

# Graph Curve Matroids

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The study of the interplay between geometry and combinatorics has proven time and time again to be an extremely fruitful endeavor. However, the study of matroids associated to curves and their algebraic geometry remains a rich and largely untapped area, offering several directions for further exploration.

We build on this idea by introducing a class of matroids, that in specific cases have a direct geometric interpretation arising from the theory of algebraic curves, see Theorem A. The construction is purely combinatorial in nature, and associates to any graph a matroid on its vertex set. Moreover, the combinatorial approach yields new ways to study properties of graph curve matroids, see e.g., Theorem B.

We will briefly go over our construction of a graph curve matroid. Let  $G = (V, E)$  be an undirected graph, that may have loops and parallel edges. We denote the rank function of the associated *graphical matroid* by  $r : E \rightarrow \mathbb{N}$ , and its *dual rank function* by  $r^*$ . For any vertex set  $A \subseteq V$ , we refer to the set of edges that are incident to at least one vertex in  $A$  by  $\delta(A)$ . We prove that the following is a set of circuits

$$\mathcal{C} = \{A \subseteq V \mid A \text{ is minimal among non-empty subsets of } V \text{ s.t. } r^*(\delta(A)) \leq |A|\},$$

and we call the corresponding matroid the *graph curve matroid*  $M_G$  of  $G$ . One remarkable observation about this new class of matroids is, that they are defined on the *vertex set* of a graph.

The name *graph curve matroid* surely requires some justification. It turns out that the above defined matroids have some intriguing algebro-geometric interpretation in some cases. Assume  $G$  to be *3-edge connected* (i.e.,  $G$  remains connected even after removing up to 2 edges) and *trivalent* (i.e., each vertex has degree 3). By [BE91], in this case there exists a unique *graph curve*  $C \subset \mathbb{P}^n$  with dual graph  $G$ . The components of  $C$  are lines and correspond to vertices of  $G$ , and their intersections to edges of  $G$ .

Any generic hyperplane  $H \subset \mathbb{P}^n$ , will intersect every component of  $C$  in exactly one point. Therefore, the point configuration  $H \cap C$  realizes a matroid on the vertex set of  $G$ , visualized in Figure 1.

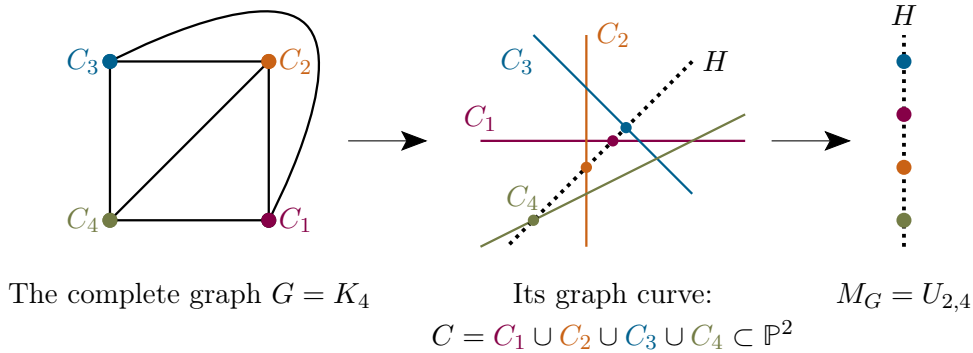


FIGURE 1. Construction of the graph curve matroid.

**Theorem A** ([GKV23]). *Let  $G$  be a trivalent, 3-edge connected graph, and  $C \subset \mathbb{P}^n$  its associated graph curve. Then the graph curve matroid of  $G$  agrees with the matroid realized by the point configuration  $C \cap H$  for any generic hyperplane  $H$ .*

This theorem fills a gap from [GHSV23], where the graph curve matroid was constructed purely in terms of the graph curve, even though both input and output of the construction are purely combinatorial objects. Our combinatorial construction has the advantage of the vast generalization to *all* graphs, not only trivalent and 3-edge connected ones.

Finally, we study some combinatorial properties of graph curve matroids. One of the most compelling ones is their *identical self-duality* (rank function equals the dual rank function):

**Theorem B** ([GKV23]). *Let  $G$  be a trivalent, connected graph. Then its graph curve matroid  $M_G$  is identically self-dual if and only if  $G$  is 2-edge-connected.*

The results are based on joint work with Alheydis Geiger and Raluca Vlad, the corresponding paper [GKV23] can be found on the arXiv.

#### REFERENCES

- [BE91] Dave Bayer and David Eisenbud, *Graph curves*, Adv. Math. **86** (1991), no. 1, 1–40, With an appendix by Sung Won Park.
- [GHSV23] Alheydis Geiger, Sachi Hashimoto, Bernd Sturmfels, and Raluca Vlad, *Self-dual matroids from canonical curves*, Experimental Mathematics **0** (2023), no. 0, 1–22.
- [GKV23] Alheydis Geiger, Kevin Kühn, and Raluca Vlad, *Graph curve matroids*, 2023, arXiv preprint arXiv:2311.08332.