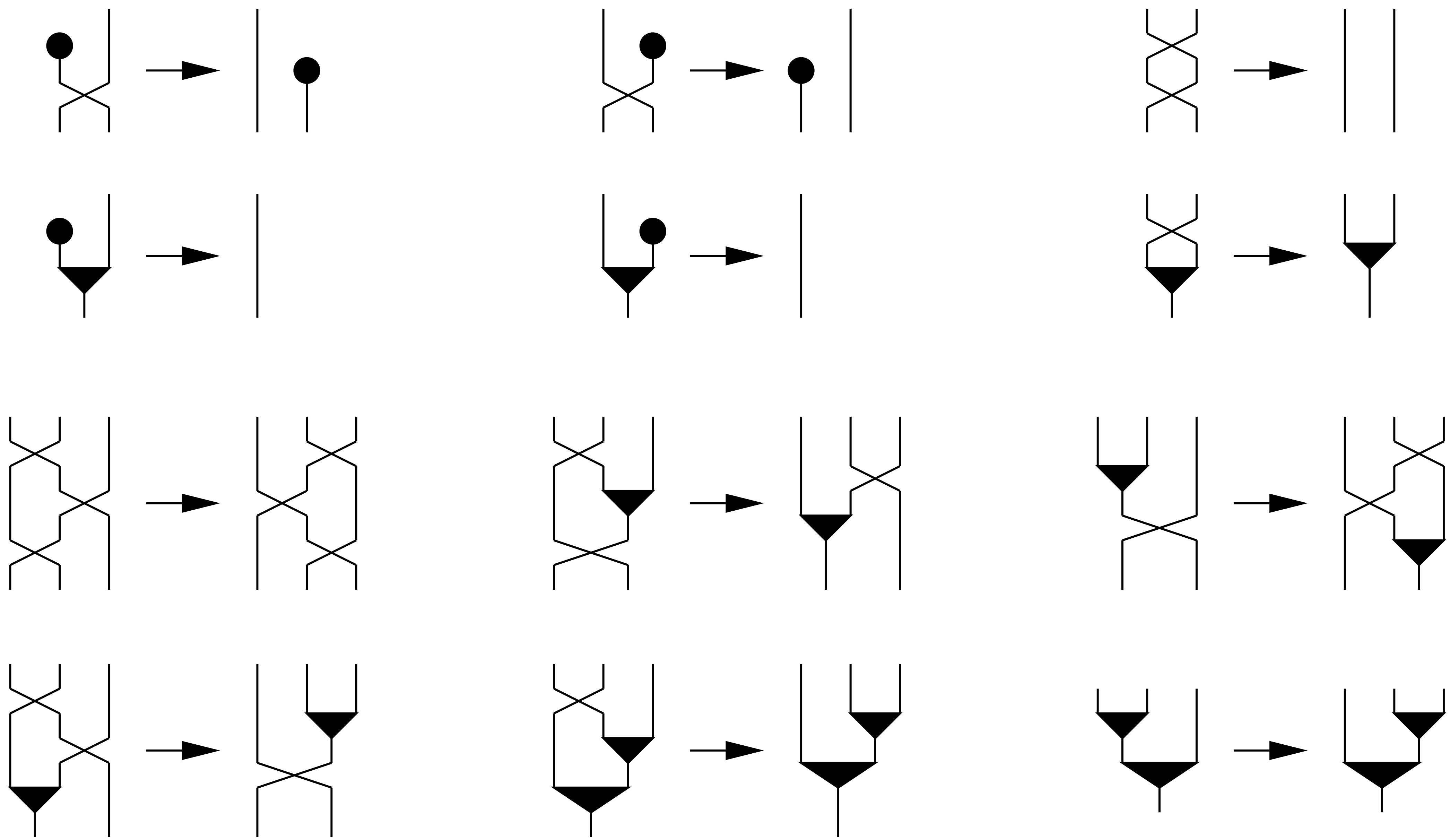
The theory of commutative monoids

3 generators: 

7 relations: 

This presentation defines the PRO of *finite maps* (Burroni).

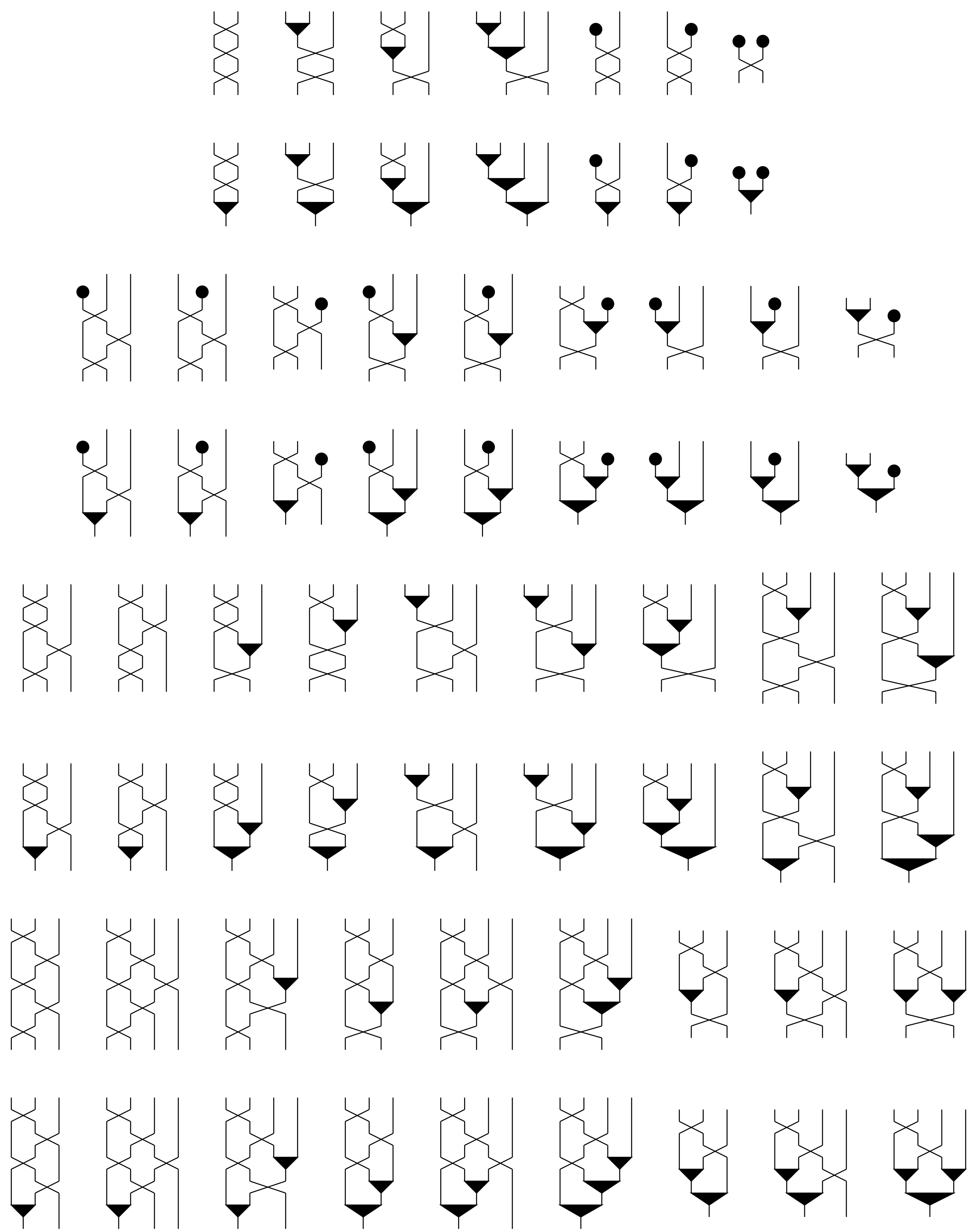
Rewrite rules



This rewrite system is convergent.

Idea: use it to prove coherence for symmetric monoidal categories.

The 68 critical peaks



Idea of the proof of coherence (unpublished)

- **Definition:** A *pseudo-symmetric monoidal category* is a monoidal category C equipped with a functor $\psi : C \times C \to C \times C$ and 3 natural isomorphisms satisfying (many) coherence conditions.

- **Example:** If ψ is the standard exchange functor, we get the notion of *symmetric monoidal category*, with much less coherence conditions, since most of the other conditions hold trivially.

- Coherence in the symmetric case follows from the pseudo-symmetric case.

Rewriting, coherence, homology and homotopy

- If a monoid M has a *finite convergent presentation*, then $H_3(M)$ is of finite type (Squier).
- Moreover, all $H_n(M)$ are of finite type (Kobayashi).
- In fact, M has a *finite derivation type* (Squier).
- This notion is extended to higher dimension by means of *polygraphic resolutions* (Métayer).
- The monoid M can be replaced by a 2-category (Guiraud & Malbos).

References

- Albert Burroni, *Higher dimensional word problems* (TCS 1993)
- Yves Lafont, *Towards an algebraic theory of Boolean circuits* (JPAA 2003)
- Yves Guiraud, *Termination Orders for 3-Dimensional Rewriting* (JPAA 2006)
- Yves Lafont, *Algebra and geometry of rewriting* (TCS 2007)
- Yves Lafont, *Diagram rewriting and operads* (Operads 2009, Séminaires et congrès, SMF 2013)
- Yves Guiraud & Philippe Malbos, *Coherence in monoidal track categories* (MSCS 2012)