

FINAL REPORT OF BIMR PROJECT ON PURE C*-algebraS

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ABSTRACT. These are part of the notes from my BIMR project on pure C*-algebras. The project was carried out under the supervision of Eduard Vilalta, whom I thank for his invaluable support. I have in particular investigated under which conditions unstable versions of pureness carry over to regular pureness. I plan to continue investigating this during the coming months.

1. PRELIMINARIES

Definition 1.1. Let A be a C*-algebra, and let $a, b \in A_+$. We say a is *Cuntz subequivalent* to b , denoted $a \precsim b$, if there is a sequence $(r_n)_n$ in A such that $a = \lim_n r_n b r_n^*$. We say a is *Cuntz equivalent* to b if $a \precsim b$ and $b \precsim a$, denoted $a \sim b$. For $a \in A_+$, we write $[a]$ for the equivalence class of a under $\sim$. The *Cuntz semigroup* of A is then defined as $(A \otimes K)_+ / \sim$ and is denoted $\text{Cu}(A)$. For elements x and y of $\text{Cu}(A)$, we write $x \leq y$ if there exists $a \in x$ and $b \in y$ such that $a \precsim b$. This is a well-defined partial order, for which $[0]$ is the smallest element. Furthermore, by fixing an isomorphism $M_2 \otimes K \cong K$, we can define addition on $\text{Cu}(A)$ through $[a] + [b] := [(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix})]$.

One may prove that this addition makes $\text{Cu}(A)$ a positively ordered monoid, that is, a set with an associative binary operation such that for any x, y and z we have $x \leq y$ implies $x + z \leq y + z$, as well as an identity element which is also the smallest element.

To get some understanding of the Cuntz relation, we started the project studying how Cuntz subequivalence behaves in the commutative case, as well as looking into Rørdam's lemma. We did this by studying the following results, which are taken from [3, 4].

Proposition 1.2. *Let X be a locally compact Hausdorff space, and let $f, g \in C_0(X)_+$. Then $f \precsim g$ if and only if $\text{supp}(f) \subseteq \text{supp}(g)$.*

Proposition 1.3. *Let A be a C*-algebra, $a \in A_+$ and $f : [0, \infty) \rightarrow [0, \infty)$ be a continuous function with $f(0) = 0$. Then the following hold:*

- (i) *We have $f(a) \precsim a$*
- (ii) *If $f(x) > 0$ for $x \neq 0$ then $f(a) \sim a$.*
- (iii) *For all $x \in A$, we have $x^* x \sim x x^*$.*

Definition 1.4. Let $a \in A$. We define $A_a := \overline{a^* A a}$, and call it the hereditary C*-algebra generated by a .

Definition 1.5. For $x, y \in \text{Cu}(A)$, choose $a \in (A \otimes K)_+$ with $y = [a]$. We write $x \ll y$ if there is some $\epsilon > 0$ such that $x \leq [(a - \epsilon)_+]$.

Theorem 1.6 (Rørdam's lemma). *Let A be a C*-algebra, and let $a, b \in A_+$. Then the following are equivalent:*

- (i) $a \precsim b$;
- (ii) For all $\epsilon > 0$, we have $(a - \epsilon)_+ \precsim b$;
- (iii) For all $\epsilon > 0$ there exists $\delta > 0$ such that $(a - \epsilon)_+ \precsim (b - \delta)_+$;
- (iv) For all $\epsilon > 0$ there exists $\delta > 0$ and $x \in A$ such that $(a - \epsilon)_+ = x^* x$ and $x x^* \in A_{(b - \delta)_+}$;
- (v) For all $\epsilon > 0$ there exists $\delta > 0$ and $r \in A$ such that $(a - \epsilon)_+ = r(b - \delta)_+ r^*$.

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2. SOME RESULTS

The project then continued with the study of scales of Cuntz semigroups (cf. [1, Paragraph 3.5]):

Definition 2.1. Let A be a C^* -algebra, and $\Sigma \subseteq \text{Cu}(A)$. We say Σ is a *scale* if:

- (1) If $x \in \text{Cu}(A)$ and $y \in \Sigma$ with $x \leq y$ then $x \in \Sigma$.
- (2) Σ is closed under suprema of increasing sequences.
- (3) For any $x, x' \in \text{Cu}(A)$ with $x' \ll x$ there exist $y_1, \dots, y_n \in \Sigma$ such that $x' \leq y_1 + \dots + y_n$.

Set $\Sigma_A := \{x \in \text{Cu}(A) : x \leq [a] \text{ for some } a \in A_+\}$. One may prove that this is a scale.

Proposition 2.2 ([5, Lemma 3.3]). *Let $x \in \Sigma_A$ with $x = [a]$ for some $a \in (A \otimes K)_+$. Then for all $\epsilon > 0$ there exists $c \in A_+$ such that $(a - \epsilon)_+ \sim c$.*

Proof. Let $\epsilon > 0$. Since $x \in \Sigma_A$ there is $b \in A_+$ such that $x \leq [b]$. By Rørdam's lemma, there exists some $c \in (A \otimes K)_+$ and $\delta > 0$ such that $(a - \epsilon)_+ \sim c$ and $c \in (A \otimes K)_{(b-\delta)_+}$. Since b is in A_+ , so is $(b - \delta)_+$ and therefore $c \in (A \otimes K)_{(b-\delta)_+} \subseteq A$. Thus $(a - \epsilon)_+ \sim c$ for $c \in A_+$ as desired. $\square$

The following are some properties of $\text{Cu}(A)$:

- (O5) For all x', x and y in $\text{Cu}(A)$ with $x' \ll x \leq y$ there exists $z \in \text{Cu}(A)$ such that $x' + z \leq y \leq x + z$. Furthermore, if for some $w, w' \in \text{Cu}(A)$ we have $x + w \leq y$, and $w' \ll w$ then z may be chosen such that $w' \ll z$.
- (O6) For every $a', a, b, c \in \text{Cu}(A)$ such that $a' \ll a \leq b + c$ there exist elements e and f in $\text{Cu}(A)$ such that $a' \leq e + f$, $e \leq a, b$ and $f \leq a, c$.
- (O7) Given $x'_1 \ll x_1 \leq w$ and $x'_2 \ll x_2 \leq w$ there exists x such that $x'_1, x'_2 \ll x \leq w, x_1 + x_2$

Definition 2.3. Let A be a C^* -algebra. We say $\text{Cu}(A)$ is *pure* if:

- (i) For all $x, y \in \text{Cu}(A)$, whenever $n \in \mathbb{N}$ is such that $(n + 1)x \leq ny$ then $x \leq y$.
- (ii) For all $k \in \mathbb{N}$ and all $x', x \in \text{Cu}(A)$ with $x' \ll x$ there is $y \in \text{Cu}(A)$ such that $x' \leq (k + 1)y$ and $ky \leq x$.

We say A is pure if $\text{Cu}(A)$ is. We say Σ_A is pure if the above conditions hold for elements of Σ_A . We say A is *unstably pure* if the first condition holds for elements of A_+ and for all $a \in A_+$, all $k \in \mathbb{N}$ and all $\epsilon > 0$ there is $b \in A_+$ such that $k[b] \leq [a]$ and $[(a - \epsilon)_+] \leq (k + 1)[b]$.

Theorem 2.4. *Let A be a C^* -algebra. Then Σ_A is pure if and only if A is unstably pure.*

Proof. Assume first that Σ_A is pure. For proving the first condition, let $a, b \in A_+$ and assume $(n + 1)[a] \leq n[b]$. Since $x \leq x$ for all $x \in \text{Cu}(A)$ we get $[a], [b] \in \Sigma_A$ and hence, by assumption, $[a] \leq [b]$. For the second condition, let $a \in A_+$, $k \in \mathbb{N}$ and $\epsilon > 0$. We have $[(a - \frac{\epsilon}{2})_+] \ll [a]$ so by assumption there is $y \in \Sigma_A$ such that $[(a - \frac{\epsilon}{2})_+] \leq (k + 1)y$ and $ky \leq [a]$. Since $[(a - \epsilon)_+] \ll [(a - \frac{\epsilon}{2})_+] \leq (k + 1)y$ we may by Proposition 2.2, get $b \in A_+$ such that $[(a - \epsilon)_+] \leq (k + 1)[b]$ and $k[b] \leq [a]$ as desired.

Now, assume instead that A is unstably pure. Let $x, y \in \Sigma_A$ and assume $(n + 1)x \leq ny$. Once more, use Proposition 2.2 to take $a' \in A_+$ with class arbitrarily Cuntz-close to x with $(n + 1)[a'] \ll (n + 1)x \leq ny$. We may then take $b' \in A_+$ such that $[b'] \leq y$ $(n + 1)[a'] \leq n[b']$. By assumption, we get $[a'] \leq [b']$ and thus $[a'] \leq y$. Since $\text{Cu}(A)$ is closed under suprema of increasing sequences, this implies $x \leq y$. Next, let $k \in \mathbb{N}$ and $x, x' \in \Sigma_A$ such that $x' \ll x$ and $x = [a]$ for $a \in (A \otimes K)_+$. We have that there is $\epsilon > 0$ such that $x' \leq [(a - \epsilon)_+] \ll [(a - \frac{\epsilon}{2})_+]$. By Proposition 2.2 we get $c \in A_+$ such that $(a - \frac{\epsilon}{2})_+ \sim c$. Further, there exists $\delta > 0$ such that $(a - \epsilon)_+ \preceq (c - \delta)_+$ so we may use our assumption to get $b \in A_+$ with $k[b] \leq [c] = [(a - \frac{\epsilon}{2})_+] \leq [a] = x$ and $[(a - \epsilon)_+] \leq [(c - \delta)_+] \leq (k + 1)[b]$. Since $x' \leq [(a - \epsilon)_+]$, we are done. $\square$

Proposition 2.5. *Let A be a C^* -algebra. If A is unstably pure and $[b] \in \Sigma_A^{(2)}$, then for all $\epsilon > 0$ and all $k \in \mathbb{N}$ there exists $x \in \Sigma_A^{(2)}$ such that*

$$kx \leq 2[b], \quad \text{and} \quad [(b - \epsilon)_+] \leq 2(k + 1)x.$$

Proof. Since $[b] \in \Sigma_A^{(2)}$, there exists $a_1, a_2 \in A_+$ such that $[b] \leq [a_1] + [a_2]$. Furthermore, since $[(b - \frac{\epsilon}{2})_+] \ll [b]$, we may use (O6) to get $[a'_1]$ and $[a'_2]$ with

$$[(b - \frac{\epsilon}{2})_+] \leq [a'_1] + [a'_2], \quad [a'_i] \leq [a_i], \quad \text{and} \quad [a'_i] \leq [b] \text{ for } i = 1, 2.$$

Then $[(b - \epsilon)_+] \ll [a'_1] + [a'_2]$ so we may choose a''_1 and a''_2 with $[a''_i] \ll [a'_i]$ for $i = 1, 2$ such that $[(b - \epsilon)_+] \leq [a''_1] + [a''_2]$. Then, by assumption, there exist $[c_1], [c_2] \in \Sigma_A$ such that for $i = 1, 2$, we have $[a''_i] \leq (k+1)[c_i]$ and $k[c_i] \leq [a'_i]$. Thus, we have

$$[b] \leq [a''_1] + [a''_2] \leq (k+1)[c_1] + (k+1)[c_2] = (k+1)([c_1] + [c_2]) \leq 2(k+1)([c_1] + [c_2])$$

and $k[c_1] + k[c_2] = k([c_1] + [c_2]) \leq [a'_1] + [a'_2] \leq 2[b]$. Since $[c_1] + [c_2] \in \Sigma_A^{(2)}$, setting $x := [c_1] + [c_2]$, gives the desired result. $\square$

Lemma 2.6. *If A is a C*-algebra with real rank zero, then for all $a \in A_+$ and $\epsilon > 0$ there is a projection $p \in A$ such that $(a - \epsilon)_+ \precsim [p] \precsim a$.*

Proof. By [2, Theorem 2.6], A having real rank zero implies that for every $a \in A_+$ the hereditary C*-subalgebra generated by a has an approximate unit of projections. For $\epsilon > 0$ we may therefore choose a projection p such that $\|a - pap\| < \epsilon$. We then have $(a - \epsilon)_+ \precsim pap \sim a^{\frac{1}{2}}pa^{\frac{1}{2}} \precsim p \leq a$. $\square$

Lemma 2.7. *If a C*-algebra A has real rank zero, $x \in \Sigma_A^{(n)}$ and $x' \ll x$ then there exist projections $r_1, \dots, r_n$ such that $x' \leq [r_1] + \dots + [r_n] \leq x$*

Proof. Take x'', x''' such that $x' \ll x'' \ll x''' \ll x$. Since $x \in \Sigma_A^{(n)}$, there exist $a_1, \dots, a_n$ such that $x'' \ll x''' \leq [a_1] + \dots + [a_n]$ and we may therefore find $\delta > 0$ such that $x' \ll x'' \leq [(a_1 - \delta)_+] + \dots + [(a_n - \delta)_+]$. We now use Lemma 2.6 to find projections $p, q_1, \dots, q_n$ such that $x' \leq [p] \leq x''$ and $[(a_i - \delta)_+] \leq [q_i] \leq [a_i]$ for $i = 1, \dots, n$. We thus have $[p] \leq [q_1] + \dots + [q_n]$, so by [6, Corollary 1.3] there exist projections $r_1, \dots, r_n$ such that $[p] = [r_1] + \dots + [r_n]$. Since $x' \leq [p] \leq x$, the lemma follows. $\square$

Theorem 2.8. *If a C*-algebra A has real rank zero and A is unstably pure, then $\Sigma_A^{(n)}$ has almost-divisibility for all $n \in \mathbb{N}$.*

Proof. Let $[b] \in \Sigma_A^{(n)}$, $\epsilon > 0$ and $k \in \mathbb{N}$. We use Theorem 2.7 to get $[(b - \frac{\epsilon}{2})_+] \leq [r_1] + \dots + [r_n] \leq [b]$. We may thus find $r'_1, \dots, r'_n$ such that $[(b - \epsilon)_+] \leq [r'_1] + \dots + [r'_n]$ and $[r'_i] \ll [r_i]$ for $i = 1, 2, \dots, n$. We may then use that A is unstably pure to find $y_1, \dots, y_n$ in Σ_A such that for $i = 1, 2, \dots, n$ we have $[r_i] \leq (k+1)y_i$ and $ky_i \leq [r_i]$.

Therefore $[(b - \epsilon)_+] \leq [r'_1] + \dots + [r'_n] \leq (k+1)y_1 + \dots + (k+1)y_n = (k+1)(y_1 + \dots + y_n)$ and $ky_1 + \dots + ky_n \leq [r_1] + \dots + [r_n] \leq [b]$. So, choosing $y := y_1 + \dots + y_n$ gives the desired result. $\square$

Question. Under which assumptions does unstable almost divisibility of A imply that $\Sigma_A^{(n)}$ is almost divisible?

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