

Final report on the harmonic measure project

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1 Introduction

During the month of the BIMR 2026 program, the project that has been completed has consisted of studying harmonic measure via Perron's method. To this aim, some introductory definitions and properties regarding harmonic functions have been established, the Dirichlet problem has been introduced and solved via different methods, fundamental results from measure theory have been stated and, finally, the concept of harmonic measure, together with some fundamental properties, has been studied. Finally, some exercises have been solved regarding the studied theory.

2 Harmonic functions

In this section, the main focus has been on harmonic functions and their properties. First of all, the definition of harmonic function has been given in terms of the Laplacian operator, and some fundamental theorems that harmonic functions satisfy have been proven, like the mean value theorem and its converse. Taking into consideration both of these theorems, an alternative definition for harmonic function has been given, and, finally, some basic properties that harmonic functions satisfy have been proven, like the maximum principle, the theorem of uniqueness and Liouville's theorem.

Furthermore, the concept of being weakly harmonic and distributionally harmonic has been introduced, and some important lemmas regarding these functions have been proven, like the Caccioppoli inequality (among other inequalities regarding different norms) and a lemma regarding the uniform convergence on compact subsets of a subsequence of some harmonic functions that satisfy special properties.

Also, Harnack's inequality has been proven, which is an inequality regarding the supremum of a positive harmonic function inside an open ball. Another version of this inequality has been studied, the asymptotic Harnack inequality, which characterises the behaviour of the bound.

Finally, the last concept introduced in this section has been the fundamental solution in $\mathbb{R}^d$. It is defined as

$$\mathcal{E}(x) = \begin{cases} \frac{|x|^{2-d}}{(d-2)\kappa_d} & \text{if } d > 2, \\ \frac{-\log|x|}{2\pi} & \text{if } d = 2. \end{cases}$$

It has been proven that $-\Delta\mathcal{E}$ is the Dirac delta distribution δ_0 and, with the help of the fundamental solution, that every harmonic distribution is a $\mathcal{C}^\infty$ and harmonic function.

3 The Dirichlet problem

In this second section, the Dirichlet problem for the Laplacian with boundary data f has been presented. Also, to be able to create a general theory of existence and uniqueness, the weak formulation of the Dirichlet problem has been presented, which consists of working in Sobolev spaces with one weak derivative. Some important theorems have been stated, such as the Riesz representation theorem for Hilbert spaces, and the uniqueness of the Dirichlet problem written in the weak formulation has been proven, and also the existence of such solution in bounded open sets.

Moreover, the weak Green function has been introduced, which is defined in terms of the fundamental solution and the solution to the weak Dirichlet problem with boundary minus the fundamental solution

with pole at a certain x in the domain. Important properties regarding the Green function have been proven, like the fact that it is symmetric and harmonic in bounded open sets with respect to its variable and its pole. Thanks to the Green function and assuming enough regularity on the boundary of our open set, the Poisson kernel can be defined and a solution to the Dirichlet problem can be obtained.

However, the weak solution to the Dirichlet problem presents some limitations. This is very clearly shown with the weak solution to the Dirichlet problem in $\Omega = B_1(0) \setminus \{0\} \subset \mathbb{R}^d$ with $d \geq 3$ (the ball of radius one and center 0 except for the point 0) with boundary values $f = 0$ in $\partial B_1(0)$ and $f(0) = 1$. The weak solution to the Dirichlet problem is $u = 0$, but it does not satisfy the boundary values at 0.

Also, the solvability of the Dirichlet problem has been thoroughly studied in the case of continuous functions f (that act as boundary data, $f \in \mathcal{C}(\partial B_1(0))$) in the unit ball, $B_1(0)$. The solution to the problem is defined in terms of the Green function and the Poisson kernel, as previously done, but in this case, both the Green function and the Poisson kernel have a very simple analytical expression. Also, the solution to the Dirichlet problem in the unit ball has some nice properties, such as being continuous inside the closed ball if f (the boundary data) is continuous.

Finally, the last step in this section has been to introduce another way of solving the Dirichlet problem for continuous boundary functions if the domain is smooth enough: via the double layer potential and the jump formulas.

4 Basic results from measure theory

In this section, some basic concepts about measure theory have been defined and some interesting properties that have to do with particular measures have been proven. First of all, the definitions of measure, being measurable (in the case of sets and also functions), a σ -algebra, the collection of Borel sets, a Borel measure, a Borel regular measure and a Radon measure have been given.

Afterwards, integration for an arbitrary measure has been defined, and some important integration results, such as the Fubini-Tonelli theorem, have been stated (for Borel measures). Moreover, some covering theorems have been stated, such as Vitali's 5r covering, or Besicovitch's covering theorem.

Another aspect that has been studied is differentiation of measures. The definition of a Radon measure being differentiable with respect to another Radon measure has been provided in terms of a limit of a quotient of the first measure applied to the ball $B_r(x)$ (for x an arbitrary point inside the adequate domain) divided by the second measure applied to that same ball, and the limit is as r tends to 0. Also, the definition of a measure being absolutely continuous with respect to another one has been given, and some interesting theorems regarding Radon measures have been stated, such as the Radon-Nikodym theorem, the Lebesgue decomposition theorem and the Lebesgue differentiation theorem.

Moreover, dyadic grids have been studied, with the most relevant result of this part being the dyadic Lebesgue differentiation theorem.

An important result proved in this section has been the Riesz representation theorem, that states that if X is a locally compact metric space and $L : C_c(X) \rightarrow \mathbb{R}$ a positive linear functional, then there is a unique Radon measure μ such that $L_f = \int f d\mu$ for $f \in C_c(X)$.

Finally, the last two aspects studied in this section have been the concept of Hausdorff measure and Hausdorff dimension (with some important lemmas regarding their properties) and Frostman's lemma (which is a very important tool used in geometric measure theory and potential theory).

5 Harmonic measure via Perron's method

In this last section, three key parts have been studied: subharmonic functions, Perron classes and resolutive functions and, finally, the concept of harmonic measure via Perron's method.

First of all, the definitions of an upper (and lower) semicontinuous function and a subharmonic (and superharmonic) function have been given.

1. A function $u : A \rightarrow [-\infty, \infty]$, where $A \subset \mathbb{R}^d$, is upper semicontinuous at $x \in A$ if $\limsup_{y \rightarrow x} u(y) \leq u(x)$ (and similarly with lower semicontinuous and $\liminf$).
2. A subharmonic (or superharmonic) function must be upper semicontinuous (or lower) and satisfy

$$u(x) \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} u \, d\mathcal{L}$$

whenever $\bar{B}_r(x) \subset A$ (or greater or equal for superharmonic).

Also, some properties have been proved, such as:

1. The sum of two subharmonic functions or the maximum between them is subharmonic. On the other hand, the sum and the minimum between two superharmonic functions is superharmonic.
2. The maximum principle, which states that if a function u is subharmonic in a bounded open set and if

$$\limsup_{x \rightarrow \xi} u(x) \leq 0 \quad \text{for every } \xi \text{ in the boundary of our set,} \quad (1)$$

then $u \leq 0$ in the set. Also, if the set is connected, then either $u \equiv 0$ or $u < 0$.

3. If u is a $\mathcal{C}^2$ function in an open set, then u is subharmonic there if and only if $\Delta u \geq 0$.
4. The locality of subharmonicity, which states that given a $\rho > 0$ and u an upper semicontinuous function in Ω satisfying

$$u(x) \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} u \, d\mathcal{L},$$

whenever $\bar{B}_r(x) \subset \Omega$ with $r < \rho$, then u is subharmonic in Ω . The relevance of this lemma is that instead of having to satisfy this second property for balls of all radius, it only must be satisfied for $r < \rho$ for a certain fixed $\rho > 0$.

Second of all, Perron classes and resolutive functions have been studied. The objective of this section has been to associate to every function f continuous in the boundary of a set a harmonic function, u_f . To do so, the lower and upper Perron classes have been defined (they are linked to a function continuous in the boundary of the given set) and, from there, two harmonic functions, called the lower and upper Perron solution, have been defined (they have been defined as the supremum of functions in the lower Perron class or the infimum of functions in the upper Perron class respectively). Furthermore, the definition of a resolutive function has been given (the lower Perron class and the upper Perron class must be the same), and some examples of resolutive functions have been given.

Lastly, the definition of harmonic measure has been given. Let $\Omega \subset \mathbb{R}^d$ be open and bounded and let $x \in \Omega$. The harmonic measure for Ω based at x is the unique Radon measure ω^x (well-defined because of the Riesz representation theorem) on $\partial\Omega$ such that

$$H_f(x) = \int_{\partial\Omega} f(\xi) \, d\omega^x(\xi)$$

for all $f \in \mathcal{C}(\partial\Omega)$.

Finally, some properties regarding harmonic measure have been introduced, such as the fact that $u(x) := \omega^x(K)$, where $K \subset \partial\Omega$ is a compact set, is harmonic, or the fact that if $x, y \in \Omega$ are in the same connected component, then ω^x and ω^y are mutually absolutely continuous.

Also, some exercises regarding this last section have been made so as to comprehend in more detail the concept of harmonic measure and its relevance.

References

- [1] Martí Prats and Xavier Tolsa. “Harmonic measure in Euclidean spaces”. In: *Forthcoming book, see the author’s webpages* (2026).