

Can algebra tell us whether we have enough data?

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1 Colored Graphical Models

We will start this section making a brief introduction to the notation we will use:

- $\text{Sym}(m)$ for $m \times m$ symmetric matrices.
- $\text{PD}(m)$ for $m \times m$ positive definite matrices.
- $\text{PSD}(m)$ for $m \times m$ positive semidefinite matrices.

We write $\text{Sym}(m, n)$ for $m \times m$ symmetric matrices with $\text{rank} \leq n$. Analogous, we write $\text{PSD}(m, n)$.

Graphical intuition: $\text{PD}(m)$ and $\text{PSD}(m)$ are convex cones with vertex $M = 0$. That's due to the fact that if A is a positive (semi)definite matrix, then λA is also a positive (semi)definite matrix for all $\lambda \in \mathbb{R}_{\geq 0}$.

- The interior of the cone is constituted by the space $\text{PD}(m)$.
- The closure of the cone is constituted by the space $\text{PSD}(m)$.

Remark. The boundary is defined by the equation $\det M = 0$.

Definition 1.1. A colored graph is $G = (V, E, \lambda)$ an undirected on vertex set $V = [m] = \{1, \dots, m\}$ and edge set E , together with a coloring function

$$\lambda : V \sqcup E \rightarrow [d] = \{1, \dots, d\}$$

such that $\lambda(V) \cap \lambda(E) = \emptyset$.

In this definition d represents the number of colors and we impose that edges and vertices cannot share the same color. This last condition is necessary for statistical interpretation.

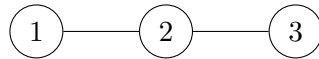
Remark. The colors of a graph will be interpreted as symmetries in the concentration matrix, which is the inverse of the covariance matrix: $K = \Sigma^{-1}$. (linear concentration models)

1.1 Where does this covariance matrix come from?

To each vertex of the graph we assign a random variable. All the random variables constitute an m -dimensional random vector $X = (X_1, \dots, X_m) \sim \mathcal{N}(0, \Sigma)$, which follows a multivariate Gaussian distribution.

1.1.1 What is the interpretation of the vertices being random variables?

We are representing statements of conditional dependence by connecting (or not) the vertices with edges. The graph encodes conditional dependence statements between these random variables. For example, consider the path



This graph represents the conditional independence relation

$$X_1 \perp X_3 \mid X_2,$$

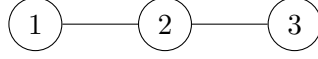
which is known as the *Markov property*. In other words, once the value of X_2 is known, the random variables X_1 and X_3 become independent.

1.2 Concentration matrices and conditional independence

The conditional independence relations encoded by a graph are reflected by the sparsity pattern of its concentration matrix

$$K = \Sigma^{-1}.$$

For instance, the path



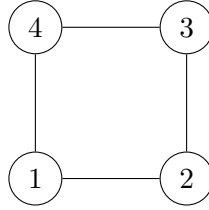
corresponds to a concentration matrix of the form

$$K = \begin{pmatrix} \lambda_1 & \lambda_4 & 0 \\ \lambda_4 & \lambda_2 & \lambda_5 \\ 0 & \lambda_5 & \lambda_3 \end{pmatrix}.$$

The zero entry in position $(1, 3)$ implies

$$X_1 \perp X_3 \mid X_2.$$

Example 1.2. We could also study the following graph.



This corresponds to the concentration matrix

$$K = \begin{pmatrix} \lambda_1 & \lambda_5 & 0 & \lambda_8 \\ \lambda_5 & \lambda_2 & \lambda_6 & 0 \\ 0 & \lambda_6 & \lambda_3 & \lambda_7 \\ \lambda_8 & 0 & \lambda_7 & \lambda_5 \end{pmatrix}.$$

With the corresponding conditional independence statements:

$$X_1 \perp X_3 \mid \{X_2, X_4\}$$

$$X_2 \perp X_4 \mid \{X_1, X_3\}$$

In the next two propositions we are following [4].

Proposition 1.3. (Prop. 4.1.9.) *The conditional independence statement $X_A \perp X_B \mid X_C$ holds for a multivariate normal random vector $X \sim \mathcal{N}(\mu, \Sigma)$ if and only if the submatrix $\Sigma_{A \cup C, B \cup C}$ of the covariance matrix Σ has rank $\#C$.*

Proof. If $X \sim \mathcal{N}(\mu, \Sigma)$, then the conditional distribution of $X_{A \cup B}$ given $X_C = x_C$ is the multivariate normal distribution

$$\mathcal{N}\left(\mu_{A \cup B} + \Sigma_{A \cup B, C} \Sigma_{C, C}^{-1} (x_C - \mu_C), \Sigma_{A \cup B, A \cup B} - \Sigma_{A \cup B, C} \Sigma_{C, C}^{-1} \Sigma_{C, A \cup B}\right),$$

by Theorem 2.4.2.

The statement $X_A \perp X_B \mid X_C$ holds if and only if

$$\left(\Sigma_{A \cup B, A \cup B} - \Sigma_{A \cup B, C} \Sigma_{C, C}^{-1} \Sigma_{C, A \cup B} \right)_{A, B} = \Sigma_{A, B} - \Sigma_{A, C} \Sigma_{C, C}^{-1} \Sigma_{C, B} = 0,$$

by Proposition 2.4.4.

The matrix

$$\Sigma_{A, B} - \Sigma_{A, C} \Sigma_{C, C}^{-1} \Sigma_{C, B}$$

is the Schur complement of the matrix

$$\Sigma_{A \cup C, B \cup C} = \begin{pmatrix} \Sigma_{A, B} & \Sigma_{A, C} \\ \Sigma_{C, B} & \Sigma_{C, C} \end{pmatrix}.$$

Since $\Sigma_{C, C}$ is always invertible (it is positive definite), the Schur complement is zero if and only if the matrix $\Sigma_{A \cup C, B \cup C}$ has rank equal to $\#C$. $\square$

Proposition 1.4. (Prop. 6.3.2.) *Let $K = \Sigma^{-1}$ be the concentration matrix of a Gaussian random vector. Then*

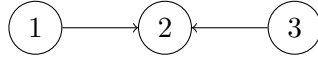
$$k_{i, j} = 0 \iff i \perp j | [m] \setminus \{i, j\}$$

Proof. Denote $C = [m] \setminus \{i, j\}$. Then, by the inversion formula of a matrix using the adjoint matrix we obtain $k_{i, j} = (-1)^{i+j} \det(\Sigma_{j \cup C, i \cup C}) / \det(\Sigma)$. Therefore, $k_{i, j} = 0$ iff $\det(\Sigma_{j \cup C, i \cup C}) = 0$ $\square$

Remark. For jointly Gaussian random variables, independence is equivalent to zero covariance.

Important: Independence statements correspond to zeros in the covariance matrix Σ , whereas conditional independence statements correspond to zeros in the concentration matrix K .

Example 1.5. For example, we can consider the following graph, which has the entry $s_{13} = 0$ which means $X_1 \perp X_3$:



All the possible orientations for the arrows implies $X_1 \perp X_3 | X_2$, except for the one we have represented, which is an example of a collider.

Remark. Conditional independent statements (zeros in the concentration matrix) can be expressed in terms of the entries of the covariance matrix, by the following:

$$S = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{12} & s_{22} & s_{23} \\ s_{13} & s_{23} & s_{33} \end{pmatrix} \implies (k_{13} = 0 \iff s_{12}s_{23} - s_{22}s_{13} = 0);$$

Here we have used $K = S^{-1}$ and the adjoint matrix to compute the entries of the inverse

1.3 Description of the model

Then, the model can be defined by describing the type of matrices that we are going to accept.

Definition 1.6. We define $\mathcal{L}_G$ as the following linear vector subspace of $\dim(\mathcal{L}_G) = d$, with d representing the number of colors (if G is uncolored, then we consider a different color for each edge and vertex).

$$\mathcal{L}_G := \{K \in \text{Sym}(m) : k_{ij} = 0 \text{ if } i - j \notin E, k_{ii} = k_{jj} \text{ if } \lambda(i) = \lambda(j), k_{ij} = k_{lw} \text{ if } \lambda(i, j) = \lambda(l, w)\}$$

For the statistical interpretation of these models, see [1]. For statistical interpretation see: Graphical Gaussian Models with Edge and Vertex Symmetries (Soren Hojsgaard, Heffen L. Lauritzen, 2008).

Definition 1.7. To be a covariance matrix Σ must be symmetric and positive definite, then:
 $\Sigma \in \text{PD}(m) \implies K = \Sigma^{-1} \in \text{PD}(m)$.

Thus, we consider:

$$\mathcal{K}_G = \mathcal{L}_G \cap \text{PD}(m)$$

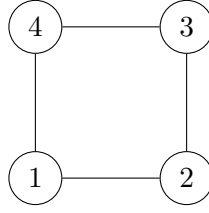
Which is known as the cone of concentration matrices.

This, can be seen in a more explicit way as:

$$\mathcal{K}_G = \{(\lambda_1, \dots, \lambda_d) \in \mathbb{R}^d : \lambda_1 k_1 + \dots + \lambda_d k_d \in \text{PD}(m)\}$$

Here, we consider $\mathcal{L}_G = \langle k_1, \dots, k_d \rangle$.

Example 1.8. 4-cycle uncolored



In this example the number of colors corresponds to: $\dim(\mathcal{L}_G) = 8$.

The generator of the first vertex 1 could be expressed as

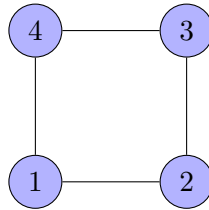
$$k_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the generator of the first edge (1,2) as

$$k_5 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Example 1.9. 4-cycle with equal colored vertices

Now, since we aim to study colored graphs which encode relations among variables, let us study the previous Example 1.8 but adding a relation between the vertices. In particular, we study the case where all the vertices are related. This relation is encoded by the blue color of the vertices.



Observe that edges are uncolored, which means that we are thinking of each edge to be a different color (different between them and also different from blue). As a consequence of this, $\dim(\mathcal{L}_g) = 5$ and we can write

$$\mathcal{L}_g = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & 0 & \lambda_5 \\ \lambda_2 & \lambda_1 & \lambda_3 & 0 \\ 0 & \lambda_3 & \lambda_1 & \lambda_4 \\ \lambda_5 & 0 & \lambda_4 & \lambda_1 \end{pmatrix} : \lambda_i \in \mathbb{R} \right\}.$$

Remark. Here, we just want to clarify that we use m to denote the number of vertices as $V = [m]$ and n to denote the number of observations $X^{(1)}, \dots, X^{(n)} \in \mathbb{R}^m$ i.i.d. drawn $\mathcal{N}(0, \Sigma)$.

Definition 1.10. Sample Covariance Matrix

$$S = \frac{1}{n} X X^T \in \mathbb{R}^{m \times m}, \text{ where } X = (X^{(1)} | \dots | X^{(n)}).$$

We will sometimes ignore the factor $\frac{1}{n}$, since it does not change the algebraic behavior. Also, it is important to notice that we can always say that $S \in \text{PSD}(m, n)$, as $v^T S v = \frac{1}{n} v^T X X^T v = \frac{1}{n} (v^T X)(v^T X)^T = \frac{1}{n} \|v^T X\|^2 \geq 0$, but it could be $\text{rank}(S) \leq m$.

Definition 1.11. Log-likelihood Function

Up to an additive constant and a positive multiplicative factor, the log-likelihood can be written as

$$\ell_s(K) := \log(\det K) - \langle S, K \rangle$$

where $\langle S, K \rangle = \text{Tr}(SK) = \sum_{1 \leq i, j \leq m} s_{ij} k_{ij}$ is the trace inner product, that behaves as the classical scalar product if we write the matrices as vectors.

This function can be also written as

$$\ell_S(\Sigma) = -\log(\det \Sigma) - \langle S, \Sigma^{-1} \rangle,$$

but the first expression has better properties.

2 Existence of the MLE

2.1 Our Convex Optimization Problem

By the above definitions and basic statistical knowledge, we have reduced the problem to

$$\text{Finding MLE of } K \text{ for } S \iff \text{maximize } K \text{ subject to } \mathcal{K}_G \{\log(\det K) - \langle S, K \rangle\}$$

Since ℓ_S is a strictly concave function on the convex set K_G , any maximum, if it exists, is unique. It is also important to remark that finding the MLE for K is equivalent to finding the MLE for Σ since $\hat{\Sigma} = \hat{K}^{-1}$, which is a well-defined inverse as we are working over $PD(m)$.

Looking at the expression we have obtained, it is important to notice that the only part that depends on the sample is $\langle S, K \rangle$. Consequently, it is interesting to study the following projection

$$\begin{aligned} \pi_G : \text{Sym}(m) &\longrightarrow \mathbb{R}^d \\ S &\longmapsto (\langle S, k_1 \rangle, \dots, \langle S, k_d \rangle). \end{aligned}$$

And due to the fact that $\langle S, K \rangle = \sum \lambda_i \langle S, k_i \rangle$, we obtain that $\langle S, k_i \rangle$ are sufficient statistics (maybe we need less depending on the color relations of the elements of the graph).

Then, we obtain

$$S_1, S_2 \in \text{Sym}(m) \text{ s.t. } \pi_G(S_1) = \pi_G(S_2) \implies \ell_{S_1}(K) = \ell_{S_2}(K).$$

Example 2.1. Exercise: Compute the projection explicitly for 1.9

First we have to notice that our dimension is $m = 4$ and the number of colors is $d = 5$, then:

$$\begin{aligned} \pi_G : \text{Sym}(4) &\longrightarrow \mathbb{R}^5 \\ S &\longmapsto (\langle S, k_1 \rangle, \dots, \langle S, k_5 \rangle). \end{aligned}$$

with the matrices k_i described in 1.9. After a straightforward calculation we obtain:

$$\begin{aligned} \langle S, k_1 \rangle &= \text{Tr}(S) \\ \langle S, k_2 \rangle &= 2s_{12} \\ \langle S, k_3 \rangle &= 2s_{23} \\ \langle S, k_4 \rangle &= 2s_{34} \\ \langle S, k_5 \rangle &= 2s_{14} \end{aligned}$$

2.2 Geometrical Interpretation

We are interested in studying the open Cone of Sufficient Statistics

$$\mathcal{C}_G := \pi_G(\text{PD}(m))$$

and its Euclidean closure

$$\text{cl}(\mathcal{C}_G) = \pi_G(\text{PSD}(m)).$$

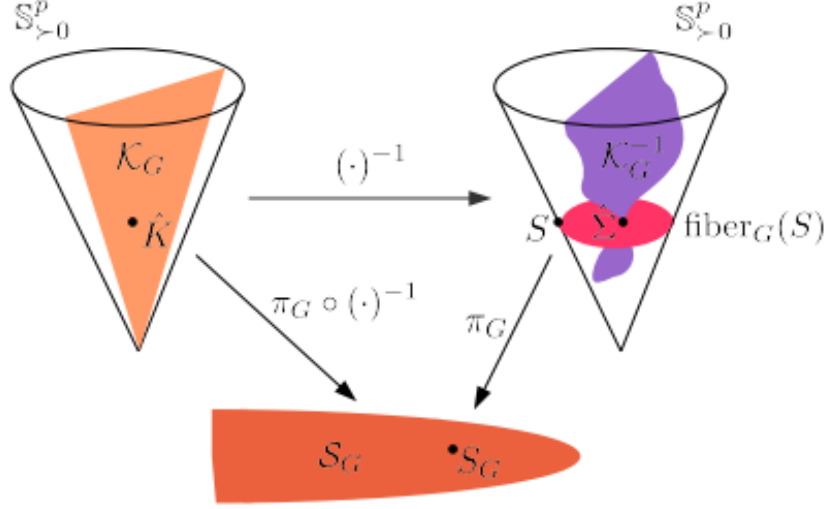


Figure 1: Geometry of maximum likelihood estimation in Gaussian graphical models. The cone $\mathcal{K}_G$ consists of all concentration matrices in the model and $\mathcal{K}_G^{-1}$ is the corresponding set of covariance matrices. The cone of sufficient statistics $\mathcal{S}_G$ is defined as the projection of $\mathcal{S}$. It is dual and homeomorphic to $\mathcal{K}_G$. Given a sample covariance matrix S , $\text{fiber}_G(S)$ consists of all positive definite completions of the G -partial matrix S_G , and it intersects $\mathcal{K}_G^{-1}$ in at most one point, namely the MLE $\hat{\Sigma}$.

Remark. $\text{fiber}_G(S) = \{S' \in \text{Sym}(m) | \pi_G(S) = \pi_G(S')\}$ are the matrices that "are the same" for the inner trace product.

Theorem 2.2. *The MLE exists for $S \iff \pi_G(S) \in \mathcal{C}_G$.*

Theorem 2.3. *The MLE $\hat{\Sigma}$ exists $\iff \text{fiber}_G(S) \cap \text{PD}(m) \neq \emptyset$.*

If it exists $\hat{\Sigma}$ is the only point in the $\text{fiber}_G(S)$ s.t. $\hat{\Sigma}^{-1} \in \mathcal{K}_G$, therefore we can write it as the following threshold:

$$\hat{\Sigma} = \text{fiber}_G(S) \cap \mathcal{K}_G^{-1}.$$

Definition 2.4. MLT and WMLT

- For $S \in \text{PSD}(m)$ MLT(G) is the minimum n s.t. $\pi_G(S) \in \mathcal{C}_G$ almost surely.
- For $S \in \text{PSD}(m)$ WMLT (G) is the minimum n s.t. $\pi_G(S) \in \mathcal{C}_G$ for some $\pi_G(S) \in \mathcal{C}_G$ (with positive probability).

2.3 Existence and Uniqueness of the Maximum Likelihood Estimator

Now, our goal is to prove the Theorem 2.3. Recall that the maximum likelihood estimation problem can be written as

$$\hat{K} = \arg \max_{K \in \mathcal{K}_G} \{\log(\det K) - \langle S, K \rangle\}.$$

Since $\mathcal{K}_G$ is an open convex cone, it is natural to ask whether this optimization problem admits a solution and whether such a solution is unique.

Proposition 2.5. *If $S \in \text{PD}(m)$, maximizing $\ell_S(K)$ has a unique solution.*

Proof. We will just prove one direction and we will divide the proof into three steps.

Step 1. $\ell_S(K)$ is strictly concave in $\text{PD}(m)$.

Remember that a function is called strictly concave if it satisfies:

$$f((1 - \alpha)x + \alpha y) > (1 - \alpha)f(x) + \alpha f(y) \text{ for any } \alpha \in (0, 1) \text{ and } x \neq y.$$

Since $\text{tr}(SK)$ is linear, it suffices to prove that $\log \det(K)$ is strictly concave over the cone domain.

Then it is enough to prove it for any matrix in a line: $Y + tV \in \mathbb{S}_{>0}^p$ s.t. $Y \in \mathbb{S}_{>0}^p$.

Doing the calculus:

$$\begin{aligned} \log \det(Y + tV) &= \log \det(Y^{1/2}(I + t(Y^{-1/2}VY^{-1/2}))Y^{1/2}) = \log(\det Y \det(I + t(Y^{-1/2}VY^{-1/2}))) \\ &= \log \det(Y) + \log \det(I + t(Y^{-1/2}VY^{-1/2})) = \log \det(Y) + \log \sum 1 + t\lambda_i \end{aligned}$$

And since the first summand is a constant and the second is concave, we conclude.

Step 2. Existence.

By the Extreme Value Theorem, we know that a continuous function in a non-empty compact set has maximum and minimum. So let us consider $k_0 \in \mathcal{K}_G$ and the following (compact) set

$$L = \{k \in \mathcal{K}_G | \ell_S(k) \geq \ell_S(k_0)\}.$$

- $L \neq \emptyset$ because $k_0 \in L$.
- L is closed because

$$\lim_{\det(k) \rightarrow 0} \ell_S(k) = -\infty$$

- L is bounded because

$$\lim_{|k| \rightarrow \infty} \ell_S(k) = -\infty$$

Consequently, by the Heine-Borel lemma it is a compact (and non-empty) set. Therefore, it exists a maximum.

Step 3. Uniqueness.

By contradiction, let us assume there exist two matrices $k_1, k_2 \in \text{PD}(m)$ s.t. they both reach the maximum. Then, by Step 1, as the function is concave:

$$\ell_S((1 - \alpha)k_1 + \alpha k_2) > (1 - \alpha)\ell_S(k_1) + \alpha\ell_S(k_2) = \ell_S(k_1) = M.$$

Ergo we reach a contradiction because $(1 - \alpha)k_1 + \alpha k_2 \in \text{PD}(m)$ is strictly bigger than the maximum. $\square$

Remark. Notice that in the step 2 to prove that L is bounded it is necessary to assume S is a positive definite matrix.

Proposition 2.6. *If $\hat{\Sigma}$ is the solution of the convex optimization problem, then $\hat{\Sigma} \in \text{fiber}_G(S)$. Equivalently,*

$$\pi_G(\hat{\Sigma}) = \pi_G(S).$$

Sketch of the proof. Recall that

$$\text{fiber}_G(S) = \{\Sigma \in \text{PD}(m) : \pi_G(\Sigma) = \pi_G(S)\}.$$

The maximum likelihood estimator is obtained by minimizing

$$-\log(\det \Sigma)$$

subject to the linear constraints

$$\langle \Sigma, k_i \rangle = \langle S, k_i \rangle, \quad i = 1, \dots, d,$$

where $k_1, \dots, k_d$ form a basis of L_G .

To solve this constrained optimization problem, we introduce the Lagrangian

$$\mathcal{L}(\Sigma, \lambda) = -\log(\det \Sigma) + \sum_{i=1}^d \lambda_i (\langle \Sigma, k_i \rangle - \langle S, k_i \rangle).$$

At an optimum, the gradient of the Lagrangian with respect to Σ must vanish. Using the identities

$$\nabla_{\Sigma}(-\log \det \Sigma) = -\Sigma^{-1}, \quad \nabla_{\Sigma} \langle \Sigma, k_i \rangle = k_i,$$

we obtain

$$-\Sigma^{-1} + \sum_{i=1}^d \lambda_i k_i = 0.$$

Therefore,

$$\Sigma^{-1} = \sum_{i=1}^d \lambda_i k_i \in L_G.$$

Since Σ is positive definite, it follows that

$$\Sigma^{-1} \in L_G \cap \text{PD}(m) = K_G.$$

On the other hand, the constraints ensure that

$$\Sigma \in \text{fiber}_G(S).$$

Hence the maximum likelihood estimator satisfies

$$\hat{\Sigma} \in \text{fiber}_G(S) \cap K_G^{-1},$$

which proves the claim. □

Corollary 2.7. *If $\text{fiber}_G(S)$ contains a PD matrix, then the MLE of Σ exists for S .*

3 Approaching the problem from algebraic geometry perspective and computational algebra

Instead of studying $\text{PSD}(m, n)$, we will study $\pi_G(\text{Sym}(m, n))$ in order to obtain equations that define geometric varieties (observe that inequations given by the (semi)positive definition are not easy to study as a geometric object).

Let us denote

$$I_n(S) = \langle \text{minors } n \times n \text{ of } S = (s_{ij}) \rangle$$

Example 3.1. If we consider the 2-cycle with both vertices colored (with $m = 2, n = 1$) and the associated matrix

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix}$$

We obtain $I_1 = \langle s_{11}s_{22} - s_{12}^2 \rangle$, by definition of the matrix S .
And by the color conditions we add

$$I_1 + \langle t_1 - (s_{11} + s_{22}), t_2 - 2s_{12} \rangle.$$

Then the elimination ideal is

$$I_{G,1} = (I_1 + \langle t_1 - (s_{11} + s_{22}), t_2 - 2s_{12} \rangle) \cap k[t_1, t_2] = 0$$

But if we consider the same graph uncolored, then $I_{G,1}$ is nonzero as we have the following expression:

$$I_{G,1} = (I_1 + \langle t_1 - s_{11}, t_2 - s_{22}, t_3 - 2s_{12} \rangle) \cap k[t_1, t_2, t_3]$$

which at least contains the polynomial $s_{11}s_{22} - s_{12}^2 = t_1t_2 - (\frac{t_3}{2})^2$.

Proposition 3.2. *If $I_{G,n} = 0$, then MLE exists with probability $p=1$. But the other implication is not true (sufficient but not necessary).*

Proof. We will not do a rigorous proof, but the idea is that □

The cases where $I_{G,n} \neq 0$ require a study of the boundary conditions of the cone of sufficient statistics to find out if the MLE exists with probability $p \in (0, 1)$ or $p = 0$. However, studying this boundary conditions can be complicated.

Definition 3.3. GCR

It can be interesting to define here $\text{GCR}(G)$ as the minimum number n such that $I_{G,n} = 0$. This definition is related with $\text{MLT}(G)$ as we have:

$$\text{MLT}(G) \leq \text{GCR}(G).$$

Remark. Notice that we can reduce the cases to study using that two graphs share the same ideal if they are isomorphic under vertices permutations. Moreover, if it has been proved the existence for some case, then the MLE also exists for all the subcases due to the fact that adding more color-symmetries can only reduce the number of observations required for the existence of the MLE.

To begin with, we will calculate the Elimination Ideals using `Macaulay2`, in order to cover the cases where we can assure the existence with $p=1$. Therefore in the following pages we will cover the study of some of the easiest cases.

3.1 Studying 3-cycles

We will begin the study of the 3-cycle graphical models depending on their colors and on the number of observations.

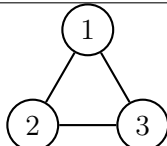
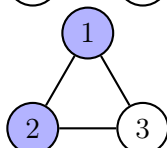
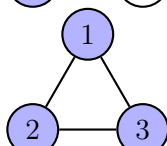
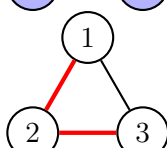
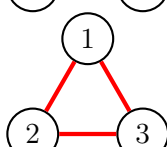
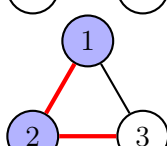
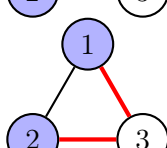
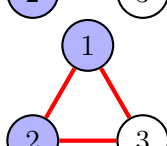
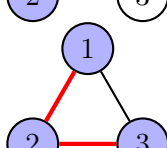
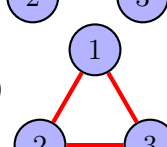
	G	K	$\pi_G : \text{Sym}(3) \longrightarrow \mathbb{R}^d$
(1)		$\begin{pmatrix} \lambda_1 & \lambda_4 & \lambda_6 \\ \lambda_4 & \lambda_2 & \lambda_5 \\ \lambda_6 & \lambda_5 & \lambda_3 \end{pmatrix}$	$(s_{11}, s_{22}, s_{33}, 2s_{12}, 2s_{23}, 2s_{13})$
(2)		$\begin{pmatrix} \lambda_1 & \lambda_3 & \lambda_5 \\ \lambda_3 & \lambda_1 & \lambda_4 \\ \lambda_5 & \lambda_4 & \lambda_2 \end{pmatrix}$	$(s_{11} + s_{22}, s_{33}, 2s_{12}, 2s_{23}, 2s_{13})$
(3)		$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_4 \\ \lambda_2 & \lambda_1 & \lambda_3 \\ \lambda_4 & \lambda_3 & \lambda_1 \end{pmatrix}$	$(s_{11} + s_{22} + s_{33}, 2s_{12}, 2s_{23}, 2s_{13})$
(4)		$\begin{pmatrix} \lambda_1 & \lambda_4 & \lambda_5 \\ \lambda_4 & \lambda_2 & \lambda_4 \\ \lambda_5 & \lambda_4 & \lambda_3 \end{pmatrix}$	$(s_{11}, s_{22}, s_{33}, 2(s_{12} + s_{23}), 2s_{13})$
(5)		$\begin{pmatrix} \lambda_1 & \lambda_4 & \lambda_4 \\ \lambda_4 & \lambda_2 & \lambda_4 \\ \lambda_4 & \lambda_4 & \lambda_3 \end{pmatrix}$	$(s_{11}, s_{22}, s_{33}, 2(s_{12} + s_{23} + s_{13}))$
(6)		$\begin{pmatrix} \lambda_1 & \lambda_3 & \lambda_4 \\ \lambda_3 & \lambda_1 & \lambda_3 \\ \lambda_4 & \lambda_3 & \lambda_2 \end{pmatrix}$	$(s_{11} + s_{22}, s_{33}, 2(s_{12} + s_{23}), 2s_{13})$
(7)		$\begin{pmatrix} \lambda_1 & \lambda_3 & \lambda_3 \\ \lambda_3 & \lambda_1 & \lambda_4 \\ \lambda_3 & \lambda_4 & \lambda_2 \end{pmatrix}$	$(s_{11} + s_{22}, s_{33}, 2(s_{12} + s_{13}), 2s_{23})$
(8)		$\begin{pmatrix} \lambda_1 & \lambda_3 & \lambda_3 \\ \lambda_3 & \lambda_1 & \lambda_3 \\ \lambda_3 & \lambda_3 & \lambda_2 \end{pmatrix}$	$(s_{11} + s_{22}, s_{33}, 2(s_{12} + s_{23} + s_{13}))$
(9)		$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_2 & \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_2 & \lambda_1 \end{pmatrix}$	$(s_{11} + s_{22} + s_{33}, 2(s_{13} + s_{23}), 2s_{12})$
(10)		$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_2 \\ \lambda_2 & \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_2 & \lambda_1 \end{pmatrix}$	$(s_{11} + s_{22} + s_{33}, 2(s_{12} + s_{23} + s_{13}))$

Table 1: Parameters configuration and projections for colored 3-cycles.

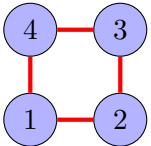
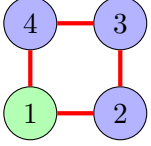
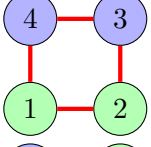
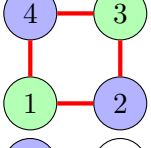
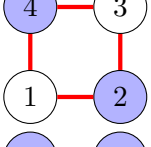
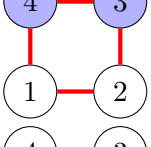
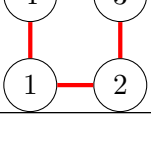
Model	$I_{G,1}$ (Minors 2×2)	$I_{G,2}$ (Determinant)	1 obs.	≥ 2 obs.
(1)	$\neq 0$	$\neq 0$	?	?
(2)	$\neq 0$	0	?	$p = 1$
(3)	$\neq 0$	0	?	$p = 1$
(4)	$\neq 0$	0	?	$p = 1$
(5)	$\neq 0$	0	?	$p = 1$
(6)	$\neq 0$	0	?	$p = 1$
(7)	$\neq 0$	0	?	$p = 1$
(8)	0	0	$p = 1$	$p = 1$
(9)	0	0	$p = 1$	$p = 1$
(10)	0	0	$p = 1$	$p = 1$

Table 2: Elimination Ideals and existence of MLE (3-cycle).

3.2 Studying 4-cycles

Now, let us do an analogous study for the 4-cycles.

Table 3: Parameters configuration and projections for colored 4-cycles.

	G	K	$\pi_G : \text{Sym}(4) \rightarrow \mathbb{R}^d$
(1)		$\begin{pmatrix} \lambda_1 & \lambda_2 & 0 & \lambda_2 \\ \lambda_2 & \lambda_1 & \lambda_2 & 0 \\ 0 & \lambda_2 & \lambda_1 & \lambda_2 \\ \lambda_2 & 0 & \lambda_2 & \lambda_1 \end{pmatrix}$	$(s_{11} + s_{22} + s_{33} + s_{44}, 2(s_{12} + s_{23} + s_{34} + s_{14}))$
(2)		$\begin{pmatrix} \lambda_1 & \lambda_3 & 0 & \lambda_3 \\ \lambda_3 & \lambda_2 & \lambda_3 & 0 \\ 0 & \lambda_3 & \lambda_2 & \lambda_3 \\ \lambda_3 & 0 & \lambda_3 & \lambda_2 \end{pmatrix}$	$(s_{11}, s_{22} + s_{33} + s_{44}, 2(s_{12} + s_{23} + s_{34} + s_{14}))$
(3)		$\begin{pmatrix} \lambda_1 & \lambda_3 & 0 & \lambda_3 \\ \lambda_3 & \lambda_1 & \lambda_3 & 0 \\ 0 & \lambda_3 & \lambda_2 & \lambda_3 \\ \lambda_3 & 0 & \lambda_3 & \lambda_2 \end{pmatrix}$	$(s_{11} + s_{22}, s_{33} + s_{44}, 2(s_{12} + s_{23} + s_{34} + s_{14}))$
(4)		$\begin{pmatrix} \lambda_1 & \lambda_3 & 0 & \lambda_3 \\ \lambda_3 & \lambda_2 & \lambda_3 & 0 \\ 0 & \lambda_3 & \lambda_1 & \lambda_3 \\ \lambda_3 & 0 & \lambda_3 & \lambda_2 \end{pmatrix}$	$(s_{11} + s_{33}, s_{22} + s_{44}, 2(s_{12} + s_{23} + s_{34} + s_{14}))$
(5)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_4 \\ \lambda_4 & \lambda_2 & \lambda_4 & 0 \\ 0 & \lambda_4 & \lambda_3 & \lambda_4 \\ \lambda_4 & 0 & \lambda_4 & \lambda_2 \end{pmatrix}$	$(s_{11}, s_{33}, s_{22} + s_{44}, 2(s_{12} + s_{23} + s_{34} + s_{14}))$
(6)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_4 \\ \lambda_4 & \lambda_2 & \lambda_4 & 0 \\ 0 & \lambda_4 & \lambda_3 & \lambda_4 \\ \lambda_4 & 0 & \lambda_4 & \lambda_3 \end{pmatrix}$	$(s_{11}, s_{22}, s_{33} + s_{44}, 2(s_{12} + s_{23} + s_{34} + s_{14}))$
(7)		$\begin{pmatrix} \lambda_1 & \lambda_5 & 0 & \lambda_5 \\ \lambda_5 & \lambda_2 & \lambda_5 & 0 \\ 0 & \lambda_5 & \lambda_3 & \lambda_5 \\ \lambda_5 & 0 & \lambda_5 & \lambda_4 \end{pmatrix}$	$(s_{11}, s_{22}, s_{33}, s_{44}, 2(s_{12} + s_{23} + s_{34} + s_{14}))$

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	G	K	$\pi_G : \text{Sym}(4) \longrightarrow \mathbb{R}^d$
(8)		$\begin{pmatrix} \lambda_1 & \lambda_3 & 0 & \lambda_2 \\ \lambda_3 & \lambda_1 & \lambda_2 & 0 \\ 0 & \lambda_2 & \lambda_1 & \lambda_2 \\ \lambda_2 & 0 & \lambda_2 & \lambda_1 \end{pmatrix}$	$(s_{11} + s_{22} + s_{33} + s_{44}, 2s_{12}, 2(s_{23} + s_{34} + s_{14}))$
(9)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_3 \\ \lambda_4 & \lambda_2 & \lambda_3 & 0 \\ 0 & \lambda_3 & \lambda_2 & \lambda_3 \\ \lambda_3 & 0 & \lambda_3 & \lambda_2 \end{pmatrix}$	$(s_{11}, s_{22} + s_{33} + s_{44}, 2s_{12}, 2(s_{23} + s_{34} + s_{14}))$
(10)		$\begin{pmatrix} \lambda_1 & \lambda_3 & 0 & \lambda_3 \\ \lambda_3 & \lambda_2 & \lambda_4 & 0 \\ 0 & \lambda_4 & \lambda_2 & \lambda_3 \\ \lambda_3 & 0 & \lambda_3 & \lambda_2 \end{pmatrix}$	$(s_{11}, s_{22} + s_{33} + s_{44}, 2s_{23}, 2(s_{12} + s_{34} + s_{14}))$
(11)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_3 \\ \lambda_4 & \lambda_1 & \lambda_3 & 0 \\ 0 & \lambda_3 & \lambda_2 & \lambda_3 \\ \lambda_3 & 0 & \lambda_3 & \lambda_2 \end{pmatrix}$	$(s_{11} + s_{22}, s_{33} + s_{44}, 2s_{12}, 2(s_{23} + s_{34} + s_{14}))$
(12)		$\begin{pmatrix} \lambda_1 & \lambda_3 & 0 & \lambda_3 \\ \lambda_3 & \lambda_1 & \lambda_4 & 0 \\ 0 & \lambda_4 & \lambda_2 & \lambda_3 \\ \lambda_3 & 0 & \lambda_3 & \lambda_2 \end{pmatrix}$	$(s_{11} + s_{22}, s_{33} + s_{44}, 2s_{23}, 2(s_{12} + s_{34} + s_{14}))$
(13)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_3 \\ \lambda_4 & \lambda_2 & \lambda_3 & 0 \\ 0 & \lambda_3 & \lambda_1 & \lambda_3 \\ \lambda_3 & 0 & \lambda_3 & \lambda_2 \end{pmatrix}$	$(s_{11} + s_{33}, s_{22} + s_{44}, 2s_{12}, 2(s_{23} + s_{34} + s_{14}))$
(14)		$\begin{pmatrix} \lambda_1 & \lambda_5 & 0 & \lambda_4 \\ \lambda_5 & \lambda_2 & \lambda_4 & 0 \\ 0 & \lambda_4 & \lambda_3 & \lambda_4 \\ \lambda_4 & 0 & \lambda_4 & \lambda_2 \end{pmatrix}$	$(s_{11}, s_{33}, s_{22} + s_{44}, 2s_{12}, 2(s_{23} + s_{34} + s_{14}))$
(15)		$\begin{pmatrix} \lambda_1 & \lambda_5 & 0 & \lambda_4 \\ \lambda_5 & \lambda_2 & \lambda_4 & 0 \\ 0 & \lambda_4 & \lambda_3 & \lambda_4 \\ \lambda_4 & 0 & \lambda_4 & \lambda_3 \end{pmatrix}$	$(s_{11}, s_{22}, s_{33} + s_{44}, 2s_{12}, 2(s_{23} + s_{34} + s_{14}))$
(16)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_4 \\ \lambda_4 & \lambda_2 & \lambda_5 & 0 \\ 0 & \lambda_5 & \lambda_3 & \lambda_4 \\ \lambda_4 & 0 & \lambda_4 & \lambda_3 \end{pmatrix}$	$(s_{11}, s_{22}, s_{33} + s_{44}, 2s_{23}, 2(s_{12} + s_{34} + s_{14}))$
(17)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_4 \\ \lambda_4 & \lambda_2 & \lambda_4 & 0 \\ 0 & \lambda_4 & \lambda_3 & \lambda_5 \\ \lambda_4 & 0 & \lambda_5 & \lambda_3 \end{pmatrix}$	$(s_{11}, s_{22}, s_{33} + s_{44}, 2s_{34}, 2(s_{23} + s_{12} + s_{14}))$
(18)		$\begin{pmatrix} \lambda_1 & \lambda_6 & 0 & \lambda_5 \\ \lambda_6 & \lambda_2 & \lambda_5 & 0 \\ 0 & \lambda_5 & \lambda_3 & \lambda_5 \\ \lambda_5 & 0 & \lambda_5 & \lambda_4 \end{pmatrix}$	$(s_{11}, s_{22}, s_{33}, s_{44}, 2s_{12}, 2(s_{23} + s_{34} + s_{14}))$

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	G	K	$\pi_G : \text{Sym}(4) \longrightarrow \mathbb{R}^d$
(19)		$\begin{pmatrix} \lambda_1 & \lambda_3 & 0 & \lambda_2 \\ \lambda_3 & \lambda_1 & \lambda_3 & 0 \\ 0 & \lambda_3 & \lambda_1 & \lambda_2 \\ \lambda_2 & 0 & \lambda_2 & \lambda_1 \end{pmatrix}$	$(s_{11} + s_{22} + s_{33} + s_{44}, 2(s_{12} + s_{23}), 2(s_{34} + s_{14}))$
(20)		$\begin{pmatrix} \lambda_1 & \lambda_3 & 0 & \lambda_2 \\ \lambda_3 & \lambda_1 & \lambda_2 & 0 \\ 0 & \lambda_2 & \lambda_1 & \lambda_3 \\ \lambda_2 & 0 & \lambda_3 & \lambda_1 \end{pmatrix}$	$(s_{11} + s_{22} + s_{33} + s_{44}, 2(s_{12} + s_{34}), 2(s_{23} + s_{14}))$
(21)		$\begin{pmatrix} \lambda_1 & \lambda_3 & 0 & \lambda_2 \\ \lambda_3 & \lambda_1 & \lambda_4 & 0 \\ 0 & \lambda_4 & \lambda_1 & \lambda_2 \\ \lambda_2 & 0 & \lambda_2 & \lambda_1 \end{pmatrix}$	$(s_{11} + s_{22} + s_{33} + s_{44}, 2s_{12}, 2s_{23}, 2(s_{34} + s_{14}))$
(22)		$\begin{pmatrix} \lambda_1 & \lambda_3 & 0 & \lambda_2 \\ \lambda_3 & \lambda_1 & \lambda_2 & 0 \\ 0 & \lambda_2 & \lambda_1 & \lambda_4 \\ \lambda_2 & 0 & \lambda_4 & \lambda_1 \end{pmatrix}$	$(s_{11} + s_{22} + s_{33} + s_{44}, 2s_{12}, 2s_{34}, 2(s_{23} + s_{14}))$
(23)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_3 \\ \lambda_4 & \lambda_2 & \lambda_4 & 0 \\ 0 & \lambda_4 & \lambda_2 & \lambda_3 \\ \lambda_3 & 0 & \lambda_3 & \lambda_2 \end{pmatrix}$	$(s_{11}, s_{22} + s_{33} + s_{44}, 2(s_{12} + s_{23}), 2(s_{34} + s_{14}))$
(24)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_3 \\ \lambda_4 & \lambda_2 & \lambda_3 & 0 \\ 0 & \lambda_3 & \lambda_2 & \lambda_4 \\ \lambda_3 & 0 & \lambda_4 & \lambda_2 \end{pmatrix}$	$(s_{11}, s_{22} + s_{33} + s_{44}, 2(s_{12} + s_{34}), 2(s_{23} + s_{14}))$
(25)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_4 \\ \lambda_4 & \lambda_2 & \lambda_3 & 0 \\ 0 & \lambda_3 & \lambda_2 & \lambda_3 \\ \lambda_4 & 0 & \lambda_3 & \lambda_2 \end{pmatrix}$	$(s_{11}, s_{22} + s_{33} + s_{44}, 2(s_{12} + s_{14}), 2(s_{34} + s_{23}))$
(26)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_3 \\ \lambda_4 & \lambda_2 & \lambda_5 & 0 \\ 0 & \lambda_5 & \lambda_2 & \lambda_3 \\ \lambda_3 & 0 & \lambda_3 & \lambda_2 \end{pmatrix}$	$(s_{11}, s_{22} + s_{33} + s_{44}, 2s_{12}, 2s_{23}, 2(s_{34} + s_{14}))$
(27)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_3 \\ \lambda_4 & \lambda_2 & \lambda_3 & 0 \\ 0 & \lambda_3 & \lambda_2 & \lambda_5 \\ \lambda_3 & 0 & \lambda_5 & \lambda_2 \end{pmatrix}$	$(s_{11}, s_{22} + s_{33} + s_{44}, 2s_{12}, 2s_{34}, 2(s_{23} + s_{14}))$
(28)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_5 \\ \lambda_4 & \lambda_2 & \lambda_3 & 0 \\ 0 & \lambda_3 & \lambda_2 & \lambda_3 \\ \lambda_5 & 0 & \lambda_3 & \lambda_2 \end{pmatrix}$	$(s_{11}, s_{22} + s_{33} + s_{44}, 2s_{12}, 2s_{14}, 2(s_{34} + s_{23}))$
(29)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_3 \\ \lambda_4 & \lambda_1 & \lambda_4 & 0 \\ 0 & \lambda_4 & \lambda_2 & \lambda_3 \\ \lambda_3 & 0 & \lambda_3 & \lambda_2 \end{pmatrix}$	$(s_{11} + s_{22}, s_{33} + s_{44}, 2(s_{23} + s_{12}), 2(s_{34} + s_{14}))$

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	G	K	$\pi_G : \text{Sym}(4) \rightarrow \mathbb{R}^d$
(30)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_3 \\ \lambda_4 & \lambda_1 & \lambda_3 & 0 \\ 0 & \lambda_3 & \lambda_2 & \lambda_4 \\ \lambda_3 & 0 & \lambda_4 & \lambda_2 \end{pmatrix}$	$(s_{11} + s_{22}, s_{33} + s_{44}, 2(s_{34} + s_{12}), 2(s_{23} + s_{14}))$
(31)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_3 \\ \lambda_4 & \lambda_1 & \lambda_5 & 0 \\ 0 & \lambda_5 & \lambda_2 & \lambda_3 \\ \lambda_3 & 0 & \lambda_3 & \lambda_2 \end{pmatrix}$	$(s_{11} + s_{22}, s_{33} + s_{44}, 2s_{12}, 2s_{23}, 2(s_{34} + s_{14}))$
(32)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_3 \\ \lambda_4 & \lambda_1 & \lambda_3 & 0 \\ 0 & \lambda_3 & \lambda_2 & \lambda_5 \\ \lambda_3 & 0 & \lambda_5 & \lambda_2 \end{pmatrix}$	$(s_{11} + s_{22}, s_{33} + s_{44}, 2s_{34}, 2s_{12}, 2(s_{23} + s_{14}))$
(33)		$\begin{pmatrix} \lambda_1 & \lambda_3 & 0 & \lambda_5 \\ \lambda_3 & \lambda_1 & \lambda_4 & 0 \\ 0 & \lambda_4 & \lambda_2 & \lambda_3 \\ \lambda_5 & 0 & \lambda_3 & \lambda_2 \end{pmatrix}$	$(s_{11} + s_{22}, s_{33} + s_{44}, 2s_{23}, 2s_{14}, 2(s_{12} + s_{34}))$
(34)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_3 \\ \lambda_4 & \lambda_2 & \lambda_4 & 0 \\ 0 & \lambda_4 & \lambda_1 & \lambda_3 \\ \lambda_3 & 0 & \lambda_3 & \lambda_2 \end{pmatrix}$	$(s_{11} + s_{33}, s_{22} + s_{44}, 2(s_{12} + s_{23}), 2(s_{34} + s_{14}))$
(35)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_3 \\ \lambda_4 & \lambda_2 & \lambda_3 & 0 \\ 0 & \lambda_3 & \lambda_1 & \lambda_4 \\ \lambda_3 & 0 & \lambda_4 & \lambda_2 \end{pmatrix}$	$(s_{11} + s_{33}, s_{22} + s_{44}, 2(s_{12} + s_{34}), 2(s_{23} + s_{14}))$
(36)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_3 \\ \lambda_4 & \lambda_2 & \lambda_5 & 0 \\ 0 & \lambda_5 & \lambda_1 & \lambda_3 \\ \lambda_3 & 0 & \lambda_3 & \lambda_2 \end{pmatrix}$	$(s_{11} + s_{33}, s_{22} + s_{44}, 2s_{12}, 2s_{23}, 2(s_{34} + s_{14}))$
(37)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_3 \\ \lambda_4 & \lambda_2 & \lambda_3 & 0 \\ 0 & \lambda_3 & \lambda_1 & \lambda_5 \\ \lambda_3 & 0 & \lambda_5 & \lambda_2 \end{pmatrix}$	$(s_{11} + s_{33}, s_{22} + s_{44}, 2s_{12}, 2s_{34}, 2(s_{23} + s_{14}))$
(38)		$\begin{pmatrix} \lambda_1 & \lambda_5 & 0 & \lambda_4 \\ \lambda_5 & \lambda_2 & \lambda_5 & 0 \\ 0 & \lambda_5 & \lambda_3 & \lambda_4 \\ \lambda_4 & 0 & \lambda_4 & \lambda_2 \end{pmatrix}$	$(s_{11}, s_{33}, s_{22} + s_{44}, 2(s_{12} + s_{23}), 2(s_{34} + s_{14}))$
(39)		$\begin{pmatrix} \lambda_1 & \lambda_5 & 0 & \lambda_4 \\ \lambda_5 & \lambda_2 & \lambda_4 & 0 \\ 0 & \lambda_4 & \lambda_3 & \lambda_5 \\ \lambda_4 & 0 & \lambda_5 & \lambda_2 \end{pmatrix}$	$(s_{11}, s_{33}, s_{22} + s_{44}, 2(s_{12} + s_{34}), 2(s_{23} + s_{14}))$
(40)		$\begin{pmatrix} \lambda_1 & \lambda_5 & 0 & \lambda_4 \\ \lambda_5 & \lambda_2 & \lambda_6 & 0 \\ 0 & \lambda_6 & \lambda_3 & \lambda_4 \\ \lambda_4 & 0 & \lambda_4 & \lambda_2 \end{pmatrix}$	$(s_{11}, s_{33}, s_{22} + s_{44}, 2s_{12}, 2s_{23}, 2(s_{34} + s_{14}))$

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	G	K	$\pi_G : \text{Sym}(4) \rightarrow \mathbb{R}^d$
(41)		$\begin{pmatrix} \lambda_1 & \lambda_5 & 0 & \lambda_4 \\ \lambda_5 & \lambda_2 & \lambda_4 & 0 \\ 0 & \lambda_4 & \lambda_3 & \lambda_6 \\ \lambda_4 & 0 & \lambda_6 & \lambda_2 \end{pmatrix}$	$(s_{11}, s_{33}, s_{22} + s_{44}, 2s_{12}, 2s_{34}, 2(s_{23} + s_{14}))$
(42)		$\begin{pmatrix} \lambda_1 & \lambda_5 & 0 & \lambda_4 \\ \lambda_5 & \lambda_2 & \lambda_5 & 0 \\ 0 & \lambda_5 & \lambda_3 & \lambda_4 \\ \lambda_4 & 0 & \lambda_4 & \lambda_3 \end{pmatrix}$	$(s_{11}, s_{22}, s_{33} + s_{44}, 2(s_{12} + s_{23}), 2(s_{34} + s_{14}))$
(43)		$\begin{pmatrix} \lambda_1 & \lambda_5 & 0 & \lambda_4 \\ \lambda_5 & \lambda_2 & \lambda_6 & 0 \\ 0 & \lambda_6 & \lambda_3 & \lambda_4 \\ \lambda_4 & 0 & \lambda_4 & \lambda_3 \end{pmatrix}$	$(s_{11}, s_{22}, s_{33} + s_{44}, 2s_{12}, 2s_{23}, 2(s_{34} + s_{14}))$
(44)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_5 \\ \lambda_4 & \lambda_2 & \lambda_5 & 0 \\ 0 & \lambda_5 & \lambda_3 & \lambda_4 \\ \lambda_5 & 0 & \lambda_4 & \lambda_3 \end{pmatrix}$	$(s_{11}, s_{22}, s_{33} + s_{44}, 2(s_{23} + s_{14}), 2(s_{12} + s_{34}))$
(45)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_6 \\ \lambda_4 & \lambda_2 & \lambda_5 & 0 \\ 0 & \lambda_5 & \lambda_3 & \lambda_4 \\ \lambda_6 & 0 & \lambda_4 & \lambda_3 \end{pmatrix}$	$(s_{11}, s_{22}, s_{33} + s_{44}, 2s_{23}, 2s_{14}, 2(s_{12} + s_{34}))$
(46)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_5 \\ \lambda_4 & \lambda_2 & \lambda_5 & 0 \\ 0 & \lambda_5 & \lambda_3 & \lambda_6 \\ \lambda_5 & 0 & \lambda_6 & \lambda_3 \end{pmatrix}$	$(s_{11}, s_{22}, s_{33} + s_{44}, 2(s_{23} + s_{14}), 2s_{12}, 2s_{34})$
(47)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_4 \\ \lambda_4 & \lambda_2 & \lambda_6 & 0 \\ 0 & \lambda_6 & \lambda_3 & \lambda_5 \\ \lambda_4 & 0 & \lambda_5 & \lambda_3 \end{pmatrix}$	$(s_{11}, s_{22}, s_{33} + s_{44}, 2s_{34}, 2s_{23}, 2(s_{12} + s_{14}))$
(48)		$\begin{pmatrix} \lambda_1 & \lambda_6 & 0 & \lambda_5 \\ \lambda_6 & \lambda_2 & \lambda_5 & 0 \\ 0 & \lambda_5 & \lambda_3 & \lambda_6 \\ \lambda_5 & 0 & \lambda_6 & \lambda_4 \end{pmatrix}$	$(s_{11}, s_{22}, s_{33}, s_{44}, 2(s_{12} + s_{34}), 2(s_{23} + s_{14}))$
(49)		$\begin{pmatrix} \lambda_1 & \lambda_6 & 0 & \lambda_5 \\ \lambda_6 & \lambda_2 & \lambda_6 & 0 \\ 0 & \lambda_6 & \lambda_3 & \lambda_5 \\ \lambda_5 & 0 & \lambda_5 & \lambda_4 \end{pmatrix}$	$(s_{11}, s_{22}, s_{33}, s_{44}, 2(s_{12} + s_{23}), 2(s_{34} + s_{14}))$
(50)		$\begin{pmatrix} \lambda_1 & \lambda_6 & 0 & \lambda_5 \\ \lambda_6 & \lambda_2 & \lambda_5 & 0 \\ 0 & \lambda_5 & \lambda_3 & \lambda_7 \\ \lambda_5 & 0 & \lambda_7 & \lambda_4 \end{pmatrix}$	$(s_{11}, s_{22}, s_{33}, s_{44}, 2s_{12}, 2s_{34}, 2(s_{23} + s_{14}))$
(51)		$\begin{pmatrix} \lambda_1 & \lambda_6 & 0 & \lambda_5 \\ \lambda_6 & \lambda_2 & \lambda_7 & 0 \\ 0 & \lambda_7 & \lambda_3 & \lambda_5 \\ \lambda_5 & 0 & \lambda_5 & \lambda_4 \end{pmatrix}$	$(s_{11}, s_{22}, s_{33}, s_{44}, 2s_{12}, 2s_{23}, 2(s_{34} + s_{14}))$

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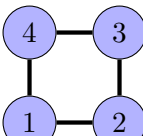
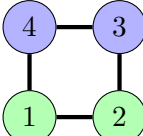
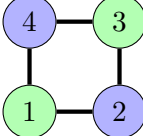
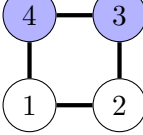
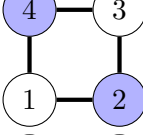
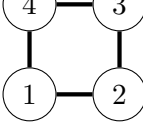
	G	K	$\pi_G : \text{Sym}(4) \longrightarrow \mathbb{R}^d$
(52)		$\begin{pmatrix} \lambda_1 & \lambda_3 & 0 & \lambda_5 \\ \lambda_3 & \lambda_1 & \lambda_2 & 0 \\ 0 & \lambda_2 & \lambda_1 & \lambda_4 \\ \lambda_5 & 0 & \lambda_4 & \lambda_1 \end{pmatrix}$	$(s_{11} + s_{22} + s_{33} + s_{44}, 2s_{12}, 2s_{34}, 2s_{23}, 2s_{14})$
(53)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_6 \\ \lambda_4 & \lambda_1 & \lambda_3 & 0 \\ 0 & \lambda_3 & \lambda_2 & \lambda_5 \\ \lambda_6 & 0 & \lambda_5 & \lambda_2 \end{pmatrix}$	$(s_{11} + s_{22}, s_{33} + s_{44}, 2s_{34}, 2s_{12}, 2s_{23}, 2s_{14})$
(54)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_6 \\ \lambda_4 & \lambda_2 & \lambda_3 & 0 \\ 0 & \lambda_3 & \lambda_1 & \lambda_5 \\ \lambda_6 & 0 & \lambda_5 & \lambda_2 \end{pmatrix}$	$(s_{11} + s_{33}, s_{22} + s_{44}, 2s_{12}, 2s_{34}, 2s_{23}, 2s_{14})$
(55)		$\begin{pmatrix} \lambda_1 & \lambda_4 & 0 & \lambda_7 \\ \lambda_4 & \lambda_2 & \lambda_6 & 0 \\ 0 & \lambda_6 & \lambda_3 & \lambda_5 \\ \lambda_7 & 0 & \lambda_5 & \lambda_3 \end{pmatrix}$	$(s_{11}, s_{22}, s_{33} + s_{44}, 2s_{34}, 2s_{23}, 2s_{12}, 2s_{14})$
(56)		$\begin{pmatrix} \lambda_1 & \lambda_5 & 0 & \lambda_7 \\ \lambda_5 & \lambda_2 & \lambda_4 & 0 \\ 0 & \lambda_4 & \lambda_3 & \lambda_6 \\ \lambda_7 & 0 & \lambda_6 & \lambda_2 \end{pmatrix}$	$(s_{11}, s_{33}, s_{22} + s_{44}, 2s_{12}, 2s_{34}, 2s_{23}, 2s_{14})$
(57)		$\begin{pmatrix} \lambda_1 & \lambda_5 & 0 & \lambda_8 \\ \lambda_5 & \lambda_2 & \lambda_6 & 0 \\ 0 & \lambda_6 & \lambda_3 & \lambda_7 \\ \lambda_8 & 0 & \lambda_7 & \lambda_4 \end{pmatrix}$	$(s_{11}, s_{22}, s_{33}, s_{44}, 2s_{12}, 2s_{23}, 2s_{34}, 2s_{14})$

Table 4: Elimination Ideals and existence of MLE (4-cycle).

Model	$I_{G,1}$	$I_{G,2}$	$I_{G,3}$	1 obs.	2 obs.	≥ 3 obs.
(1)	0	0	0	$p = 1$	$p = 1$	$p = 1$
(2)	0	0	0	$p = 1$	$p = 1$	$p = 1$
(3)	0	0	0	$p = 1$	$p = 1$	$p = 1$
(4)	0	0	0	$p = 1$	$p = 1$	$p = 1$
(5)	0	0	0	$p = 1$	$p = 1$	$p = 1$
(6)	0	0	0	$p = 1$	$p = 1$	$p = 1$
(7)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(8)	0	0	0	$p = 1$	$p = 1$	$p = 1$
(9)	0	0	0	$p = 1$	$p = 1$	$p = 1$
(10)	0	0	0	$p = 1$	$p = 1$	$p = 1$
(11)	0	0	0	$p = 1$	$p = 1$	$p = 1$
(12)	0	0	0	$p = 1$	$p = 1$	$p = 1$
(13)	0	0	0	$p = 1$	$p = 1$	$p = 1$
(14)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(15)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(16)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(17)	$\neq 0$	0	0	?	$p = 1$	$p = 1$

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Model	$I_{G,1}$	$I_{G,2}$	$I_{G,3}$	1 obs.	2 obs.	≥ 3 obs.
(18)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(19)	0	0	0	$p = 1$	$p = 1$	$p = 1$
(20)	0	0	0	$p = 1$	$p = 1$	$p = 1$
(21)	0	0	0	$p = 1$	$p = 1$	$p = 1$
(22)	0	0	0	$p = 1$	$p = 1$	$p = 1$
(23)	0	0	0	$p = 1$	$p = 1$	$p = 1$
(24)	0	0	0	$p = 1$	$p = 1$	$p = 1$
(25)	0	0	0	$p = 1$	$p = 1$	$p = 1$
(26)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(27)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(28)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(29)	0	0	0	$p = 1$	$p = 1$	$p = 1$
(30)	0	0	0	$p = 1$	$p = 1$	$p = 1$
(31)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(32)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(33)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(34)	0	0	0	$p = 1$	$p = 1$	$p = 1$
(35)	0	0	0	$p = 1$	$p = 1$	$p = 1$
(36)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(37)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(38)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(39)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(40)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(41)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(42)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(43)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(44)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(45)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(46)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(47)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(48)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(49)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(50)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(51)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(52)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(53)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(54)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(55)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(56)	$\neq 0$	0	0	?	$p = 1$	$p = 1$
(57)	$\neq 0$	$\neq 0$	0	?	?	$p = 1$

3.3 Studying the boundary condition

In this section we aim to study what happens when the ideal $I_{G,n} \neq 0$. As we have mentioned before, this does not necessarily implies the MLE does not exist. In fact, as we will see it is possible to have $0 < p$ probability although the ideal it is not zero.

There is no standard method for proving this, in this section we aim to exemplify some strategies we can use in some specific cases.

3.3.1 A method for non-existence: Sum of Squares

Suppose that $I_{G,n} \neq 0$ for some particular model. Then the projection does not necessarily cover all the cone of all sufficient statistics as, in fact, the projection defines a non-trivial geometric variety by the polynomials of the ideal.

Then, if we prove that some generator of the ideal can be expressed as a sum of squares, it follows that the condition can not be satisfied by a positive definite matrix. As a consequence, the variety will just intersect the boundary of the cone, which means that the MLE does not exist for this particular number of observations and model, so we can write $\text{WMLT}(G) > n$.

Example 3.4. To illustrate this method, let us consider the uncolored 4-cycle from Table 4, Model (57). From our computations, we know that the elimination ideal for one observation ($n = 1$) is not zero: $I_{G,1} \neq 0$.

In particular, the ideal $I_{G,1}$ contains the generator corresponding to the principal minor of the vertices 3 and 4:

$$f = s_{33}s_{44} - s_{34}^2 = 0$$

To check if this condition can be satisfied by a positive definite matrix, we can parameterize a general symmetric positive semidefinite matrix $S \in \text{PSD}(4)$ using the Cholesky decomposition $S = LL^T$, where L is a lower triangular matrix with entries u_{ij} .

By substituting the entries of LL^T into the negation of our generator ($-f$) and using the `SumsOfSquares` package in `Macaulay2`, we can express the polynomial as follows:

$$\begin{aligned} -f(LL^T) = & 4(u_{13}u_{24} - u_{14}u_{23})^2 \\ & + 4(u_{13}u_{34} - u_{14}u_{33})^2 \\ & + 4(u_{13}u_{44})^2 \\ & + 4(u_{23}u_{34} - u_{24}u_{33})^2 \\ & + 4(u_{23}u_{44})^2 \\ & + 4(u_{33}u_{44})^2 \end{aligned}$$

Since $-f(LL^T)$ is explicitly written as a sum of squares, it follows that $-f(LL^T) \geq 0$ for any real matrix L , which implies that $f(S) \leq 0$ for any $S \in \text{PSD}(4)$.

However, any matrix belonging to the variety defined by the elimination ideal must satisfy $f(S) = 0$. For the sum of squares to be identically zero, each individual squared term must be zero. In particular, we must have:

$$u_{33}u_{44} = 0$$

Since L is a lower triangular matrix, its determinant is the product of its diagonal entries (which includes u_{33} and u_{44}). The condition $u_{33}u_{44} = 0$ implies that $\det(L) = 0$, and consequently, $\det(S) = \det(LL^T) = 0$.

Therefore, any matrix satisfying the elimination ideal condition must lie on the boundary of the cone, meaning that $S \notin \text{PD}(4)$. We conclude that the MLE does not exist for $n = 1$, and consequently, $\text{WMLT}(G) > 1$.

3.3.2 A method for (non-)existence: Studying the minors

Sometimes, the condition imposed by the ideal's generators can be expressed as a function of the variables s_{ij} . If the equation corresponds to a principal minor of the matrix this implies that the MLE will not exist, as any positive definite matrix will satisfy the equation. However, if the minor is not principal it can be possible to find a positive definite matrix which lies in the variety.

Example 3.5. For example, the model (52) of the 4-cycles has $I_{G,1} = (t_2t_3 - t_4t_5) \neq 0$, where $t_2 = 2s_{12}$, $t_3 = 2s_{34}$, $t_4 = 2s_{23}$ and $t_5 = 2s_{14}$.

Consequently, as we are in characteristic different from 2, the ideal can be expressed as $I_{G,1} = (s_{12}s_{34} - s_{23}s_{14}) \neq 0$, which corresponds to the following minor

$$\begin{pmatrix} s_{12} & s_{14} \\ s_{23} & s_{34} \end{pmatrix}.$$

Since it is not a principal minor, its vanishing is not automatically incompatible with positive definiteness. For example, the matrix

$$\begin{pmatrix} 4 & 1 & 0 & 1 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 1 & 0 & 1 & 4 \end{pmatrix}$$

is positive definite (using Sylvester criterion or strict diagonal dominance).

Conclusion

In this project, we have studied the existence of the maximum likelihood estimator for colored Gaussian graphical models from both a convex-geometric and an algebraic perspective. The computations for 3-cycles and 4-cycles show how elimination ideals can provide sufficient conditions for almost sure existence of the MLE. When the elimination ideal is non-trivial, we have seen that additional geometric information about the boundary of the cone is needed. This suggests several natural directions for further study, including a systematic analysis of the remaining cases.

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