Root polytopes of matroids, and their connections to greedoids and Alexander polynomials

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Let G be an Eulerian digraph. It is an intriguing fact (which follows from the BEST theorem) that the graph G has the same number of arborescences rooted at v for any fixed vertex v. But even more is true. Greedoids were introduced by Korte and Lovász as a generalization of matroids. An important example is the directed branching greedoid, that is associated to a digraph G rooted at r, and the bases are the arborescences rooted at r. Björner, Korte and Lovász introduced the greedoid polynomial (which is a one-variable generalization of the Tutte polynomial). For a branching greedoid, the evaluation of this polynomial at 1 gives the number of arborescences of G rooted at r. Swee Hong Chan proved that not only the number of arborescences, but even the greedoid polynomial of an Eulerian digraph is independent of the choice of root. We give a new, geometric proof for this fact by defining a suitable polytope: the root polytope of a (regular) matroid. For an Eulerian digraph G, the root polytope of the cographic matroid has a volume equal to the number of arborescences rooted at any given vertex. Moreover, each spanning tree of G has a corresponding unimodular simplex in the polytope, and the set of all arborescences rooted at a given vertex gives a triangulation of the polytope. The greedoid polynomial can be expressed as the h^*-polynomial of the polytope. Another application of the root polytope of matroids is in connection with the Alexander polynomial. The classical Alexander polynomial is associated to links. Murasugi and Stoimenov generalized the Alexander polynomial of special alternating links as a polynomial associated to Eulerian digraphs. Elena Hafner, Karola Mészáros, and Alexander Vidinas showed that the Murasugi-Stoimenov polynomial can be expressed using volumes of root polytopes of cographic matroids.