

Exact Johnson solids in OSCAR

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Some polytopes have properties that require nonrational algebraic numbers in their vertex and facet descriptions. These might be combinatorial properties, as for the 8-dimensional polytope found by Perles [Gr3], or geometric properties, e.g. related to regularity. While the study of polytopes benefits greatly from the use of computational tools, nonrational numbers are often only approximated using floats. In **OSCAR** [OSC25, DEF⁺25] the required functionality for exact coordinates exists together with a whole machinery for polyhedral geometry. For example, mixed integer programs can be solved over any polytope with algebraic coordinates, which has recently been employed by Breuer, Joswig and Malle [BJM25] to solve a problem in representation theory. The Johnson solids are a family of 3-polytopes, some of which do not admit a representation by rational numbers or even quadratic field extensions thereof. Previously there was no freely accessible implementation of their exact vertex or facet descriptions. We encoded them in **OSCAR**, and made the corresponding exact data available on Zenodo [GJJ⁺24].

The Johnson solids are the 92 3-dimensional polytopes whose facets are regular polygons but whose group of symmetries does not act transitively on their vertices. A list of them was computed by Johnson in 1966 [Joh66] and proven to be complete by Zalgaller in 1969 [Zal69]. Together with the 5 platonic solids, the 13 Archimedean solids, and the infinitely many regular prisms and antiprisms, they make up all 3-dimensional polytopes with regular faces.

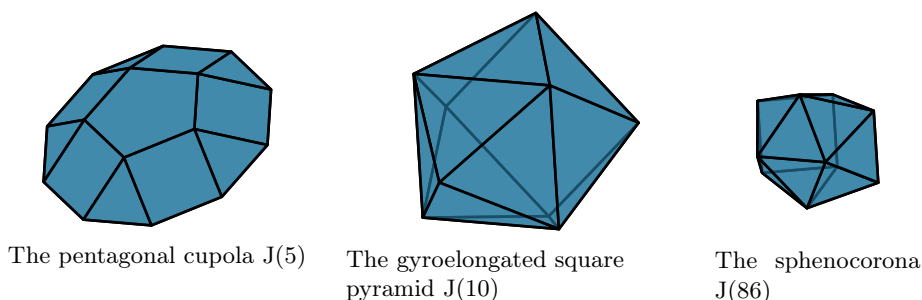


Figure 1: Three Johnson solids

While some Johnson solids, such as the pentagonal cupola, are easily representable with rational vertices, some others, such as the gyroelongated square pyramid and the sphenocorona, require coordinates in non-quadratic or non-simple field extensions over the rationals. For these solids see Figure 1. The exact coordinates of the noncomposite Johnson solids requiring non-quadratic field extensions have been computed by Timofeenko in 2009 [Tim09].

OSCAR is a computer algebra system in which, since version 1.0, all Johnson solids that are representable by simple quadratic field extensions over the rationals are implemented. Now it also provides functionality to create non-simple and non-quadratic field extensions. Using both the work of Timofeenko and the functionality of **OSCAR**, we implemented the required concrete field extensions and stored the remaining Johnson solids with exact coordinates. They can be called in an active **OSCAR** session via the function `johnson_solid(i)` with an index $1 \leq i \leq 92$.

Loading a Johnson solid employs the MaRDI file format, which is JSON-based and was

developed for the use in and beyond OSCAR [DVJL24]. In order to make our data as widely accessible as possible, the resulting data set is also available on Zenodo [GJJ⁺24]. Among the exact vertex and facet representations, which are currently specific to OSCAR, we also included approximate vertex and facet descriptions via floating points, and crucially the descriptions of the respective field extensions needed to produce the exact data. For the sake of completeness, we also included the combinatorial data of vertex-facet-incidence relations. The data can be used outside of OSCAR, most simply by employing our additional `julia` script with which one can load the data using only `julia`'s `JSON` package.

This is joint work with Alexej Jordan, Michael Joswig, Bernd Sturmfels, Marta Panizzut and Olivia Röhrig.

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