Contributed Talk Proposal

for the Closing conference on Combinatorial Geometries and Geometric Combinatorics

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Title

Solving puzzles of shellable simplicial spheres

Abstract

Reconstructing polytopes and simplicial complexes from partial information has been a problem of interest for decades. In the 1980s, Blind and Mani proved an unpublished conjecture of Perles asserting that the graph of a simple polytope determines the entire combinatorial structure of the polytope. A year later, Kalai gave a much simpler proof of this result.

Given a triangulated d-dimensional sphere, its facet-ridge graph, or "puzzle," encodes the incidence relationship between its d-dimensional and $(d-1)$-dimensional faces. Stated in the dual form, Blind and Mani's result asserts that the combinatorial structure of the boundary of a simplicial polytope (and therefore the polytope itself) is determined by its facet-ridge graph. This naturally leads to the following question: can we uniquely recover the entire combinatorial structure of a triangulated sphere from its facet-ridge graph? We prove that the answer is yes when the sphere is shellable.

Our proof utilizes the notion of good acyclic orientations from Kalai's proof. Instead of inspecting many of them as in the case of polytopes, we show that one such orientation suffices to begin our reconstruction. From this orientation, we can partially recover the face lattice of the shellable sphere by listing all its faces as they are added to the complex during shelling. Then, for every minimal new face at each shelling step, we determine exactly which facets contain it. This is done with the language of k-frames and k-systems introduced by Joswig, Kaibel, and Körner. We finish the proof by induction on the dimension of the sphere. (Here one relies on an easy fact that links of shellable spheres are also shellable spheres.)

We will start the talk by reviewing several basic definitions, including that of shellability and related notions. This will be followed by discussing asymptotic bounds on the number of shellable simplicial spheres. We will then outline the main steps of our reconstruction procedure, illustrating them with 2- and 3-dimensional examples. We will conclude the talk with several related open problems.