A Model of Type Theory in Cubical Sets

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Univalent Foundations

- Vladimir Voevodsky formulated the *Univalence Axiom* (UA) in Martin-Löf Type Theory as a strong form of the Axiom of Extensionality.
- UA is *classically* justified by the interpretation of types as *Kan simplicial sets*
- However, this justification uses non-constructive steps. Hence this does not provide a way to compute with univalence.
- Goal: give a model of univalence in a constructive metatheory.

Outline

1. Cubical Sets
2. Constructive Kan Cubical Sets
3. Kan completion, Spheres, Propositional Reflection
4. Universe

A Category of Names and Substitutions

We define the category of names and substitutions $\mathcal{C}$ as follows. Fix a countable set of *names* $x, y, z, \ldots$ distinct from $0, 1$.

$\mathcal{C}$ is given by:

- objects are finite (decidable) sets of names $I, J, K, \ldots$
- a morphism $f : I \to J$ is given by a set map

  $$f : I \to J \cup \{0, 1\}$$

  such that if $f(x), f(y) \in J$, then $f(x) = f(y)$ implies $x = y$ (f is injective on its *defined* elements.)
  This represents a substitution: assign values 0 or 1 to variables or rename them.

A Category of Names and Substitutions

- Composition of $f : I \to J$ and $g : J \to K$ defined by

$$(g \circ f)(x) = \begin{cases} g(fx) & f \text{ defined on } x, \\ fx & \text{otherwise;} \end{cases}$$

  We write fg for $g \circ f$.
- For each I we assume a selected fresh name $x_I \notin I$.

Cubical Sets

Definition

A *cubical set* X is a functor $X : \mathcal{C} \to \mathbf{Set}$.

So a cubical set X is given by sets $X(I)$ for each I, and maps $X(I) \to X(J)$, $a \mapsto af$ for $f : I \to J$ with

$$a1 = a \quad \text{and} \quad (af)g = a(fg).$$

Call an element of $X(I)$ and *I-cube*.

Cubical Sets

Remark

- Kan's original approach (1955) to combinatorial homotopy theory used cubical sets
- Close to the presentation of cubical sets as in Crans' thesis
- Our notion is equivalent to nominal sets with 01-substitions (Pitts, Staton)

Cubical Sets

Think of names x as a name for a "dimension" and

- $X(\emptyset)$ as points,
- $X(\{x\})$ as lines in dimension x,
- $X(\{x, y\})$ as squares in the dimensions x, y,
- $X(\{x, y, z\})$ as cubes,
- $\ldots$

Cubical Sets: Faces

For $x \in I$ the maps $(x = 0), (x = 1) \colon I \to I - x$ sending x to 0 and 1 respectively are called the face map.

An I-cube θ of X connects its two *faces* $\theta(x = 0)$ and $\theta(x = 1)$:

$$\theta(x = 0) \xrightarrow[x]{\theta} \theta(x = 1)$$

Cubical Sets: Degeneracies

$f: I \to J$ is a degeneracy map if f is defined on all elements in I and J has more elements than I.

If $x \notin I$, consider the inclusion $(x): I \to I, x$. We have $(x)(x = 0) = 1 = (x)(x = 1)$, and so for an I-cube α of X:

$$\alpha \xrightarrow[x]{\alpha(x)} \alpha$$

If $\beta = \alpha(x)$ is such a degenerate I, x-cube, we can think of β to be *independent of the dimension* x.

Cubical Sets as a Category with Families

Cubical sets form (as any presheaf category) a model of type theory:

- The category of contexts $\Gamma \vdash$ and substitutions $\sigma \colon \Delta \to \Gamma$ is the category of cubical sets.
- Types $\Gamma \vdash A$ are given by

$$
\begin{aligned}
&A\alpha \text{ a set,} && \text{for } \alpha \in \Gamma(I), I \in \mathcal{C}, \\
&A\alpha \to A\alpha f \text{ a map,} && \text{for } f \colon I \to J \text{ in } \mathcal{C}, \\
&\quad a \mapsto af &&
\end{aligned}
$$

  such that $a1 = a$, $(af)g = a(fg)$.
- Terms $\Gamma \vdash t \colon A$ are given by $t\alpha \in A\alpha$ such that $(t\alpha)f = t(\alpha f)$.

Cubical Sets as a Category with Families

- For $\Gamma \vdash A$ the context extension $\Gamma.A \vdash$ is defined as

$$(\alpha, a) \in (\Gamma.A)(I) \text{ iff } \alpha \in \Gamma(I) \text{ and } a \in A\alpha,$$
$$(\alpha, a)f = (\alpha f, af).$$

  We can define the projections $\mathsf{p} \colon \Gamma.A \to \Gamma$ and $\Gamma.A \vdash \mathsf{q} : A\,\mathsf{p}$ by

$$\mathsf{p}(\alpha, a) = \alpha,$$
$$\mathsf{q}(\alpha, a) = a.$$

This gives a model of Π and Σ but will not get us the identity type we want!

Identity Types

The degeneracy operations give us a natural interpretation of the identity type $\Gamma \vdash \mathsf{Id}_A\ a\ b$ for $\Gamma \vdash a : A$ and $\Gamma \vdash b : A$:
For $\alpha \in \Gamma(I)$ we define $\omega \in (\mathsf{Id}_A\ a\ b)\alpha$ if

$$\omega \in A\alpha(x_I) \text{ s.t. } \omega(x_I = 0) = a\alpha \text{ and } \omega(x_I = 1) = b\alpha.$$

(Recall: x_I is a fresh name; $x_I \notin I$)
We can extend $f : I \to J$ to $(f, x_I = x_J) : I, x_I \to J, x_J$, and define the map $(\mathsf{Id}_A\ a\ b)\alpha \to (\mathsf{Id}_A\ a\ b)\alpha f$ by

$$\omega f =_{\text{def}} \omega(f, x_I = x_J) \quad \in A\alpha f(x_J).$$

Identity Types

This immediately justifies the introduction rule

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash \mathsf{Ref}\, a : \mathsf{Id}_A\ a\ a}$$

by setting $(\mathsf{Ref}\, a)\alpha = a\ \alpha(x_I)$.

But the elimination rule is *not* justified! We have to strengthen our notion of types!

Kan condition, classically

Classically, the Kan condition can be stated as:

any open box can be filled

Effectivity Problems

There are two main *effectivity problems* with the Kan condition:

- Closure of the Kan condition under exponentiation seems to essentially use *decidability* of degeneracy.
- A Kripke counter-model shows that Kan fibrations need not have equivalent fibres in a constructive setting (M. Bezem & T. Coquand).

So we have to refine this notion!

Kan condition, revisited

Let X be a cubical set; we first define the notion of an *open box*. Let $J, x \subseteq I$ with $x \notin J$. Set

$$O^+(J,x) = \{(x,0)\} \cup \{(y,c) \mid y \in J \wedge c \in \{0,1\}\}.$$

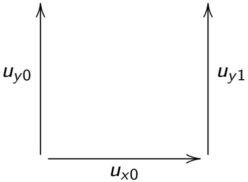
An open box $\vec{u}$ is given by $u_{yc} \in X(I - y)$ for $(y,c) \in O^+(J,x)$ s.t.

$$u_{yc}(z = d) = u_{zd}(y = c) \quad \text{for } (y,c),(z,d) \in O^+(J,x), y \neq z$$

(Similar: boxes given by $O^-(J,x)$ which contains $(x,1)$ instead of $(x,0)$)

Open Box

For example, a box $\vec{u} = u_{x0}, u_{y0}, u_{y1}$ has the shape:



Note that $\vec{u}$ may also depend on other variables (i.e., may consist of higher cubes).

The Uniform Kan Condition

X is *constructive Kan cubical set* if we have operations $X{\uparrow}$ such that for any open box $\vec{u}$ in $X(I)$ indexed by $O^+(J,x)$ (where $J,x \subseteq I$) we have fillers

$$X{\uparrow}\vec{u} \in X(I)$$

such that for $(y,c) \in O^+(J,x)$

$$(X{\uparrow}\vec{u})(y = c) = u_{yc}$$

and (!) for $f : I \to K$ defined on J,x

$$(X{\uparrow}\vec{u})f = X{\uparrow}(\vec{u}\ f)$$

where $\vec{u}f$ is the open box given by the $u_{yc}(f - y) \in X(K - fy)$ where $(f - y) \colon I - y \to K - fy$.

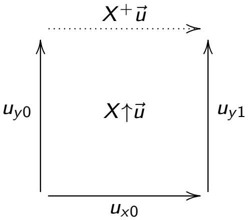
The Uniform Kan Condition

(Similarly, we require $X{\downarrow}$ operations for O^{-}-indexed open boxes.)

We set

$$X^{+}\vec{u} = (X{\uparrow}\vec{u})(x = 1),$$
$$X^{-}\vec{u} = (X{\downarrow}\vec{u})(x = 0).$$

Example:



The Uniform Kan Condition

Similar operations were already considered in an approach using semi-simplicial sets (B. Barras, T. Coquand, SH).

In a classical metatheory, the uniform Kan condition follows from the ordinary Kan condition.

Theorem

If a Kan cubical set X has decidable degeneracies, it also has the uniform Kan operations.

Constructive Kan Fibrations

A type $\Gamma \vdash A$ is a *(constructive) Kan fibration* if for all $\alpha \in \Gamma(I)$ we have operations

$$A\alpha{\uparrow}\vec{u} \ \in \ A\alpha \qquad \text{for open boxes } \vec{u}$$

where $u_{yc} \in A\alpha(y = c), (y, c) \in O^{+}(J, x)$ such that $(A\alpha{\uparrow}\vec{u})(y = c) = u_{yc}$ and for $f : I \to K$ defined on J, x

$$(A\alpha{\uparrow}\vec{u})f = (A\alpha f){\uparrow}(\vec{u}\ f).$$

(Similarly we require operations $A\alpha{\downarrow}\vec{u}$.)

Model of Type Theory

By restricting types $\Gamma \vdash A$ to constructive Kan fibrations, we get an effective model of type theory.

Theorem

Constructive Kan fibrations are closed under Π-, Σ- *and* Id-*types.*

Adding this extra conditions *solves* the effectivity problem!

Identity Type (cont.)

Theorem

If $\Gamma.A \vdash P$ is Kan fibration, then there is a term J s.t.

$$\frac{\Gamma \vdash A \quad \Gamma \vdash a : A \quad \Gamma \vdash b \quad \Gamma \vdash p : \mathsf{Id}_A\ a\ b \quad \Gamma \vdash u : P[a]}{\Gamma \vdash \mathsf{J}(p, u) : P[b]}$$

Proof.

Let α be an I-cube of Γ; then $p\alpha$ connects $a\alpha$ and $b\alpha$ in dimension x with $x \notin I$. So we get an I, x-cube in $\Gamma.A$:

$$[a]\alpha \xrightarrow[x]{(\alpha(x), p\alpha)} [b]\alpha$$

We define $\mathsf{J}(p, u)\alpha = P(\alpha(x), p\alpha)^{+}(u\alpha)$. □

Identity Type (cont.)

Note that we have a line:

$$u\alpha \xrightarrow{P(\alpha(x),p\alpha)\uparrow(u\alpha)} \mathsf{J}(p,u)\alpha$$

In particular, if $p = \mathsf{Ref}\, a$ this gives a term of

$$\Gamma \vdash \mathsf{Id}_{P[a]}\ u\ (\mathsf{J}(\mathsf{Ref}\, a, u)). \tag{1}$$

One can also show that the singleton type $\Sigma x : A\ \mathsf{Id}_A\ a\ x$ is contractible.
This suffices to develop basic properties of univalent mathematics (N.A. Danielsson).
(To get (1) as definitional equality $\mathsf{J}(\mathsf{Ref}\, a, u) = u$ one has to consider *regular* fibrations.)

Kan Completion

We can “complete” any cubical set X to a Kan cubical set X'.

Add operations $X^+, X{\uparrow}, X{\downarrow}, X^-$ in a *free* way, i.e., by considering these operations as *constructors*.

The uniformity conditions determine how a morphism acts on the new constructors.

This defines a Kan cubical set such that for any morphism $X \to Y$ with Y Kan can be extended to $X' \to Y$.

The Circle S^1

S^1 is the Kan completion of the cubical set generated by a point **base** and a line **loop** connecting **base** to **base**.

For a type $S^1 \vdash P$ with $\vdash a : P\ \textbf{base}$ and $\vdash l : P\ \textbf{loop}$ we can define $S^1 \vdash E : P$ satisfying

$$E\ \textbf{base} = a \qquad \text{and} \qquad E\ \textbf{loop} = l.$$

Propositional Reflection

For a Kan cubical set X we define $\mathrm{inh}(X)$.

$\mathrm{inh}(X)$ is a *h-proposition* that states that X is inhabited.

To X we add a constructor $\alpha_x(a_0, a_1)$ for an I, x-cube ($x \notin I$) for I-cubes a_0, a_1 and set

$$\begin{aligned} \alpha_x(a_0, a_1)(x = d) &= a_d && \text{for } d = 0, 1 \\ (\alpha_x(a_0, a_1))f &= \alpha_{fx}(a_0(f - x), a_1(f - x)) && f \text{ def. on } x \end{aligned}$$

Additionally we have constructors for the Kan operations, and get a Kan cubical set $\mathrm{inh}(X)$ as before.

If Y is a h-proposition, then $X \to Y$ gives $\mathrm{inh}(X) \to Y$.

Universe

The universe U of Kan cubical sets is intuitively as follows:

- points of U are (small) Kan cubical sets
- a line in U between A and B can be seen as a "heterogeneous" notion of lines, cubes, ..., $a \to b$ where a and b are I-cubes of A and B respectively, where we can fill all open boxes
- ...

Universe

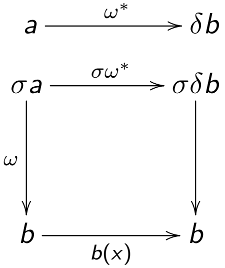
More formally, $A \in U(I)$ is given by

- a family of (small) sets A_f with $f : I \to J$;
- maps $A_f \to A_{fg}$, $a \mapsto ag$ if $g : J \to K$, satisfying $a1 = a$ and $(ag)h = a(gh)$;
- operations $A_f{\uparrow}$, $A_f{\downarrow}$ analogous to the uniform Kan fillings, e.g., $A_f{\uparrow}\vec{a} \in A_f$ if $a_{yc} \in A_{f(y=c)}$ for $(y, c) \in O^+(K, x)$, $K, x \subseteq J$, is an open box.

This defines a cubical set (with $(Af)_g = A_{fg}$).

Equivalence of Types

A map $\sigma\colon A \to B$ between to Kan cubical sets A and B is an equivalence if there is a map $\delta\colon B \to A$ and a map $\sigma\delta b \to b$ and a transformation of any equality $\omega\colon \sigma a \to b$ (where a and b are I-cubes of A and B resp.) to a I, x-cube in A and an I, x, y-cube in B:



From Equivalence to Equality of Types

We can transform an equivalence $\sigma : A \to B$ into a line C in $U(x)$ between A and B. Define the sets C_f for $f : \{x\} \to I$ as follows.

- If $fx = 0$, set $C_f = A(I)$.
- If $fx = 1$, set $C_f = B(I)$.
- If f is defined on x, $fx = y$, we set C_f to consist of pairs (a, b) where

$$a \in A(I - y) \text{ and } b \in B(I) \text{ s.t. } b(y = 0) = \sigma a.$$

From the fact that σ is an equivalence one can check elementary that C has the uniform Kan properties.

Conclusion and Further Work

- Cubical sets are suitable for modeling type theory, especially Id-types
- The uniform Kan condition gives a well-behaved notion in a constructive setting; all results are concrete and effective
- We only checked a weak corollary of the Axiom of Univalence but we expect Univalence to hold in the model
- Close connections to nominal sets (Pitts) and internal parametricity (Bernardy, Moulin); should give rise to an implementation
- Connections to other work on cubical sets?

Thank you!