Advances in vortex dynamics via KAM theory

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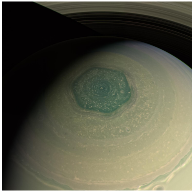
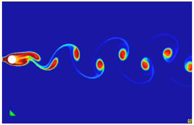
Coherent structures in turbulent flows

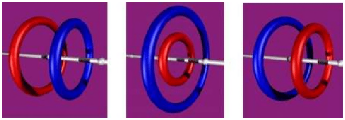
1. Rotating vortices : Saturn's hexagon,



2. Kármán vortex street :



3. Leapfrogging of two coaxial rings :



Structure

1. Generalities on Euler equations.
2. Vortex patch problem.
3. Point vortex system.
4. Desingularization of rigid configuration.
5. Desingularization of non-rigid periodic motion.

Euler equations 1755

$$\begin{cases} \partial_t v + (v\cdot\nabla)v + \nabla p = 0, & x\in\mathbb{R}^d,\, t\geq 0 \\ \text{div } v = 0, & \\ v_{|t=0} = v_0. & \end{cases}$$

- Velocity field : $(t,x)\in[0,T]\times\mathbb{R}^d \mapsto v = (v^1,..,v^d)\in\mathbb{R}^d$
- The operator $v\cdot\nabla$ is defined by

$$v\cdot\nabla = \sum_{j=1}^{d} v^j\partial_j.$$

- The pressure p is a scalar satisfying the elliptic equation :

$$-\Delta p = \text{div}\,(v\cdot\nabla v).$$

- Kato : For $v_0\in H^s, s>\frac{d}{2}+1$ there is a unique maximal solution $v\in C([0,T^\star), H^s)$.

Vorticity formulation in 2d

- The vorticity $\omega = \partial_1 v^2 - \partial_2 v^1$ satisfies

$$(E) \begin{cases} \partial_t \omega + v \cdot \nabla \omega = 0, & t \geq 0, x \in \mathbb{R}^2 \\ v = \nabla^\perp \psi \\ \omega_{|t=0} = \omega_0 \end{cases}$$

- Biot-Savart law. Stream function ψ is defined by

$$\psi(x) = \Delta^{-1}\omega = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log(|x-y|)\omega(t,y)dy$$

and

$$v(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(t,y)dy, \quad x^\perp = \mathrm{i}x$$

- Characteristic method : $\omega(t,x) = \omega_0(\phi^{-1}(t,x))$ with ϕ being the flow map :

$$\begin{cases} \partial_t \phi(t,x) = v(t,\phi(t,x)) \\ \phi(0,x) = x. \end{cases}$$

- Conservation laws : since $\phi(t)$ preserves Lebesgue measure, then

$$\forall p \in [1,\infty], \forall t \geq 0 \quad \|\omega(t)\|_{L^p} = \|\omega_0\|_{L^p}$$

- Classical solutions are global.

Yudovich solutions

- Yudovich (1963) : If $\omega_0 \in L^1 \cap L^\infty$ then (E) has a unique global solution $\omega \in L^\infty(\mathbb{R}_+; L^1 \cap L^\infty)$ and

$$\omega(t,x) = \omega_0(\phi^{-1}(t,x))$$

- The flow ϕ is uniquely defined and continuous in (t,x). For each t, $\phi(t)$ is a homeomorphism preserving Lebesgue measure. It is a diffeomorphism for classical solutions.
- In general, $\phi(t) \in C^{e^{-\alpha t}}$, degenerate regularity with t.
- Less information can be said about the boundary regularity.

Vortex patch problem

- A patch is $\omega_0 = \mathbf{1}_D$, with D a bounded domain.

$$\omega(t) = \mathbf{1}_{D_t}, \qquad D_t = \phi(t, D).$$

- Contour dynamics problem : What about the regularity of the boundary ?
- If $s \in \mathbb{T} \mapsto \gamma_0(s)$ is a parametrization of ∂D_0, then $\gamma_t(s) = \phi(t, \gamma_0(s))$ is a parametrization of ∂D_t, called Lagrangian parametrization,

$$\partial_t \gamma_t = v(t, \gamma_t)$$

- Let $s \in [0, 2\pi] \mapsto z_t(s)$ be any smooth parametrization of ∂D_t, then

$$\big(\partial_t z_t(s) - v(t, z_t(s))\big) \cdot \vec{n}(z_t(s)) = 0$$

- Contour dynamics equation (Deem Zabusky 1978) :

$$\begin{aligned} \partial_t \gamma_t(s) &= -\frac{1}{2\pi} \int_{\partial D_t} \log\big|\gamma_t(s) - z\big|\, dz \\ &= -\frac{1}{2\pi} \int_0^{2\pi} \log\big|\gamma_t(s) - \gamma_t(s')\big|\, \partial_{s'} \gamma_t(s') ds' \end{aligned}$$

We have assumed that the initial domain D is simply connected.

- Persistance regularity. Chemin(1993), Bertozzi-Constantin (1993).

$$\partial D \in C^{1+\varepsilon} \Longrightarrow \forall t \geq 0 \quad \partial D_t \in C^{1+\varepsilon}.$$

- The cases C^1 and Lip are open even locally in time.
- Other contributions : Bertozzi, Constantin, Cordoba, Danchin, Depauw, Dutrifoy, Gamblin-Saint-Raymond, Gancedo,Garnet, Kiselev, Luo, Elgindi, H., Cantero, Mateu, Orobitg, Verdera,...

Conservation laws

- Recall

$$\begin{cases} \partial_t \omega + v \cdot \nabla \omega = 0, \\ v(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(t,y) dy, \end{cases}$$

- Mass conservation :

$$m(t) = \int_{\mathbb{R}^2} \omega(t,x) dx = m(0)$$

Indeed,

$$m'(t) = \int_{\mathbb{R}^2} \partial_t \omega(t,x) dx = -\int_{\mathbb{R}^2} \mathrm{div}\, (v(t,x)\omega(t,x)) dx = 0$$

- First moments :

$$\xi(t) = \int_{\mathbb{R}^2} x\omega(t,x) dx = \xi(0)$$

Indeed,

$$\begin{aligned} \xi'(t) &= -\int_{\mathbb{R}^2} x \mathrm{div}\, (v(t,x)\omega(t,x)) dx \\ &= \int_{\mathbb{R}^2} v(t,x)\omega(t,x) dx = \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(t,y)\omega(t,x)) dx dy = 0 \end{aligned}$$

- Second Order moment :

$$I(t) = \int_{\mathbb{R}^2} |x|^2 \omega(t,x)dx = I(0)$$

Indeed,

$$\begin{aligned} I'(t) &= -\int_{\mathbb{R}^2} |x|^2 \text{div } (u(t,x)\omega(t,x))dx \\ &= \frac{-1}{\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{x \cdot (x-y)^\perp}{|x-y|^2} \omega(t,y)\omega(t,x))dxdy \\ &= \frac{-1}{\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(x-y) \cdot (x-y)^\perp}{|x-y|^2} \omega(t,y)\omega(t,x))dxdy = 0 \end{aligned}$$

- Kinetic energy

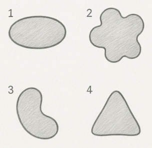
$$E(t) = \frac{1}{2} \int_{\mathbb{R}^2} |v(t,x)|^2 dx = E(0)$$

- Modified kinetic energy : $\Delta\psi = \omega$

$$E_m(t) = -\frac{1}{2} \int_{\mathbb{R}^2} \psi(t,x)\omega(t,x)dx = E_m(0)$$

Point vortex system

Dynamics of isolated vortices



- Assume that

$$\omega(t,x)=\sum_{k=1}^{n}\omega_k(t,x),\quad v=\sum_{k=1}^{n}v_k,\quad v_k(t,x)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^2}\omega_k(t,y)dy$$

then

$$\partial_t\omega_k+v\cdot\nabla\omega_k=0$$

- Define

$$\gamma_j(t)=\int_{\mathbb{R}^2}\omega_j(t,x)dx=\gamma_j(0),\quad z_j(t)=\frac{1}{\gamma_j}\int_{\mathbb{R}^2}x\omega_j(t,x)dx$$

- Differentiating in time

$$\dot{z}_j(t) = \frac{-1}{\gamma_j}\int_{\mathbb{R}^2} x\text{div }(v(t,x)\omega_j(t,x))dx = \frac{1}{\gamma_j}\int_{\mathbb{R}^2} v(t,x)\omega_j(t,x))dx$$
$$= \frac{1}{\gamma_j}\sum_{k\neq j}\int_{\mathbb{R}^2} v_k(t,x)\omega_j(t,x))dx$$

- If the vorticity ω_k is concentrated around its center z_k then

$$v_k(x) \approx \frac{\gamma_k}{2\pi}\frac{(x-z_k)^\perp}{|x-z_k|^2}, x \in \text{supp }\omega_j$$

- Thus we get the approximation

$$\dot{z}_j(t) = \sum_{k\neq j}\frac{\gamma_k}{2\pi}\frac{(z_j-z_k)^\perp}{|z_j-z_k|^2}$$
$$= \frac{i}{2\pi}\sum_{k\neq j}\frac{\gamma_k}{\overline{z_j}-\overline{z_k}}$$

Point vortex system

- **Helmholtz** (1856) : If $\omega_0 = \sum_{j=1}^{N} \gamma_j \delta_{z_j}$, $z_j \in \mathbb{R}^2, \gamma_j \in \mathbb{R}^\star$ then formally
$$\omega(t,x) = \sum_{j=1}^{N} \gamma_j \delta_{z_j(t)},$$
with
$$\frac{d\overline{z_j(t)}}{dt} = \frac{1}{2i\pi} \sum_{k\neq j} \frac{\gamma_k}{z_j - z_k}, \qquad j = 1, ..., N$$
- Kirchhoff (1876) : the system is Hamiltonian with
$$\gamma_j \frac{d\overline{z_j(t)}}{dt} = i\partial_{z_j} H, \quad H(z_1, ..z_N) = -\frac{1}{\pi} \sum_{1\leqslant j\neq k\leqslant N} \gamma_j \gamma_k \log|z_j - z_k|$$
- H, $I = \sum_j \gamma_j |z_j|^2$ and $\sum_j \gamma_j z_j$ are three independent first integrals in involution.
- Gröbli (1877)-Poincaré (1893) : this system is integrable for $N \leqslant 3$.
- If all the γ_j have the same sign then there is no collision in finite time and the points remain in a planar compact set.

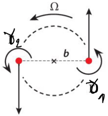
- The equations are given by

$$\frac{d\overline{z_1(t)}}{dt} = \frac{1}{2i\pi}\frac{\gamma_1}{z_1 - z_2}, \qquad \frac{d\overline{z_2(t)}}{dt} = \frac{1}{2i\pi}\frac{\gamma_2}{z_2 - z_1}$$

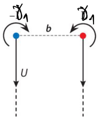
- Thus the vector $Z(t) = z_1(t) - z_2(t)$ satisfies

$$\frac{d\overline{Z(t)}}{dt} = \frac{\gamma_1 + \gamma_2}{2i\pi}\frac{1}{Z(t)}$$

- We distinguish two scenarios :
  1. Case $\gamma_1 + \gamma_2 \neq 0$. The pairs rotate uniformly about the center of mass, with $\Omega = \frac{\gamma_1+\gamma_2}{2\pi d^2}$

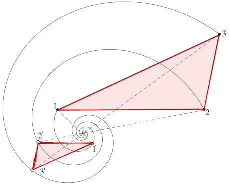


  2. Case $\gamma_1 + \gamma_2 = 0$. The pairs translate uniformly with $U = \frac{\gamma_1}{2\pi d}$.



Triple vortices.

- Gröbli (1877), Synge (1949),Novikov (1975), Aref (1979-2010),.. :
  1. Remind that 3 vortices form an integrable system.
  2. Classification of rigid motion (Equilateral triangles and collinear configuration)
  3. Sufficient and necessary condition of Self-similar collapse



Relative equilibria : link with polynomials

▶ Rotating configurations : $z_j(t) = e^{i\Omega t} z_j(0), j = 1, ..., N$. By taking $\gamma_j = 1$ and rescaling the time we find the system

$$\overline{z_j} = \sum_{k \neq j} \frac{1}{z_j - z_k}, j = 1, ..., N$$

• Stieltjes [Acta Math. 6–7, 321–326 (1885)] : Collinear vortices on the real line rotates iff they correspond to the zeros of Hermite polynomial H_N.

- Consider the generating polynomial $P(z) = \prod_{j=1}^{N} (z - z_j)$, then

$$P'(z) = P(z) \sum_{j=1}^{N} \frac{1}{z - z_j},$$

$$P''(z) = P'(z) \sum_{j=1}^{N} \frac{1}{z - z_j} - P \sum_{j=1}^{N} \frac{1}{(z - z_j)^2} = P(z) \sum_{j \neq k = 1}^{N} \frac{1}{(z - z_j)(z - z_k)}$$

$$= P(z) \sum_{j \neq k = 1}^{N} \left(\frac{1}{z - z_j} - \frac{1}{z - z_k} \right) \frac{1}{z_j - z_k} = 2P(z) \sum_{j \neq k = 1}^{N} \frac{1}{z - z_j} \frac{1}{z_j - z_k}$$

$$= 2P(z) \sum_{j=1}^{N} \frac{z_j - z + z}{z - z_j} = -2NP + 2zP'$$

- Thus P satisfies

$$P'' - 2zP' + 2NP = 0$$

- P is a Hermite polynomial : $P(z) = \lambda H_N(z)$, with

$$H_N(z) = (-1)^N e^{z^2} \left(e^{-z^2}\right)^{(N)}$$

- The points $\{z_1, z_2, .., z_N\}$ are located on the zeroes of Hermite polynomial H_N. The configuration is symmetric with respect to the origin.

• Thomson (1883) : The regular polygon $z_j = e^{i\frac{2\pi j}{N}} \in \mathbb{T}$ rotates uniformly.

Some references

- P. Newton, The N-Vortex problem, analytical techniques, 2001. Book
- Saffman, Vortex Dynamics, 1992. Book
- Aref and all, Vortex Crystals : Adv. Appl. Mech, 2003.
- Aref, Point vortex dynamics : A classical mathematics playground, J. Math. Phys. (2007).
- Tkachenko, On vortex lattices. Sov. Phy. 1966.
- Demina and Kudryashov, Vortices and polynomials (2011-2012)
- O'neil, Stationary configurations of point vortices, Trans. AMS 1987.

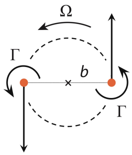
Desingularization of rigid configuration

Desingularization of relative equilibria

- Marchioro-Pulvirenti 1992 : Vortex localization around vortex point system (short time description).
- Problem statement : Is it possible to desingularize a rigid configuration ?
  Find classical solutions to Euler equations that replicate the same dynamics as the point vortex system ?

Pairs of vortices

- Rotating vortex pairs :



- Deem- Zabusky(1978), Saffman-Szeto 1980, Pierrehumbert (1980) : numerical existence of pairs of symmetric rotating and translating patches.
- Turkington (1985), Keady (1985) gave proofs using variational principles. The topological structure of each patch is not explored.

Contour dynamics approach

- Let $0 < \varepsilon < 1, d > 1$ and

$$\omega_{0,\varepsilon} = \frac{1}{\varepsilon^2}\chi_{D_1^\varepsilon} + \frac{1}{\varepsilon^2}\chi_{D_2^\varepsilon}$$

with D_1^ε be a small simply connected domain containing the origin,

$$D_2^\varepsilon = -D_1^\varepsilon + 2d, \quad D_1^\varepsilon = \varepsilon D^\varepsilon$$

with D^ε a small perturbation of the unit disc.

Theorem (H-Mateu, Comm. Math. Phys. 2017)

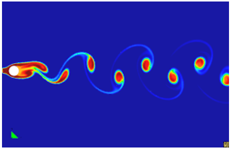
There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ there exists a strictly convex smooth domain D^ε such that $\omega_{0,\varepsilon}$ generates a co-rotating vortex pair for Euler equations

Remark

Actually, we obtained a more genral result : counter-rotating patches and for the (SQG_α) model.

- This approach is flexible and has been used in different configurations :

1. H.-Hassainia (Discrete Contin. Dyn. Syst. 2021) : construction of asymmetric pairs for Euler equations confirming the numerical simulations of Dritschel (1995).
2. García(Nonlinearity 2020) : periodic pattern of Kármán vortex street for (SQG_α).



3. Hassainia-Miles (SIAM 2022) : general case covering the nested N-gons.

- For $\varepsilon \in (0,1)$ and $d > 2$ we define the domains

$$D_1^\varepsilon = \varepsilon D^\varepsilon \quad \text{and} \quad D_2^\varepsilon = -\varepsilon D^\varepsilon + 2d.$$

- Set

$$\omega_{0,\varepsilon} = \frac{1}{\varepsilon^2}\chi_{D_1^\varepsilon} + \frac{1}{\varepsilon^2}\chi_{D_2^\varepsilon}$$

It gives rise to a rotating pairs about $(d,0)$ and with angular velocity Ω iff

$$\omega(t,z) = \omega_{0,\varepsilon}(e^{it\Omega}(z-d))$$

- It is a solution to Euler equations if and only if

$$\mathrm{Re}(-i\Omega\left(\overline{z}-d\right)\vec{n}) = \mathrm{Re}(\overline{v(z)}\,\vec{n}), \quad \forall z \in \partial D_1^\varepsilon.$$

Therefore

$$\mathrm{Re}\Big\{\Big(2\Omega\big(\overline{z}-d\big) + I(z)\Big)\,\vec{\tau}\Big\} = 0, \quad \forall z \in \partial D_1^\varepsilon,$$

with $\vec{\tau}$ a tangent vector to ∂D_1^ε and by Green-Stokes theorem,

$$I(z) = \frac{1}{\varepsilon^2}\fint_{\partial D_1^\varepsilon} \frac{\overline{\xi}-\overline{z}}{\xi - z}d\xi - \frac{1}{\varepsilon^2}\fint_{\partial D_1^\varepsilon} \frac{\overline{\xi}}{\xi + z - 2d}d\xi.$$

- Remind that $D_1^\varepsilon = \varepsilon D^\varepsilon$ and D^ε a perturbation of the unit disc. Rescaling,

$$\mathrm{Re}\Big\{\Big(2\Omega\big(\varepsilon\overline{z}-d\big)+I_\varepsilon(z)\Big)\,\vec{\tau}\Big\}=0,\quad \forall z\in\partial D^\varepsilon,$$

and

$$I_\varepsilon(z)=\frac{1}{\varepsilon}\fint_{\partial D^\varepsilon}\frac{\overline{\xi}-\overline{z}}{\xi-z}d\xi-\fint_{\partial D^\varepsilon}\frac{\overline{\xi}}{\varepsilon\xi+\varepsilon z-2d}d\xi.$$

Take the conformal parametrization : $\phi_\varepsilon:\mathbb{T}\to\partial D^\varepsilon$

$$\phi_\varepsilon(w)=w+\varepsilon\, f(w),\quad f(w)=\sum_{n\geq 1}a_n w^{-n},\quad a_n\in\mathbb{R}$$

then the boundary equation becomes : $\forall w\in\mathbb{T}$

$$G(\varepsilon,\Omega,f(w))\equiv \mathrm{Im}\Big\{\Big(2\Omega\big[\varepsilon\,\overline{\phi_\varepsilon(w)}-d\big]+I_\varepsilon(\phi_\varepsilon(w))\Big)\,w\,\phi_\varepsilon'(w)\Big\}=0$$

Easy computations

$$
\begin{aligned}
I_\varepsilon(\phi_\varepsilon(w)) &= -\frac{1}{\varepsilon}\overline{w} + \fint_{\mathbb{T}} \frac{\overline{A}+\varepsilon\overline{B}}{A+\varepsilon B} f'(\tau)d\tau + \fint_{\mathbb{T}} \frac{A\overline{B}-\overline{A}B}{A(A+\varepsilon B)}d\tau \\
&- \fint_{\mathbb{T}} \frac{(\overline{\tau}+\varepsilon f(\overline{\tau}))(1+\varepsilon f'(\tau))}{\varepsilon(\tau+w)+\varepsilon^2\big(f(\tau)+f(w)\big)-2d}d\tau.
\end{aligned}
$$

with $A = \tau - w$ and $B = f(\tau) - f(w)$. Hence

$$
G(\varepsilon,\Omega,f(w)) = \operatorname{Im}(F(\varepsilon,\Omega,f(w))) = 0
$$

with

$$
\begin{aligned}
F(\varepsilon,\Omega,f(w)) &= 2\Omega\Big(\varepsilon\overline{w}+\varepsilon^2 f(\overline{w})-d\Big)w\big(1+\varepsilon f'(w)\big) - f'(w) \\
+ & \left(\fint_{\mathbb{T}} \frac{\overline{A}+\varepsilon\overline{B}}{A+\varepsilon B} f'(\tau)d\tau + \fint_{\mathbb{T}} \frac{A\overline{B}-\overline{A}B}{A(A+\varepsilon B)}d\tau\right)w\big(1+\varepsilon f'(w)\big) \\
- & \left(\fint_{\mathbb{T}} \frac{\big(\overline{\tau}+\varepsilon f(\overline{\tau})\big)\big(1+\varepsilon f'(\tau)\big)}{\varepsilon(\tau+w)+\varepsilon^2\big(f(\tau)+f(w)\big)-2d}d\tau\right)w\big(1+\varepsilon f'(w)\big)
\end{aligned}
$$

- We can check that
$$F(0,\Omega,0)(w)=\Big(-2\Omega d+\frac{1}{2d}\Big)w$$
Let $\Omega_\infty \equiv \frac{1}{(2d)^2}$, then $F(0,\Omega_\infty,0)=0$.
- Function spaces : let $0<\beta<1$
$$X=\Big\{f\in C^{1+\beta}(\mathbb{T}), f(w)=\sum_{n\geq 1}a_n w^{-n}, a_n\in\mathbb{R}\Big\},$$
$$Y=\Big\{f\in C^{\beta}(\mathbb{T}), f=\sum_{n\geq 1}a_n e_n,\ a_n\in\mathbb{R}\Big\},\quad e_n(w)=\operatorname{Im}(w^n)$$
$$Y_1=\Big\{f\in Y, a_1=0\Big\}.$$

The rest of the proof follows the following steps :

- Step 1 : The function $G : (-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R} \times B_1 \to Y$ is well-defined and is of class C^1, where B_1 is the open unit ball of X.
- Step 2 : Let $L \equiv \partial_f G(0, \Omega, 0)$ then

$$Lh(w) = -\text{Im}(h'(w)).$$

and $L : X \to Y_1$ is an isomorphism, (and not onto Y, $Y_1 \subset Y$).

- Step 3. Ω as a Lagrange multiplier : $\Omega = \Omega(\varepsilon, f)$ in such way

$$H(\varepsilon, f) \equiv G(\varepsilon, \Omega(\varepsilon, f), f)$$

is well-defined from $(-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R} \times B_1$ to Y_1. The constraint on Ω is

$$\int_{\mathbb{T}} F(\varepsilon, \Omega, f(w))\big(\overline{w}^2 - 1\big)\,dw = 0$$

from which we get

$$\Omega = \Omega_\infty + \varepsilon \mathcal{N}(\varepsilon, f)$$

Moreover,

$$\partial_f H(0,0)h(w) = -\mathrm{Im}(h'(w)).$$

and therefore $\partial_f H(0,0) : X \to Y_1$ is an isomorphism.

- Step 4 : We conclude by using the implicit function theorem.

Euler equations in bounded domains

- Let $\mathrm{D} \subset \mathbb{R}^2$ be a bounded simply connected domain. Euler equation writes

$$\partial_t \omega + v(t,x) \cdot \nabla \omega = 0, \, x \in \mathrm{D}, \qquad v = \nabla^\perp \psi = (-\partial_2 \psi, \partial_1 \psi)$$

  with

$$\psi(t,x) = \int_{\mathrm{D}} G_{\mathrm{D}}(x,y)\omega(t,y)dy.$$

  and $G_{\mathrm{D}} : \mathrm{D} \times \mathrm{D} \to \mathbb{R}$ is the Green function

- We have the decomposition

$$G_{\mathrm{D}}(x,y) = \tfrac{1}{2\pi} \log|x-y| + \tfrac{1}{2\pi} K(x,y), \quad x,y \in \mathrm{D}$$

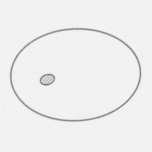
  with K being smooth in $\mathrm{D} \times \mathrm{D}$.

- Robin function is defined by

$$\mathcal{R}_{\mathrm{D}}(x) = K(x,x), x \in \mathrm{D}$$

  It is smooth in D and

$$\lim_{x \to \partial \mathrm{D}} \mathcal{R}_{\mathrm{D}}(x) = +\infty.$$



- Define

$$\gamma(t) = \int_{\mathbb{D}} \omega(t,x)dx = \gamma(0), \quad \xi(t) = \frac{1}{\gamma}\int_{\mathbb{R}^2} x\omega(t,x)dx$$

- Then, we have

$$v(t,x) = \frac{1}{2\pi}\int_{\mathbb{D}} \frac{(x-y)^\perp}{|x-y|^2}\omega(t,y)dy + \frac{1}{2\pi}\nabla_x^\perp \int_{\mathbb{R}^2} K(x,y)\omega(t,y)dy$$
$$\approx \frac{1}{2\pi}\int_{\mathbb{D}} \frac{(x-y)^\perp}{|x-y|^2}\omega(t,y)dy + \frac{\gamma}{2\pi}(\nabla_x^\perp K)(x,\xi(t))$$

- Differentiating

$$\dot{\xi}(t) = \frac{-1}{\gamma}\int_{\mathbb{D}} x\mathrm{div}\,(v(t,x)\omega(t,x))dx = \frac{1}{\gamma}\int_{\mathbb{D}} v(t,x)\omega(t,x))dx$$
$$= \frac{\gamma}{2\pi}(\nabla_x^\perp K)(\xi(t),\xi(t)) = \frac{\gamma}{4\pi}\nabla_x^\perp \mathcal{R}_{\mathbb{D}}(\xi(t))$$

- A single vortex $\omega(t) = \gamma\delta_{\xi(t)}$ obeys to the Hamiltonian equation

$$\frac{d\xi(t)}{dt} = \frac{\gamma}{4\pi}\nabla_z^{\perp}\mathcal{R}_{\mathtt{D}}(\xi(t)).$$

- When $\mathtt{D} = \mathbb{D}$ the unit disc, then

$$G_{\mathbb{D}}(z,w) = \frac{1}{2\pi}\log\left|\frac{z-w}{1-z\overline{w}}\right|, \quad \mathcal{R}_{\mathbb{D}}(z) = -\log(1-|z|^2),$$

- General domains. Let $\phi : \mathtt{D} \to \mathbb{D}$ be a conformal map, then

$$G_{\mathtt{D}}(z,w) = G_{\mathbb{D}}(\phi(z),\phi(w)), \quad z,w \in \mathtt{D}$$

and

$$\mathcal{R}_{\mathtt{D}}(z) = \log\left(\frac{|\phi'(z)|}{1-|\phi(z)|^2}\right) := -\log(\underbrace{r_{\mathtt{D}}(z)}_{\text{conformal radius}})$$

Classification of a single vortex motion

- The orbits $\mathcal{E}_\lambda = \{t \mapsto \xi(t)\} \subset \{z \in \mathbb{C}, \mathcal{R}_{\mathrm{D}}(z) = \lambda\}$
- Almost all the orbits are periodic.
- Identify the stationary points and the geometry of the orbits ?
- At least, one critical point exists.
- Caffarelli-Friedman 1985- Gustaffsson 1990 : For convex bounded domains, Robin function is strictly convex and all the orbits

$$\mathcal{E}_\lambda = \big\{z, \mathcal{R}_{\mathrm{D}}(z) = \lambda\big\}, \lambda > \lambda_\star := \inf_{z\in\mathrm{D}} \mathcal{R}_{\mathrm{D}}(z)$$

are time periodic and enclose convex regions.

Main result

- Given $\lambda_\star < a < b$ such that for any $\lambda \in [a,b]$ the orbit $\mathcal{E}_\lambda$ is periodic with minimal period $\mathtt{T}(\lambda)$ and parametrized by $t \in \mathbb{R} \mapsto \xi_\lambda(t)$.
- Consider the $T(\lambda)-$periodic matrix :

$$\mathbb{A}_\lambda(t) = \begin{pmatrix} \mathtt{u}_\lambda(t) & \mathtt{v}_\lambda(t) \\ \overline{\mathtt{v}_\lambda(t)} & \overline{\mathtt{u}_\lambda(t)} \end{pmatrix}, \mathtt{u}_\lambda(t) = -\frac{\mathbf{i}}{2r_{\mathbf{D}}^2(\xi_\lambda(t))}, \mathtt{v}_\lambda(t) = \frac{\mathbf{i}}{4}\Big[\partial_z \mathcal{R}_{\mathbf{D}}\big(\xi_\lambda(t)\big)\Big]^2$$

- We consider the fundamental matrix $\mathscr{M}_\lambda$:

$$\partial_t \mathscr{M}_\lambda(t) = \mathbb{A}_\lambda(t)\mathscr{M}_\lambda(t), \qquad \mathscr{M}_\lambda(0) = \mathrm{Id}.$$

- The monodromy matrix is $\mathscr{M}_\lambda(\mathtt{T}(\lambda))$

Theorem (Hassainia-H.-Roulley '24)

Let $\mathtt{D}$ *be a simply connected bounded domain and assume that :*

1. *Non-degeneracy of the period :*

$$\min_{\lambda\in[a,b]}\left|\mathtt{T}'(\lambda)\right|>0.$$

2. *Spectral assumption :*

$$\forall\lambda\in[a,b],\quad 1\notin sp(\mathscr{M}_\lambda(\mathtt{T}(\lambda))).$$

Then, $\exists\varepsilon_0>0$ *such that* $\forall\varepsilon\in(0,\varepsilon_0)$*, there exists a Cantor set* $\mathscr{C}_\varepsilon\subset[a,b]$*, with*

$$\lim_{\varepsilon\to0}|\mathscr{C}_\varepsilon|=b-a,$$

and for any $\lambda\in\mathscr{C}_\varepsilon$*, there exists a solution to Euler equation taking the form*

$$\forall t\in\mathbb{R},\quad \omega(t)=\frac{1}{\varepsilon^2}\mathbf{1}_{D_t^\varepsilon},\quad D_t^\varepsilon=\xi_\lambda(t)+\varepsilon O_t^\varepsilon,$$

with

$$\forall t\in\mathbb{R},\quad D^\varepsilon_{t+T(\lambda)}=D_t^\varepsilon,\qquad \xi_\lambda(t+T(\lambda))=\xi_\lambda(t).$$

Corollary

The main Theorem holds true under the following assumptions

1. *Robin function admits only one critical point* ξ_0 *(satisfied for convex bounded domains)*
2. *The conformal mapping* $F : \mathbb{D} \to \mathtt{D}$ *with* $F(0) = \xi_0$ *satisfies*

$$\left|\frac{F^{(3)}(0)}{F'(0)}\right| \notin \left\{2\sqrt{1-\frac{1}{n^2}},\ n \in \mathbb{N}^*\right\}.$$

Corollary applies with

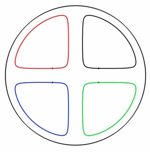
- Almost all the rectangles and ellipses.
- Sectors of type $\left\{z \in \mathbb{D} \quad \text{s.t.} \quad 0 < \arg(z) < \frac{\pi}{m}\right\}, m \in \mathbb{N}^\star$

General Remarks

1. For $\mathtt{D} = \mathbb{D}$ we get better result : we can desingularize all the orbits with rigid time periodic patches.
2. In general the solutions are non-rigid time periodic.
3. This is the first construction of this type of solutions near point vortices in bounded domains.
4. Hassainia-H.-Masmoudi (2023) :Similar construction for the leapfrogging with 4 symmetric vortices in the plane

Application to 4-point vortices in the disc

1. The motion of 4 symmetric points in a disc reduces to a single point in a quarter disc.



2. Our Theorem works in a quarter disc and we can desingularize into concentrated periodic patches.

Main ideas of the proof

1. Contour dynamics equation
2. Construction of a good periodic approximation without Cantor sets
3. Nash Moser scheme
4. KAM tools

Contour dynamics equation

- Ansatz

$$\omega(t)=\tfrac{1}{\varepsilon^2}\mathbf{1}_{D_t^\varepsilon},\quad \text{with}\quad D_t^\varepsilon \triangleq \varepsilon O_t^\varepsilon+\xi(t),$$

- Let $\theta\in\mathbb{T}\mapsto\gamma(t,\theta)$ be any smooth parametrization of the domain O_t^ε. Then the contour dynamics equation writes

$$\varepsilon^2\mathrm{Im}\Big\{\partial_t\overline{\gamma(t,\theta)}\partial_\theta\gamma(t,\theta)\Big\}-\tfrac{\varepsilon}{2}\mathrm{Re}\Big\{\partial_z\mathcal{R}_{\mathbf{D}}\big(\xi(t)\big)\partial_\theta\gamma(t,\theta)\Big\}$$
$$+\tfrac{1}{2\pi}\partial_\theta\int_{O_t^\varepsilon}\log(|\gamma(t,\theta)-\zeta|)dA(\zeta)$$
$$+\tfrac{1}{2\pi}\partial_\theta\int_{O_t^\varepsilon}K\big(\varepsilon\gamma(t,\theta)+\xi(t),\varepsilon\zeta+\xi(t)\big)dA(\zeta)=0.$$

- We look for time periodic solutions

$$\theta\in\mathbb{T}\mapsto\sqrt{1+2\varepsilon r(\omega_0(\lambda)t,\theta)}\,e^{\mathrm{i}\theta},\quad \omega_0(\lambda)=\frac{2\pi}{T(\lambda)}$$

with $r:(\varphi,\theta)\in\mathbb{T}^2\mapsto r(\varphi,\theta)\in\mathbb{R}$. Hence

$$F(r)(\varphi,\theta)\triangleq\varepsilon^2\omega_0(\lambda)\partial_\varphi r+\partial_\theta\Big[F_0(\varepsilon,\xi_\lambda(\varphi),r)\Big]=0.$$

Formal Nash-Moser scheme

- Newton scheme : To construct a solution to $F(r) = 0$ we use the scheme :

  $$r_0 \quad \text{is given such that } F(r_0) \text{ is small enough, } r_{n+1} = r_n + h_n, h_n := -F'(r_n)^{-1}F(r_n)$$

  To do that, it is enough that $F : X \to Y$ is C^1 and $F'(r_0) : X \to Y$ is an isomorphism.
- In our context, $F'(r_0)$ is not an isomorphism !
- Nash-Moser scheme is a regularization of Newton scheme where we require that $F'(r_n)$ admits a right inverse (with a loss of regularity+ suitable tame estimates)

Linearization

- First, $F(0)=O(\varepsilon)$.
- By linearization at any small state r, we get

$$\partial_r F(r)[h]=\varepsilon^2\omega_0(\lambda)\partial_\varphi h+\partial_\theta\Big[\Big(\tfrac12-\tfrac{\varepsilon}{2}r+\varepsilon^2\mathrm{g}+O(\varepsilon^3)\Big)h\Big]$$
$$-\tfrac12\mathcal{H}[h]+\varepsilon^2\partial_\theta Q_0[h]+O(\varepsilon^3),$$

with $\mathcal{H}$ the Hilbert transform in the toroidal case

$$\mathrm{g}(\varphi,\theta)\triangleq\tfrac12\mathrm{Re}\Big\{\Big(\big(\partial_z\mathcal{R}_{\mathrm{D}}(\xi_\lambda(\varphi))\big)^2+\tfrac13 S(\Phi)(\xi_\lambda(\varphi))\Big)e^{2\mathrm{i}\theta}\Big\},$$

$$Q_0[h](\varphi,\theta)\triangleq\int_{\mathbb{T}}h(\varphi,\eta)\Big(\frac{\cos(\theta-\eta)}{r_{\mathrm{D}}^2(\xi_\lambda(\varphi))}+\tfrac16\mathrm{Re}\Big\{e^{\mathrm{i}(\theta+\eta)}S(\Phi)\big(\xi_\lambda(\varphi)\big)\Big\}\Big)d\eta,$$

- For $\varepsilon=0$, the operator degenerates (in time),

$$\partial_r F(r)[h]=\tfrac12\big(\partial_\theta-\mathcal{H}\big)h$$

The spatial modes ±1 are trivial resonances !

A toy model (Resonance and loss of regularity)

Consider the operator :

$$L_0 h = \varepsilon^2 \omega_0(\lambda)\partial_\varphi h + \partial_\theta h$$

- To solve $L_0 h = f$, with $\langle f\rangle_{\varphi,\theta} = 0$, we use Fourier expansion

$$h(\varphi,\theta) = \sum_{(k,n)\neq(0,0)} h_{k,n} e^{i(k\varphi+n\theta)}, \qquad h_{k,n} = -\mathrm{i}\frac{f_{k,n}}{\varepsilon^2\omega_0(\lambda)k+n}$$

- In the Cantor set

$$\mathcal{C}_0 = \Big\{\lambda \in [a,b], \forall (k,n)\neq(0,0), |\varepsilon^2\omega_0(\lambda)k+n| \geqslant \tfrac{\varepsilon^{2+\delta}}{(1+|n|)^\tau}\Big\},$$

we get

$$\|L_0^{-1} f\|_{H^s} \leqslant \varepsilon^{-2-\delta}\|f\|_{H^{s+\tau}}$$

- We know that $\lambda \mapsto \omega_0(\lambda)$ does not degenerate,

$$\inf_{\lambda\in[a,b]} |\omega'(\lambda)| > 0.$$

Hence for $\tau > 1$

$$|\mathcal{C}_0| \geqslant b - a - C\varepsilon^\delta$$

Good approximation and new scaling

- We cannot start from $r_0 = 0$ because

$$F(0) = O(\varepsilon), \quad (\partial_r F)^{-1}(0) = O(\varepsilon^{-2-\delta}), \quad (\partial_r F)^{-1}(0)F(0) = O(\varepsilon^{-1-\delta})$$

- We have to find a good approximation. Actually we obtain the following result : there exists $\overline{r_\varepsilon}$ such that

$$\overline{r_\varepsilon} = O(\varepsilon) \quad \text{and} \quad F(\overline{r_\varepsilon}) = O(\varepsilon^4)$$

- The functional that we will use is ($\mu \in (0,1)$)

$$G(\rho) = \frac{1}{\varepsilon^{1+\mu}} F(\overline{r_\varepsilon} + \varepsilon^{1+\mu}\rho), \quad G(0) = O(\varepsilon^{3-\mu})$$

- We show that in a suitable Cantor set

$$(\partial_\rho G)^{-1}(0) = O(\varepsilon^{-2-\delta}), \quad (\partial_\rho G)^{-1}(0)G(0) = O(\varepsilon^{1-\delta-\mu})$$

Invertibility of the linearized operator and strategy

- The linear operator is given by

$$\partial_\rho G(\rho)[h] = \varepsilon^2\omega_0(\lambda)\partial_\varphi h + \partial_\theta\left[\mathbf{V}_1^\varepsilon(\rho)h\right] - \tfrac{1}{2}\mathcal{H}[h] + \varepsilon^2\partial_\theta Q_0[h] + O(\varepsilon^3)$$

- With

$$\mathbf{V}_1^\varepsilon(\rho) = \tfrac{1}{2} + \varepsilon^2\mathtt{g} - \tfrac{\varepsilon^{2+\mu}}{2}\rho + O(\varepsilon^3),$$

- Is it possible to invert the operator $\partial_\rho G(\rho)$, for ρ and ε small enough ?

- Difficulties :

1. The operator is quasi-linear (variable coefficients at the main order).
2. Small divisor problems.
3. Trivial resonance of the spatial modes ± 1.
4. Degeneracy in ε in the time direction

- Tools :
  1. KAM techniques in the spirit of the works of Berti-Montalto and Feola-Giuliani-Procesi, to conjugate the linear operator into a Fourier multiplier.
  2. Monodromy matrix to handle the modes ± 1.
  3. Nash Moser scheme to construct solutions to the nonlinear problem.
  4. Measure of the Cantor set.

Reduction of the linearized operator

- In the spirit of Baldi-Berti-Montalto [2014], Feola-Giuliani-Montalto-Procesi [2019], we construct an isomorphism $\mathscr{B}(\lambda) : H^s(\mathbb{T}^2) \to H^s(\mathbb{T}^2)$ in the form

$$\mathscr{B}h = (1+\partial_\theta\beta)h(\varphi,\theta+\beta(\varphi,\theta))$$

- There exists a change of coordinates transform $\mathscr{B}$ such that on the Cantor set

$$\mathcal{C}(\rho) = \bigcap_{\substack{(k,n)\in\mathbb{Z}^2\\ |n|\geqslant 1}} \left\{\lambda\in(a,b);\ \left|\varepsilon^2\omega(\lambda)k + n\,\mathtt{c}(\varepsilon,\lambda,\rho)\right| \geqslant \frac{\varepsilon^{2+\delta}}{|n|^\tau}\right\}$$

we have

$$\mathscr{B}^{-1}\partial_\rho G(\rho)\mathscr{B} = \varepsilon^2\omega(\lambda)\partial_\varphi + \mathtt{c}(\varepsilon,\lambda)\partial_\theta - \tfrac{1}{2}\mathcal{H} - \varepsilon^2\partial_\theta Q + \varepsilon^{2+\mu}\mathcal{R}$$

with

$$Q[h](\varphi,\theta) = \int_{\mathbb{T}} h(\varphi,\eta)K\big(\xi_\lambda(\varphi),\theta,\eta\big)\,d\eta,$$

$$K(\xi_\lambda,\theta,\eta) = \frac{\cos(\theta-\eta)}{r_{\mathrm{D}}^2(\xi_\lambda)} - \tfrac{1}{2}\mathrm{Re}\left\{\big(\partial_z\mathcal{R}_{\mathrm{D}}(\xi_\lambda)\big)^2 e^{\mathrm{i}(\theta+\eta)}\right\}$$

$$\mathtt{c}(\varepsilon,\lambda,\rho) = \tfrac{1}{2} + O(\varepsilon^3).$$

- The operator $\mathcal{R}$ is smoothing in space.

Thank you so much for your attention !