Towards a constructive view at simplicial sets

(work in progress)

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The standard definitions of semi-simplicial and simplicial sets

A simplicial set is a presheaf on the simplex category made of order-preserving maps over non-empty finite ordinals.

A semi-simplicial set is a presheaf on the category of order-preserving injective maps over non-empty finite ordinals.

Part I

A dependently-typed inductively-generated construction of simplicial sets

Awodey-Lumsdaine's dependently-typed inductively-generated construction of *semi*-simplicial sets

$$
\begin{aligned}
Y_0 &: \mathsf{Type} \\
Y_1 &: \Pi ab : Y_0.\ \mathsf{Type} \\
Y_2 &: \Pi abc : Y_0.\ \Pi x : Y_1(a,b).\ \Pi y : Y_1(a,c).\ \Pi z : Y_1(b,c).\ \mathsf{Type} \\
&\vdots
\end{aligned}
$$

Awodey-Lumsdaine's dependently-typed inductively-generated construction of semi-simplicial sets: an effective construction

[HH 2013] gives a precise dependently-typed inductively-generated construction of (augmented) semi-simplicial sets. We give here a variant for (non-augmented) semi-simplicial sets independently due to Voevodsky.

Let's rephrase the types of the Y_i's as follows:

$$\begin{array}{lcl}
Y_0 : T_0 & \triangleq & \text{Type} \\
Y_1 : T_1 & \triangleq & \Pi \underbrace{(a : Y_0, b : Y_0)}_{F^{0,1}(Y_0)} . \text{Type} \\
Y_2 : T_2 & \triangleq & \underbrace{\Pi[\Sigma \underbrace{(a : Y_0, b : Y_0, c : Y_0)}_{F^{0,2}(Y_0)} . (x : Y_1(a,b), y : Y_1(a,c), z : Y_1(b,c))]}_{F^{1,2}(Y_0, Y_1)} . \text{Type} \\
\vdots & &
\end{array}$$

and let's set

$$sst_n \triangleq (T_0, T_1, ..., T_n)$$

The construction is then shown on next slide

Awodey-Lumsdaine's dependently-typed inductively-generated construction of semi-simplicial sets: the construction

$$
\begin{array}{lll}
sst_n & & : \ \mathsf{Type}_2 \\
sst_0 & & \triangleq \ \mathsf{Type}_1 \\
sst_{n+1} & & \triangleq \ \Sigma X : sst_n . F^{n,n+1}(X) \to \mathsf{Type}_1
\end{array}
$$

$$
\begin{array}{lll}
F^{n,k} & (X : sst_n) & : \ \mathsf{Type}_1 \\
F^{0,k} & X & \triangleq \ [k] \to X \\
F^{n+1,k} & (X, Y) & \triangleq \ \Sigma x : F^{n,k}(X) . \Pi \sigma \in [n+1] \hookrightarrow [k] . Y(\underline{d}^{n,n+1,k}_{\sigma}(x))
\end{array}
$$

$$
\begin{array}{lll}
\underline{d}^{n,j,k}_{\sigma:[j]\hookrightarrow[k]} & (X : sst_n)\ (x : F^{n,k}(X)) & : \ F^{n,j}(X) \\
\underline{d}^{0,j,k}_{\sigma} & X \ x & \triangleq \ x \circ \sigma \\
\underline{d}^{n+1,j,k}_{\sigma} & (X, Y)\ (x, y) & \triangleq \ (\underline{d}^{n,j,k}_{\sigma}(X)(x), \lambda \sigma' \in [n+1] \hookrightarrow [j] . \texttt{subst } \underline{\alpha}^{n,n+1,j,k}_{\sigma,\sigma'} \texttt{ in } y(\sigma \circ \sigma'))
\end{array}
$$

$$
\begin{array}{lll}
\underline{\alpha}^{n,i,j,k}_{\sigma:[j]\hookrightarrow[k],\sigma':[i]\hookrightarrow[j]} & (X : sst_n)\ (x : F^{n,k}(X)) & : \ \underline{d}^{n,i,k}_{\sigma\circ\sigma'}(x) \ = \ (\underline{d}^{n,i,j}_{\sigma'} \circ \underline{d}^{n,j,k}_{\sigma})(x) \\
\underline{\alpha}^{0,i,j,k}_{\sigma,\sigma'} & \mathsf{unit}\ \mathsf{unit} & \triangleq \ \mathsf{refl} \\
\underline{\alpha}^{n+1,i,j,k}_{\sigma,\sigma'} & (X, Y)\ (x, y) & \triangleq \ (\underline{\alpha}^{n,i,j,k}_{\sigma,\sigma'}(x), \ldots \text{ a proof using assoc. of } \circ \text{ and strictness of } = \ldots)
\end{array}
$$

Awodey-Lumsdaine's dependently-typed inductively-generated construction of semi-simplicial sets: comments

Note 1: [HH2013]'s construction for augmented semi-simplicial sets is based on atomic face maps d_i rather than on "long-range" face maps.

Note 2: The construction scales to semi-simplicial objects at arbitrary types if strict equality is available in the logic (in concurrency with a univalent equality).

Note 3: Products and exponentials can be defined recursively (no proof yet that they coincide to the categorical constructions though).

Awodey-Lumsdaine's dependently-typed inductively-generated construction of semi-simplicial sets

What we learned so far: the well-founded structure of semi-simplicial sets can be exploited to provide with a recursive definition of them.

Can we do the same for simplicial sets?

Dependently-typed inductively-generated construction of *simplicial* sets

Yes, it is possible (though no properties of the correctness of the construction have been proved yet): inject degeneracies on the fly by replacing $Y(x)$ in $F^{n+1,k}$ above by

$$\begin{array}{lll}\widehat{Y} & (x : F^{n,n+1}(X)) & : \mathsf{Type}_1 \\ \widehat{Y} & x & \triangleq Y(x) \vee \underbrace{\Sigma(p,\sigma,y).\ \Pi(q,\delta).\ \mathtt{connect}^{n,p,q}(X)(\sigma)(y)(x)(\delta)}_{\uparrow}\end{array}$$

these are the added degenerate simplices
one for each non-trivial $\sigma \in [n+1] \xrightarrow{s} [p+1]$ and $y \in F^{p,p+1}(X)$
i.e. y non-degenerate at some level p

For each extra degenerate simplex $s_\sigma(y)$,
and for each face map $\delta \in [n+1] \hookrightarrow [q+1]$ coming out of it,
a simplicial equation canonically expressing $d_\delta(s_\sigma(y))$ in term of some $s_{\sigma'}(d_{\delta'}(y))$ has to hold

(this latter equation is expressed by the recursively-defined predicate $\mathtt{connect}$)

Part II
Injecting degeneracies in the context of the presheaf definition

Sorting out degenerate and non-degenerate simplices

Key lemma:
Let

(X, d, s) be a simplicial set

and

$|X_n| \triangleq \{x \in X_n | n = 0 \vee \forall i \, \forall x' : X_{n-1}. \, s_i^{n-1}(x') \neq x\}$

be the subset of non-degenerate simplices of X_n.

Then, for all $x \in X_n$, there is
a unique $p \leq n$,
a unique $\sigma_{i_0} \circ ... \circ \sigma_{i_{p-1}} \in [n] \xrightarrow{s} [n-p]$
and a unique $x' \in |X_{n-p}|$ such that $x = s_{i_{p-1}}^{n-1}(...s_{i_0}^{n-p}(x')...)$.

A constructive definition of simplicial sets

(extension with degeneracies)

For n given, let Y_i be a finite family of sets indices over $0 \leq i \leq n$.

We define the *extension of Y_n with degeneracies from* $(Y_i)_{0 \leq i < n}$ to be the set

$$\widehat{Y}_n \triangleq \{(\sigma, y) | \sigma \in [n] \xrightarrow{s} [m] \wedge y \in Y_m\} \; .$$

The case $n = m$ gives an embedding of Y_n into $\widehat{Y}_n$. When $n > m$, this gives as many copies of Y_m into $\widehat{Y}_n$ as there are surjections from $[n]$ to $[m]$. We call such embeddings degeneracies.

Degeneracy maps can be defined over the $\widehat{Y}_n$'s by composition of atomic degeneracy maps $\tilde{s}_i^n$, which themselves go from $\widehat{Y}_n$ to $\widehat{Y}_{n+1}$. The *atomic degeneracy map* $\tilde{s}_i^n$ is defined by:

$$\tilde{s}_i^n(\sigma, y) \;\triangleq\; (\sigma \circ \sigma_i^n, y)$$

where σ ranges over $[n] \xrightarrow{s} [n-p]$ and y over Y_{n-p}.

A constructive definition of simplicial sets

(extension of face maps to degeneracies)

Let us assume in addition that $d_\delta^{n,m}$ is a family of maps from Y_n to $\widehat{Y}_m$ indexed over $\delta \in [m] \hookrightarrow [n]$.

We define the extension of $d_\delta^{n,m}$ from $\widehat{Y}_n$ to $\widehat{Y}_m$ to be the function $\widehat{d_\delta^{n,m}}$ recursively defined by well-founded induction on n by the clause

$$\widehat{d_\delta^{n,m}}(\sigma, y) \triangleq (\sigma', d_{\delta'}^{p,q}(y))$$

where

- $y \in Y_p$ and $\sigma \in [n] \xrightarrow{s} [p]$

- q and $\sigma' \in [m] \xrightarrow{s} [q]$ and $\delta' \in [q] \hookrightarrow [p]$ are unique such that $\sigma \circ \delta = \delta' \circ \sigma'$

A constructive definition of simplicial sets

(main definition)

A *constructive simplicial set* is given by

- a family Y_n of *pure simplices*,
- a family of *face maps* $d_\delta^{n,p}$ from Y_n to $\widehat{Y}_p$, with δ ranging over $[p] \hookrightarrow [n]$,

satisfying for all $\delta' \in [m] \hookrightarrow [p]$ the following family of equations:

$$\widehat{d}_{\delta'}^{p,m} \circ d_\delta^{n,p} = d_{\delta \circ \delta'}^{n,m}$$

A constructive definition of simplicial sets

(further work)

We can define morphisms as families of maps from X_n to $\widehat{Y}_n$ that commute with the $d_\sigma^{n,m}$ and $\widehat{d}_\sigma^{n,m}$ when canonically extended into maps from $\widehat{X}_n$ to $\widehat{Y}_n$.

We conjecture to have an equivalence of categories between such constructive simplicial sets and (ordinary) simplicial sets.

Constructively defining exponential is tricky but we believe it is feasible, though it requires to modify the above definition of morphisms.