Allowing Room for Error: The Quasi-Shortest Path Betweenness

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In network science, the concept of centrality refers to the importance of its constituent elements within the network. Since what is important usually depends on the problem at hand, many centrality measures exist. Among them, the concept of shortest path betweenness stands out as one of the most commonly used metrics in recent literature. It quantifies how often a node or link acts as a bridge along the shortest paths between all pairs of nodes, thereby capturing its importance in facilitating efficient communication or flow through the network. Formally, the shortest path betweenness centrality of a node v, $C_B(v)$, is defined as:

$$C_B(v) = \sum_{s \neq v \neq t \in V} \frac{\sigma_{st}(v)}{\sigma_{st}} \tag{1}$$

where σ_{st} is the number of shortest paths from node s to t and $\sigma_{st}(v)$ is the amount of shortest paths from s to t that passing through v.

This concept has been broadly used in transport networks to estimate traffic and network load, interpreting the number of shortest paths through a vertex as an approximation to the frequency of use of that node . Using this framework, it is well-known that the nodes of largest betweenness will be the first to show congestion, and the critical traffic rate determining the onset of the congestion transition, λ_c, is related to B_{max} as $\lambda_c \sim \frac{1}{B_{max}}$ [1, 2].

Despite this, in some cases, other factors besides distance may also influence routing and decision-making. Pedestrian mobility is a paradigmatic example of this behaviour, where the most popular path may be influenced by the presence of safer, accessible, or pleasurable detours. Indeed, [3] found deviations on average close to 10-15% in distance relative to the optimal in pedestrian paths in Boston and San Francisco, and [4] showed that a major number of the studied vehicle routes did not follow the shortest path.

In our work, we extend the shortest path betweenness metric to account for these deviations, considering all quasi-shortest simple paths that fall within a given tolerance, ε, of the shortest distance ($\delta^P_{s,t} \leq (1+\varepsilon)\delta^*_{st}$). Consequently, we investigate the relevance and behaviour of an understudied metric, the Quasi-Shortest Path Betweenness (QSP-BW), and explore its potential with a focus on traffic congestion dynamics.

We define the QSP-BW as:

$$C_B^{QSP}(v, \varepsilon) = \sum_{s \neq v \neq t \in V} \frac{\sigma^{\varepsilon}_{st}(v)}{\sigma^{\varepsilon}_{st}}, \tag{2}$$

where $\sigma^{\varepsilon}_{st}$ is the number of quasi-shortest paths between s and t and $\sigma^{\varepsilon}_{st}(v)$ corresponds to the number of those which pass through v as in Eq. (1). To compute the quasi-shortest paths involved in Eq. (2), we employed a modified version of Yen's K-shortest paths algorithm [5].

By analyzing both spatially embedded and non-spatial networks, we observed that the number of quasi-shortest paths grows exponentially with the tolerance parameter, following the relation $\sigma^{\epsilon}_{st} = e^{\gamma \delta^*_{st} \varepsilon}$. We validated this assumption analytically in the case of Erdős–Rényi graphs. Using this exponential growth model, we derived an analytical approximation for the QSP-BW of a planar graph in the limit of high node density, effectively treating the graph as a continuous 2D plane. Specifically, we computed the QSP-BW for a disk of radius R. As shown in Fig. 1 **a** and **b**, the radial profile of the QSP-BW remains qualitatively unchanged: it peaks at the center of the disk and decreases toward the boundary. However, the overall magnitude of betweenness increases with the tolerance. This result implies that in planar, homogeneous graphs, adding tolerance to routing increases overall traffic load, as vehicles use a greater extent of the network to reach their destinations. Additionally, despite adding tolerance, congestion will still emerge at the center at even lower input rates.

Consequently, reductions in betweenness, and thus traffic load, due to increased tolerance are primarily limited to networks with uneven structures or scenarios where certain roads, such as high-capacity routes, are significantly faster than the rest. As a simple illustrative example, we added a ring road to a Delaunay triangulation graph with a homogeneous spatial distribution of nodes, Fig. 1 **c**. The links in the ring had twice the speed of those in the rest of the graph. In this configuration, the nodes with the highest betweenness were located along the ring. However, when tolerance was introduced, the betweenness of the ring decreased, while it increased across the rest of the network, Fig. 1 **d-f**. This result suggests that, in such cases, adding tolerance redistributes the load away from the main high-capacity routes and onto the broader network, which, at the same time, might not be prepared to receive larger traffic loads.

Future work will focus on computing quasi-shortest path betweenness in real-world networks, identifying the onset of the transition, and exploring its implications for the Macroscopic Fundamental Diagram of traffic flow.

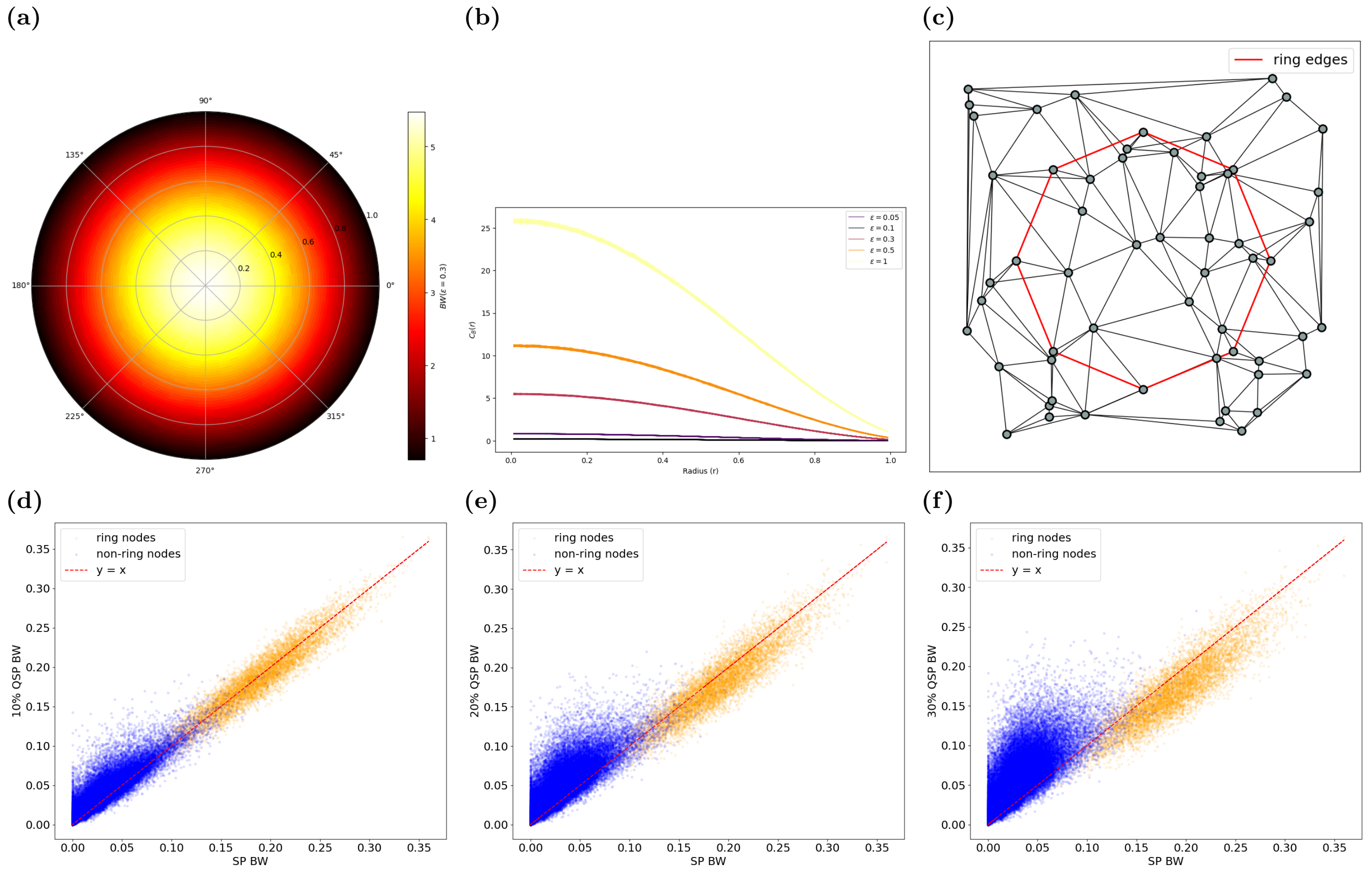

FIG. 1: **(a)** Monte Carlo integration of the QSP-BW with tolerance $\varepsilon = 0.3$ and $\gamma = 10$ in a disk of radius $R = 1$. (b) Radial profile of the QSP-BW in the disk for different tolerance levels. (c) Toy model consisting of a Delaunay network with an added ring road. (d–f) Comparison of shortest path (SP) and quasi-shortest path (QSP) betweenness for tolerances of 10%, 20%, and 30%, respectively, in 1000 different spatial configurations of the graph in **(c)**. Ring nodes (highlighted in orange) show a decrease in betweenness, while non-ring nodes exhibit an increase as tolerance increases.

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