Shellings of tropical hypersurfaces

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Shellability is a desirable property in combinatorial topology. It is a way to build a regular cell complex cell by cell, e.g. a polytopal complex, ensuring that the topology of cell complex stays controlled after a new cell has been added. The order in which the cells are glued together is called a *shelling*. We show that the boundary of one-point compactifications of unbounded polyhedra are shellable. Due to the polyhedral nature of tropical hypersurfaces we apply this result to show that tropical hypersurfaces are shellable.

Shellability of bounded polyehdra has previously been studied by Brugisser and Mani who showed that the boundary of a bounded polyhedron is shellable. Moreover, for any two facets H_1, H_2 of a bounded polyhedron there exists a shelling such that H_1 comes first and H_2 last.

Before we explain our setup, note that we choose min as tropical addition. We define a tropical hypersurface defined by a tropical polynomial F as the orthogonal projection of the codimension-1-skeleton of the unbounded polyhedron

$$\mathcal{D}(F) \ = \ \{(x,s) \in \mathbb{R}^d \times \mathbb{R} \mid s \leq F(x)\} \ .$$

In general, the challenge to define shellability for unbounded polyhedra as well as tropical hypersurfaces arises from their non-compactness as the notion of shellability applies to regular complexes. We consider two different compactifications.

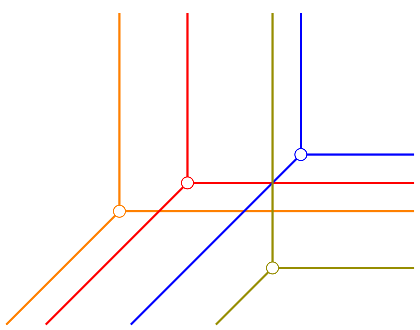
We begin with a one-point compactification.

Theorem 1. *The one-point compactification of a tropical hypersurfaces is shellable.*

To show the theorem above we embed $\mathcal{D}(F)$ in a suitable projective space. This allows us to view the polyhedron as a polytope $\overline{\mathcal{D}(F)}$ with an additional facet at infinity. We pick a suitable shelling of the boundary of $\overline{\mathcal{D}(F)}$ which induces a shelling of the one point compactification of $\mathcal{D}(F)$.

Second, we consider tropical hypersurfaces defined by tropical homogeneous polynomials, i.e., for every $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^d$ the tropical polynomial F satisfies $F(\lambda\mathbb{1} + x) = F(x) + \lambda\mathbb{1}$. This kind of tropical hypersurface is embedded in the tropical torus $\mathbb{R}^d/\mathbb{R}\mathbb{1}$. One of its natural compactifications is the max-*tropical projective space*

$$\mathbb{TP}^{d-1} \ := \ \Big((\mathbb{R} \cup \{-\infty\})^d \setminus \{-\infty\mathbb{1}\}\Big)/\mathbb{R}\mathbb{1} \ .$$



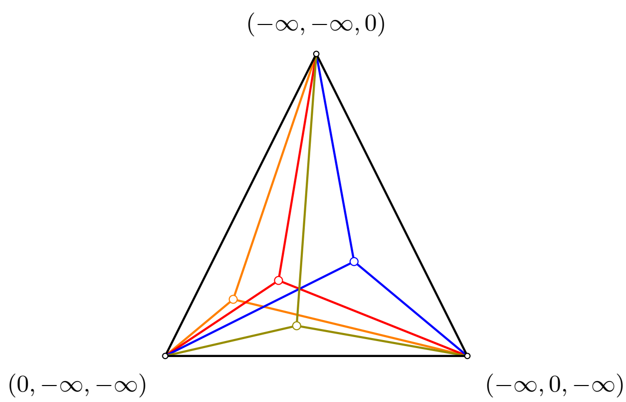


Figure 1. A tropical hyperplane arrangement in the planar tropical torus $\mathbb{R}^2/\mathbb{R}\mathbb{1}$ (left) and the max-tropical projective plane $\mathbb{TP}^2$ with four tropical hyperplanes.

We restrict ourselves to the case of *tropical hyperplane arrangements*, e.g., the union of tropical hyperplanes which are tropical hypersurfaces defined by tropical linear forms. A tropical hyperplane arrangement has *full support* if all coefficients of the respective linear forms are finite. See figure 1 for an example of a tropical hyperplane arrangement in $\mathbb{R}^3/\mathbb{R}\mathbb{1}$ and its compactification in $\mathbb{TP}^2$.

A tropical hyperplane arrangement subdivides the tropical torus $\mathbb{R}^d/\mathbb{R}\mathbb{1}$ into a polyhedral complex, called the *covector decomposition*.

Theorem 2. *The compactification of the covector decomposition in* $\mathbb{TP}^{d-1}$ *is combinatorially equivalent to the Schlegel diagram of* $\mathcal{D}(F)$ *based at the facet at infinity*

Moreover, from there we conclude the following.

Theorem 3. *The max-tropical compactification of a min-tropical hyperplane arrangement is shellable.*

Last, by translating our setup into the language of discrete Morse theory we can conclude from the results above that tight spans of regular subdivisions are contractible.

This is joint work with George Balla and Michael Joswig and can be found on the arXiv under the following link: `https://arxiv.org/abs/2506.07241`.