

Unimodular triangulation and Ehrhart non-positivity for s-lecture hall simplices

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Abstract

Given a sequence of positive integers $\mathbf{s} = (s_1, \dots, s_n)$, the s-lecture hall simplex is defined as:

$$\mathcal{P}_n^{\mathbf{s}} := \left\{ \mathbf{x} \in \mathbb{R}^n : 0 \leq \frac{x_1}{s_1} \leq \dots \leq \frac{x_n}{s_n} \leq 1 \right\}.$$

Introduced by Savage and Schuster [SS12], s-lecture hall simplices have been widely studied in discrete geometry and combinatorics.

A central question in discrete geometry is whether $\mathcal{P}_n^{\mathbf{s}}$ admits a unimodular triangulation, *i.e.*, a triangulation into lattice simplices of unit volume. It is conjectured that $\mathcal{P}_n^{\mathbf{s}}$ admits a unimodular triangulation for every $\mathbf{s}$ [HOT16, Conj. 5.2]. While some progress has been made toward resolving this conjecture, significant gaps remain. In particular, [HOT16, Thm. 3.3] proves that $\mathcal{P}_n^{\mathbf{s}}$ admits a unimodular triangulation if $s_{i+1} = k_i s_i$ for some $k_i \geq 1$, and [BS20, Cor. 3.5] shows the existence of regular-unimodular triangulations when $0 \leq s_{i+1} - s_i \leq 1$. In this talk, I present a significant extension of these results by establishing a more general (and recursive) criterion for the existence of unimodular triangulations of $\mathcal{P}_n^{\mathbf{s}}$: Let $\mathbf{s} = (s_1, \dots, s_{n-1})$ and $s_n = k s_{n-1} + \varepsilon$ with $k \geq 1$ and $\varepsilon \in \{0, 1\}$. If $\mathcal{P}_{n-1}^{\mathbf{s}}$ admits a unimodular triangulation, then $\mathcal{P}_n^{(\mathbf{s}, s_n)}$ also admits one.

Another central topic in discrete geometry is Ehrhart theory, which studies the Ehrhart polynomial $\mathcal{L}_P(t)$ of a lattice polytope P . A polytope is said to be Ehrhart positive if all coefficients of $\mathcal{L}_P(t)$ are nonnegative. It is known that there exist sequences $\mathbf{s}$ such that $\mathcal{P}_n^{\mathbf{s}}$ is *not* Ehrhart positive. More precisely, Liu and Solus showed that certain $\mathbf{s}$ -lecture hall simplices are not Ehrhart positive for some specific $\mathbf{s}$ [LS19]. Motivated by this, Olsen [Ols19, Question 6.8] asked for conditions on $\mathbf{s}$ that determine whether $\mathcal{P}_n^{\mathbf{s}}$ is Ehrhart positive or not. In this talk, I show that for sequences of the form $\mathbf{s} = (a, \dots, a, a+1) \in \mathbb{Z}^n$ with $a \in \mathbb{Z}^+$ and $n > 4$, the simplex $\mathcal{P}_n^{\mathbf{s}}$ is not Ehrhart positive when a is large enough. This is unexpected, given the similarities to (a unimodular transform of) the dilated standard simplex, which is known to be Ehrhart positive. Moreover, since these sequences $\mathbf{s}$ are weakly increasing, their lecture hall simplices admit regular unimodular triangulations [BS20].

This is joint work with Martina Juhnke and Germain Poullot.

References

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