

The Proximity Conjecture on Group-Labeled Matroid Bases

Tamás Schwarcz

Based on joint works [1, 2] with D. Garamvölgyi, F. Hörsch, A. Imolay, R. Mizutani, T. Oki, Y. Yamaguchi

Consider a matroid M whose ground set E is equipped with a labeling $\psi: E \rightarrow \Gamma$ to an abelian group Γ . For a finite subset $F \subseteq \Gamma$, a basis of the matroid is called F -avoiding if $\sum_{e \in B} \psi(e) \notin F$. We conjecture the following.

Conjecture 1. *For any basis B_1 of M , there exists an F -avoiding basis B_2 of M such that $|B_1 \setminus B_2| \leq |F|$, provided that at least one F -avoiding basis exists.*

Conjecture 1 generalizes a recent conjecture of Liu and Xu [3] who studied the case when Γ is a finite group and $F = \Gamma \setminus \{0\}$. They showed that their conjecture holds if $|\Gamma|$ is a prime power or the product of two primes. Besides algorithmic implications, these conjectures are also motivated by connections to notorious conjectures such as the one of Schrijver and Seymour on the number of bases with different labels and Gabow's conjecture on symmetric exchanges in matroids.

We showed the following partial results towards Conjecture 1.

Theorem 2. *Conjecture 1 holds if at least one of the following conditions is satisfied.*

- (i) M is a strongly base orderable or a sparse paving matroid,
- (ii) $|F| \leq 4$,
- (iii) $\Gamma = \mathbb{Z}$ or Γ has prime order.

By studying novel relaxations of strong base orderability, we proved the following relaxation of Conjecture 1 for graphic matroids and for matroids representable over fixed, finite fields.

Theorem 3. *Assume that the matroid M has at least one F -avoiding basis.*

- (i) *If M is a graphic matroid, then for any basis B_1 , there is an F -avoiding basis B_2 with $|B_1 \setminus B_2| = O(|F|^3)$.*
- (ii) *If M is representable over the q -element field, then for any basis B_1 , there is an F -avoiding basis B_2 with $|B_1 \setminus B_2| \leq f(|F|, q)$ for some function f .*

References

- [1] D. Garamvölgyi, R. Mizutani, T. Oki, T. Schwarcz, and Y. Yamaguchi. Towards the proximity conjecture on group-labeled matroids. In *Proceedings of the 52nd International Colloquium on Automata, Languages, and Programming (ICALP '25)*, pages 85:1–85:17. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2025.
- [2] F. Hörsch, A. Imolay, R. Mizutani, T. Oki, and T. Schwarcz. Problems on group-labeled matroid bases. In *Proceedings of the 51st International Colloquium on Automata, Languages, and Programming (ICALP '24)*, pages 86:1–86:20, 2024.
- [3] S. Liu and C. Xu. On the congruency-constrained matroid base. In *Proceedings of the 25th Integer Programming and Combinatorial Optimization (IPCO '24)*, pages 280–293, 2024.