ORDER POLYTOPES OF CROWN POSETS

Order polytopes were introduced by Richard P. Stanley in 1986 as a geometric representation of partially ordered sets (posets). They play a significant role in combinatorial geometry, optimization, and algebraic combinatorics. The *zigzag* (or the *fence*) *poset* Z_n on $[n]$ is defined by the cover relations $1 \prec 2 \succ 3 \prec \cdots \succ n-1 \prec n$ if n is even and $1 \prec 2 \succ 3 \prec \cdots \prec n-1 \succ n$ if n is odd, and it has been extensively studied in the literature, but the same cannot be said about the crown poset. The *crown poset* $\mathcal{C}_{2n}$ on $[2n]$ is defined by the cover relations $1 \prec 2 \succ 3 \prec \cdots \succ 2n-1 \prec 2n \succ 1$, that is, it corresponds to the zigzag poset Z_{2n} with the extra order relation $2n \succ 1$.

In this project we study the order polytope of the crown poset $\mathcal{C}_{2n}$. The crown poset $\mathcal{C}_{2n}$ is the poset of proper faces of a polygon with n vertices.

We start by studying the f-vector of $\mathcal{O}(\mathcal{C}_{2n})$ and provide an explicit formula for each of its entries. These formulas are obtained by using the well-known connection between faces of the order polytope and the so called connected and compatible partitions of the underlying poset. We count these partitions using elementary combinatorial methods.

Theorem. *For all $k \geq 0$,*

$$f_k(\mathcal{O}(\mathcal{C}_{2n})) = \delta_k + \sum_{i=2}^{2n} \sum_{m=1}^{\lfloor i/2 \rfloor} \frac{2n}{i} \binom{i}{2m} \binom{n+m-1}{i-1} \binom{2m}{i-k}$$

where

$$\delta_k = \begin{cases} 2, & \text{if } k = 0, \\ 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1. \end{cases}$$

We also investigate the Ehrhart polynomial of $\mathcal{O}(\mathcal{C}_{2n})$. We provide a recursive formula for the order polynomial of $\mathcal{C}_{2n}$ in terms of order polynomials of zigzags, which gives a recursive expression for the Ehrhart polynomial of $\mathcal{O}(\mathcal{C}_{2n})$. This provides a counterpart to the formulas found in the zigzag case by Petersen–Zhuang (2025). This also allows us to simplify the computation for the linear coefficient of $\Omega_{\mathcal{C}_{2n}}(t)$ made by Ferroni–Morales–Panova (2025).

Theorem.

$$\begin{aligned} \Omega_{\mathcal{C}_{2n}}(t) &= \Omega_{\mathcal{C}_{2n}}(t-1) + n \cdot \Omega_{Z_{2n-1}}(t-1) + \Omega_{Z_{2n-3}}(t) + \sum_{1 \leq i < j \leq n} \Omega_{Z_{2(n-j+i)-1}}(t-1) \cdot \Omega_{Z_{2(j-i)-1}}(t) \\ &= \Omega_{\mathcal{C}_{2n}}(-t) - n \cdot \Omega_{Z_{2n-1}}(-t) + t \cdot \Omega_{Z_{2n-3}}(t) - \sum_{\substack{1 \leq i < j \leq n \\ (i,j) \neq (1,n)}} \Omega_{Z_{2(n-j+i)-1}}(-t) \cdot \Omega_{Z_{2(j-i)-1}}(t) \end{aligned}$$

We finish by studying the h^*-polynomial of $\mathcal{O}(\mathcal{C}_{2n})$. We give a new combinatorial interpretation of its coefficients in terms of a new permutation statistic that we call *cyclic swap* and discuss some combinatorial properties of this statistic. This provides a circular version of a result for the zigzag poset studied by Coons–Sullivant (2023).

Theorem. *For any $n \geq 1$,*

$$h^*(\mathcal{O}(\mathcal{C}_{2n}))(t) = \sum_{\sigma \in \mathrm{CA}_{2n}} t^{\mathrm{cswap}(\sigma)}.$$