# On Lattice Diameters

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We study *lattice diameter segments*, which are segments in convex bodies that contain a maximal number of lattice points. The number of lattice points on such a segment is called the *lattice diameter* of the convex body. This is a special case of studying lattice points in slices of convex bodies, i.e., intersections with affine subspaces.

Slices of convex bodies have been studied for many years both from the analytic and combinatorial points of view. Our knowledge about lattice diameters is still very limited, most known results are the following works: [1], [2], [3].

For lattice polytopes, the lattice diameter bounds the total number of lattice points in the polytope [4], and in dimension two they also bound the lattice width [2]. They are also related to the covering problem, i.e., covering lattice points in a lattice polygon by lines [3]. Moreover, it is easy to see that even in dimension two a lattice polygon $P$ can have multiple lattice diameter segments, they can exist in different directions, and can be contained in the interior of the polygon.

In dimension two we have the following contributions:

**Theorem 1:** There exists a polynomial time algorithm that computes a lattice diameter segment of a lattice polygon, for every direction in which a lattice diameter exists. (The input of the algrithm is a list of vertices.)

Let $f_P(k) := \#\{\text{lattice diameter segments in } kP\}$, a counting function, that counts the number of lattice diameter segments of a lattice polygon $P$.

**Theorem 2:** For a lattice polygon $P$, $f_P$ eventually agrees with a quasi-polynomial.

For $S \subseteq \mathbb{Z}^2$ define its lattice diameter segments as those of $\text{conv}(S)$ with endpoints in $S$. Let $\beta_{\mathbb{Z}}(S)$ be the minimal number of sets in a partition of $S$, such that each part in the partition has smaller lattice diameter than $S$ does. This is a discrete version of the classical *Borsuk number*.

**Theorem 3:** For a lattice polygon $P$, $\beta_{\mathbb{Z}}(P \cap \mathbb{Z}^2) \leq 4$ and if the lattice diameter of $P$ is 2 then equality is attained if and only if $P$ is affine unimodular equivalent to a square.

In higher dimensions we have the following contributions:

**Theorem 4:** Computing lattice diameters of semi-algebraic bodies in any fixed dimension $d \geq 3$ is NP-hard.

**Theorem 5:** Let $P$ be a lattice polytope with lattice diameter two. Then $P$ has at most $\binom{2^d}{2}$ diameter directions and this bound is tight.

## References

[1] E. G. Alarcon II, *Convex lattice polygons*, University of Illinois at Urbana-Champaign, 1987.

[2] I. Bárány and Z. Füredi, *On the lattice diameter of a convex polygon*, Discrete Mathematics, 241 (2001).

[3] C. E. Corzatt, *Some extremal problems of number theory and geometry.*, UIUC, 1974.

[4] S. Rabinowitz, *A theorem about collinear lattice points*, Utilitas Mathematica, 36 (1989), pp. 93–95.