



Proof of Taylor’s Conjecture on Magnetic Helicity Conservation

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Abstract: We prove the mathematical version of Taylor’s conjecture which says that in 3D MHD, magnetic helicity is conserved in the ideal limit in bounded, simply connected, perfectly conducting domains. When the domain is multiply connected, magnetic helicity depends on the vector potential of the magnetic field. In that setting we show that magnetic helicity is conserved for a large and natural class of vector potentials but not in general for all vector potentials. As an analogue of Taylor’s conjecture in 2D, we show that mean square magnetic potential is conserved in the ideal limit, even in multiply connected domains.

1. Introduction

Magnetohydrodynamics (MHD in short) couples Navier–Stokes equations with Maxwell’s equations to study the macroscopic behaviour of electrically conducting fluids such as plasmas and liquid metals (see [30,48]). Given a domain $\Omega \subset \mathbb{R}^3$ and a time interval $[0, T)$, the Cauchy problem for the *incompressible, viscous, resistive MHD equations* consists of the equations

$$\partial_t u + (u \cdot \nabla)u - (b \cdot \nabla)b - \nu \Delta u + \nabla \Pi = 0, \quad (1.1)$$

$$\partial_t b + \operatorname{curl}(b \times u) + \mu \operatorname{curl} \operatorname{curl} b = 0, \quad (1.2)$$

$$\operatorname{div} u = \operatorname{div} b = 0, \quad (1.3)$$

$$u(\cdot, 0) = u_0, \quad b(\cdot, 0) = b_0, \quad (1.4)$$

where b is the magnetic field, u is the velocity field, Π is the total pressure, $\nu > 0$ is the coefficient of viscosity, $\mu > 0$ is the coefficient of resistivity and the initial data u_0 and

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b_0 are divergence-free. The Navier–Stokes equations are a special case of MHD where $b \equiv 0$. Furthermore, setting $\mu = \nu = 0$ one obtains the *ideal MHD equations*, while in the case $\mu = 0 < \nu$, (1.1)–(1.3) are called the *non-resistive MHD equations*.

In this work we consider Leray–Hopf solutions of (1.1)–(1.4) in a bounded domain Ω of \mathbb{R}^3 that has a $\mathcal{C}^{1,1}$ boundary Γ . As we want to incorporate Tokamaks and other laboratory plasma configurations, it is mandatory to consider multiply connected domains (see Assumptions 2.1–2.2 for the exact conditions on Ω). We use the standard *no-slip* and *perfect conductivity* boundary conditions

$$u|_{\Gamma} = 0, \tag{1.5}$$

$$b \cdot n|_{\Gamma} = 0 \quad \text{and} \quad (\text{curl } b) \times n|_{\Gamma} = 0, \tag{1.6}$$

(see Sect. 2.4 for precise definitions).

The existence of Leray–Hopf solutions in smooth simply connected domains goes back to [20, 48], and in [55], existence is shown under the slip without friction conditions on u . The more complicated case of smooth multiply connected domains is covered in the doctoral dissertation [35]. Since [35] is not readily available, we present our version of the proof for $\mathcal{C}^{1,1}$ multiply connected domains in the Appendix. For local-in-time existence and uniqueness of strong solutions as well as weak solutions in suitable Besov spaces for ideal MHD see [39, 46], and [47], and for the case of non-resistive MHD see [12, 24, 25], and [37]. For further references see [30, p. 57].

In ideal 3D MHD, smooth solutions conserve the *total energy* $2^{-1} \int_{\Omega} (|u(x, t)|^2 + |b(x, t)|^2) dx$ and the *cross helicity* $\int_{\Omega} u(x, t) \cdot b(x, t) dx$ in time. In simply connected domains the *magnetic helicity*

$$\int_{\Omega} \psi(x, t) \cdot b(x, t) dx,$$

where ψ is a vector potential of b (that is, $\text{curl } \psi = b$), is also conserved by smooth solutions and is independent of the choice of ψ .

Recently obtained numerical evidence points, however, towards *anomalous energy dissipation*, that is, the rate of total energy dissipation in viscous, resistive MHD *does not tend to zero* when $\mu, \nu \rightarrow 0$ (when the Reynolds number and magnetic Reynolds number tend to infinity); see [17, 38, 40]. Thus, if ideal MHD equations are to be a good model for magnetohydrodynamic turbulence at very high Reynolds number and magnetic Reynolds number, then the equations must possess (physically realistic) energy dissipative solutions. This is in analogy to the celebrated Onsager Conjecture on Euler equations (see [9, 15, 21, 33, 43]). In ideal MHD, bounded non-vanishing weak solutions with compact support in time (thus violating total energy conservation) were found in [7], while non-vanishing smooth strict subsolutions with compact support in space-time were constructed in [22].

In stark contrast to total energy, magnetic helicity has proved to be a very robust time invariant of ideal MHD. First, Caffisch, Klapper and Steele showed in [10] that magnetic helicity is conserved whenever $u \in C([0, T]; B_{3,\infty}^{\alpha_1}(\mathbb{T}^3, \mathbb{R}^3))$ and $b \in C([0, T]; B_{3,\infty}^{\alpha_2}(\mathbb{T}^3, \mathbb{R}^3))$ with $\alpha_1 + 2\alpha_2 > 0$, and next Kang and Lee ([34]) and Aluie and Eyink ([1, 2]) showed independently that magnetic helicity is conserved for $u, b \in L^3(0, T; L^3(\mathbb{T}^3, \mathbb{R}^3))$. In [22], the authors extended conservation to subsolutions and weak limits of solutions in $L^3(0, T; L^3(\mathbb{T}^3, \mathbb{R}^3))$.

It is still open whether magnetic helicity is conserved if u and b belong to the energy space $L^\infty(0, T; L^2(\mathbb{T}^3, \mathbb{R}^3))$. However, a straightforward adaptation of our Theorem 1.2

to the torus implies that conservation occurs if $u, b \in L^\infty(0, T; L^2(\mathbb{T}^3, \mathbb{R}^3))$ are a weak ideal limit of Leray–Hopf solutions (see Definition 1.1 and Corollary 1.3), which is arguably the physically relevant case.

It has been conjectured in the physics literature that magnetic helicity is approximately conserved at very low resistivities (see [49] where the conjecture was first formulated by Taylor, and for bounds at small resistivities see [4]). Mathematically, the conjecture says that magnetic helicity is conserved in the ideal limit $\mu, \nu \rightarrow 0$ (see [10, p. 444]). Taylor’s conjecture has been successful in explaining magnetic structures in laboratory plasmas, e.g., in the prediction of the relaxed state of a reversed field pinch, and lies at the heart of Taylor relaxation theory (for reviews with numerous further references see [8, 50]).

In Theorem 1.2 we prove Taylor’s conjecture under weak and natural assumptions. We consider arbitrary weak limits of Leray–Hopf solutions when $\mu_j, \nu_j \rightarrow 0$ (which exist, up to a subsequence, whenever the L^2 norms of the initial datas are uniformly bounded). In particular, *we do not assume that the weak limits satisfy the ideal MHD equations*. Recall that

$$L^2_\sigma(\Omega, \mathbb{R}^3) := \{v \in L^2(\Omega, \mathbb{R}^3) : \operatorname{div} v = 0, v \cdot n|_\Gamma = 0\}.$$

Definition 1.1. Given Leray–Hopf solutions (u_j, b_j) of (1.1)–(1.6) with $\mu_j, \nu_j > 0$ and initial datas $u_{j,0}, b_{j,0} \in L^2_\sigma(\Omega, \mathbb{R}^3)$ suppose that $\mu_j, \nu_j \rightarrow 0$ and that $u_{j,0} \rightharpoonup u_0$ and $b_{j,0} \rightharpoonup b_0$ in $L^2_\sigma(\Omega, \mathbb{R}^3)$. Assume that $u_j \overset{*}{\rightharpoonup} u$ and $b_j \overset{*}{\rightharpoonup} b$ in $L^\infty(0, T; L^2_\sigma(\Omega, \mathbb{R}^3))$. We then say that (u, b) is a *weak ideal limit* of (u_j, b_j) .

If instead $\mu_j \rightarrow 0$ and $\nu_j = \nu > 0$ for every $j \in \mathbb{N}$, we say that (u, b) is a *weak non-resistive limit* of (u_j, b_j) .

Taylor’s conjecture concerns the case where magnetic helicity is *gauge invariant* (i.e. independent of the choice of the vector potential of b), that is, simply connected domains. The following theorem proves Taylor’s conjecture.

Theorem 1.2. *Suppose Ω is simply connected and (u, b) is a weak ideal limit of Leray–Hopf solutions (u_j, b_j) with $\mu_j, \nu_j \rightarrow 0$. Then $\int_\Omega \psi(x, t) \cdot b(x, t) \, dx$ is a.e. constant in t for every vector potential $\psi \in L^\infty(0, T; W^{1,2}(\Omega, \mathbb{R}^3))$ of b .*

Although in Theorem 1.2 we do not assume that u and b satisfy the ideal MHD equations, we present a corollary on solutions of ideal MHD. If a solution (u, b) lies in the energy space $L^\infty(0, T; L^2_\sigma(\Omega, \mathbb{R}^3))$, then we may choose representatives $u, b \in C_w([0, T]; L^2_\sigma(\Omega, \mathbb{R}^3))$; this can be proved by slightly modifying [28, Lemmas 2.1–2.2].

Corollary 1.3. *Suppose Ω is simply connected and $u, b \in C_w([0, T]; L^2_\sigma(\Omega, \mathbb{R}^3))$ form a weak solution of ideal MHD. If (u, b) is a weak ideal limit of Leray–Hopf solutions (u_j, b_j) , then b conserves magnetic helicity in time.*

Notice that for general weak solutions of ideal MHD, magnetic helicity conservation has only been proved under $L^3_t L^3_x$ integrability, and so there is an apparent scaling gap. Actually, as pointed out to us by the anonymous referee, the same phenomenon has been observed in other equations of hydrodynamics, see [13] for Euler and [16] for SQG, and the proof of Theorem 1.2 is built on similar principles.

While simply connected domains (and especially the torus \mathbb{T}^3) allow a relatively neat mathematical treatment, we also cover the much more difficult case of multiply

connected domains in order to incorporate plasma containers in typical laboratory settings. The topology of multiply connected domains leads, however, to mathematical complications starting with the very definition of magnetic helicity.

Consider an arbitrary weak ideal limit (u, b) of Leray–Hopf solutions (u_j, b_j) . If the domain Ω is multiply connected, then $\int_{\Omega} \psi(x, t) \cdot b(x, t) dx$ depends on the choice of the vector potential ψ . The basic reason behind this gauge dependence is the fact that when Ω is multiply connected, the orthogonal complement of $\ker(\operatorname{curl})$ in $L^2(\Omega, \mathbb{R}^3)$ is a *strict* subspace of $L^2_{\sigma}(\Omega, \mathbb{R}^3)$ —in other words, the set of harmonic *Neumann vector fields*

$$L^2_H(\Omega, \mathbb{R}^3) := \{v \in L^2_{\sigma}(\Omega, \mathbb{R}^3) : \operatorname{curl} v = 0\}$$

is non-trivial. For a physical interpretation of $L^2_H(\Omega, \mathbb{R}^3)$ see e.g. [11, pp. 428–430].

We write

$$L^2_{\sigma}(\Omega, \mathbb{R}^3) = L^2_{\Sigma}(\Omega, \mathbb{R}^3) \oplus L^2_H(\Omega, \mathbb{R}^3); \tag{1.7}$$

a useful intrinsic characterisation of $L^2_{\Sigma}(\Omega, \mathbb{R}^3)$ was given in [27] (see Theorem 3.2). For the purposes of this article, it is also illuminating to use a characterisation familiar from Hodge–Friedrichs–Morrey decomposition theory (see e.g. [41]),

$$L^2_{\Sigma}(\Omega, \mathbb{R}^3) = \{\operatorname{curl} \psi : \psi \in W^{1,2}(\Omega, \mathbb{R}^3), \psi \times n|_{\Gamma} = 0\}. \tag{1.8}$$

In fact, we will need slightly more refined versions of (1.8), see Theorem 3.8 and Remark 3.9.

Bearing in mind (1.7), we decompose b uniquely as

$$b = b_{\Sigma} + b_H \quad (b_{\Sigma} \in L^{\infty}(0, T; L^2_{\Sigma}(\Omega, \mathbb{R}^3)) \text{ and } b_H \in L^{\infty}(0, T; L^2_H(\Omega, \mathbb{R}^3))) \tag{1.9}$$

and use similar notation for every b_j . In multiply connected domains, we prove that $\int_{\Omega} \psi(x, t) \cdot b(x, t) dx$ is conserved for all vector potentials $\psi \in L^{\infty}(0, T; W^{1,2}(\Omega, \mathbb{R}^3))$ of b if and only if the harmonic part $b_H = 0$. There exist, however, weak ideal limits (u, b) of Leray–Hopf solutions with $b_H \neq 0$ (see Proposition 3.5 for both claims).

We are thus led to the following question in multiply connected domains:

$$\text{Is } \int_{\Omega} \psi(x, t) \cdot b(x, t) dx \text{ conserved for some natural class of potentials } \psi? \tag{1.10}$$

We give a positive answer to (1.10) in Corollary 1.6. First, in Theorem 1.5 we compute the magnetic helicity dissipation rate for arbitrary Leray–Hopf solutions and arbitrary vector potentials. In (1.11) we are able to compute the dissipation rate also for weak ideal limits and all their potentials. Corollary 1.6 then gives a condition on potentials that is coherent with (1.8) and ensures magnetic helicity conservation.

We use the decomposition in (1.9) in order to give a formula for the time evolution of magnetic helicity. The components b_{Σ} and b_H of b behave in rather differing ways; in particular, b_H is constant in time (see Proposition 4.3). Because of difficulties described in Sect. 4.1, we also need to decompose ψ in order to take advantage of the different features of b_{Σ} and b_H :

$$\psi = \psi^{\Sigma} + \psi^H \quad (\operatorname{curl} \psi^{\Sigma} = b_{\Sigma} \text{ and } \operatorname{curl} \psi^H = b_H).$$

The decomposition $\psi = \psi^\Sigma + \psi^H$ is not unique, and a judicious choice of the components $\psi^\Sigma, \psi^H \in L^\infty(0, T; W^{1,2}(\Omega, \mathbb{R}^3))$ is a fundamental part of the proof of Theorem 1.5. In fact, we end up performing a further decomposition of ψ^Σ , and the whole decomposition of ψ is described in Sects. 3.2 and 4.1.

In order to state Theorem 1.5 we already note below that given ψ , there exists a canonical choice of ψ^H , and we use it for all the vector potentials in this article. In particular, with this choice, $\partial_t b_H = 0$ implies that $\partial_t \psi^H = 0$.

Definition 1.4. Suppose that $v = v_\Sigma + v_H \in L^\infty(0, T; L^2_\sigma(\Omega, \mathbb{R}^3))$ and that $\psi \in L^\infty(0, T; W^{1,2}(\Omega, \mathbb{R}^3))$ satisfies $\text{curl } \psi = v$. We denote by ψ^H the unique mapping in $L^\infty(0, T; W^{1,2}(\Omega, \mathbb{R}^3) \cap L^2_\Sigma(\Omega, \mathbb{R}^3))$ such that $\text{curl } \psi^H = b_H$ (see Theorem 3.7), and we furthermore denote $\psi^\Sigma := \psi - \psi^H$.

We are now in a position to state our main theorem; the strategy of the proof is described in Sects. 4.1–4.2, and the details are presented in Sects. 4.3–4.6.

Theorem 1.5. *Suppose a domain $\Omega \subset \mathbb{R}^3$ satisfies Assumption 2.1, and assume that (u, b) is a weak ideal limit or weak non-resistive limit of Leray–Hopf solutions (u_j, b_j) , $j \in \mathbb{N}$. Then any vector potentials ψ_j and $\psi_{j,0}$ of b_j and $b_{j,0}$ satisfy*

$$\begin{aligned} \int_\Omega \psi_j(x, t) \cdot b_j(x, t) \, dx &= \int_\Omega \psi_{j,0}(x) \cdot b_{j,0}(x) \, dx \\ &\quad - 2\mu_j \int_0^t \int_\Omega b_j(x, \tau) \cdot \text{curl } b_j(x, \tau) \, dx \, d\tau \\ &\quad - \int_\Gamma [\psi_j^\Sigma(x, t) - \psi_{j,0}^\Sigma(x)] \times n \cdot \psi_{j,0}^H(x) \, dx \end{aligned}$$

for all $j \in \mathbb{N}$ and $t \in [0, T)$. Furthermore,

$$\int_\Omega \psi(x, t) \cdot b(x, t) \, dx = \int_\Omega \psi_0(x) \cdot b_0(x) \, dx - \int_\Gamma [\psi^\Sigma(x, t) - \psi_0^\Sigma(x)] \times n \cdot \psi_0^H(x) \, dx \tag{1.11}$$

for a.e. $t \in (0, T)$ and all vector potentials ψ and ψ_0 of b and b_0 .

Formula (1.11) allows us to show magnetic helicity conservation for a large class of vector potentials. The class is specified in (1.12), and its naturality is apparent from (1.8) and (1.11).

Corollary 1.6. *Suppose the assumptions of Theorem 1.5 hold. If*

$$\psi_j^\Sigma \times n|_\Gamma = \psi^\Sigma \times n|_\Gamma = 0 \quad \text{and} \quad \psi_{j,0}^\Sigma \times n|_\Gamma = \psi_0^\Sigma \times n|_\Gamma = 0, \tag{1.12}$$

then

$$\int_\Omega \psi(x, t) \cdot b(x, t) \, dx = \int_\Omega \psi_0(x) \cdot b_0(x) \, dx = \lim_{j \rightarrow \infty} \int_\Omega \psi_{j,0}(x) \cdot b_{j,0}(x) \, dx \tag{1.13}$$

for a.e. $t \in (0, T)$. In particular, under condition (1.12), the magnetic helicity of b is independent of the choice of ψ .

In Sect. 5 we prove a two-dimensional analogue of Theorem 1.5: in bounded, multiply connected Lipschitz domains, mean-square magnetic potential is conserved in the weak ideal limit. In 2D, there exists a canonical choice of potentials, and so we can follow the philosophy of [22] which is based on \mathcal{H}^1 –BMO duality and compensated compactness. In fact, we also show that in multiply connected domains, all solutions of ideal MHD in the energy space conserve magnetic mean-square potential, extending a similar result on the torus \mathbb{T}^2 from [22].

In three dimensions, when magnetic field lines are allowed to cross Γ , that is, the assumption $b \cdot n|_{\Gamma} = 0$ is dropped, magnetic helicity is no longer gauge invariant even for smooth solutions of ideal MHD in simply connected domains. In such a setting the so-called *relative magnetic helicity*, defined in [5, 26], can be studied instead. We defer a treatment of relative magnetic helicity to a subsequent work.

2. Background

In this chapter we review tools and results needed in this article. We first fix our assumptions on the domain Ω in Sect. 2.1 and recall basic material on boundary traces of Sobolev and L^p functions in Sects. 2.2 and 2.3 reviews some standard results on time-dependent mappings in Bochner spaces, and in Sect. 2.4 we discuss Leray–Hopf solutions of viscous, resistive 3D MHD equations and the notion of ideal limit.

2.1. Assumptions on the domain. We start by fixing our assumptions on the domain Ω , and we illustrate the assumptions in Fig. 1. Our exposition follows [3, pp. 835–836] (see also [51]).

Assumption 2.1. The domain $\Omega \subset \mathbb{R}^3$ is bounded and its boundary Γ is of class $\mathcal{C}^{1,1}$ and has a finite number of connected components denoted by $\Gamma_1, \dots, \Gamma_K$.

Another assumption is introduced in order to produce a simply connected domain by making cuts into Ω . The cuts will, however, only play an implicit role in this article.

Assumption 2.2. There exist connected open surfaces Σ_j , $1 \leq j \leq N$, contained in Ω and satisfying the following conditions:

- (i) Each surface Σ_j is an open subset of a smooth manifold \mathcal{M}_j .
- (ii) The boundary of each Σ_j is contained in $\partial\Omega$.
- (iii) $\bar{\Sigma}_i \cap \bar{\Sigma}_j = \emptyset$ whenever $i \neq j$.
- (iv) The open set $\dot{\Omega} := \Omega \setminus \cup_{j=1}^N \Sigma_j$ is simply connected and pseudo-Lipschitz (see Definition 2.3 below).

The sets Σ_j are called *cuts*.

The notion of a pseudo-Lipschitz domain is a generalization of a Lipschitz domain that allows the domain to locally lie on both sides of its boundary.

Definition 2.3. A bounded domain $\Delta \subset \mathbb{R}^3$ is called *pseudo-Lipschitz* if for every $x \in \partial\Delta$ there exists an integer $r(x) \in \{1, 2\}$ and a radius $\rho_0 > 0$ such that whenever $0 < \rho < \rho_0$, the intersection $\Delta \cap B(x, \rho)$ has $r(x)$ connected components, each one with a Lipschitz boundary.

Assumptions 2.1–2.2 are standard in the study of fluid dynamics in multiply connected domains (see e.g. [3, 27, 51]) and will remain in place for the rest of this article (except Sect. 5 where we discuss the two-dimensional setting). In particular, a solid torus clearly satisfies Assumptions 2.1–2.2.

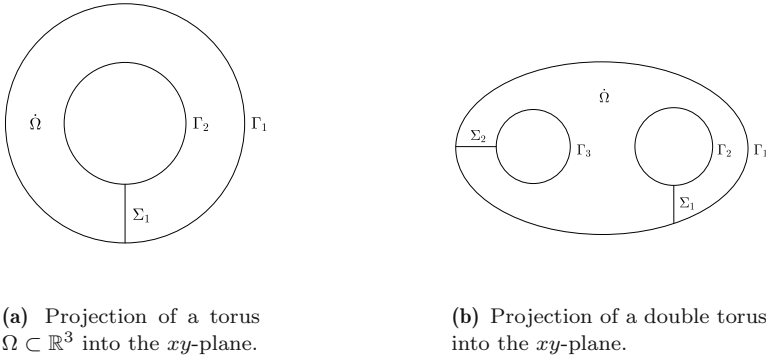


Fig. 1. Examples of domains satisfying Assumptions 2.1 and 2.2

2.2. *Traces of Sobolev functions.* We recall results on boundary traces, normal traces and tangential traces and refer to [29,31,42] for the proofs. In Theorems 2.4–2.7 the assumption that Γ is $\mathcal{C}^{1,1}$ can in fact be relaxed to Γ being Lipschitz regular. The first trace theorem we present is a special case of [29, Theorem II.4.1].

Theorem 2.4. *Let $1 \leq p < \infty$. Then there exists a unique, continuous linear map $\gamma : W^{1,p}(\Omega) \rightarrow L^p(\Gamma)$ such that for every $u \in C^\infty(\bar{\Omega})$ we have $\gamma(u) = u$ on Γ .*

We denote by $W^{1-1/p,p}(\Gamma)$ the subspace of $L^p(\Gamma)$ of functions for which

$$\|u\|_{W^{1-1/p,p}(\Gamma)} := \|u\|_{L^p(\Omega)} + \left(\int_{\Gamma} \int_{\Gamma} \frac{|u(x) - u(y)|^p}{|x - y|^{1+p}} dS(x) dS(y) \right)^{\frac{1}{p}} < \infty.$$

The space $W^{1-1/p,p}(\Gamma)$ is dense in $L^p(\Gamma)$ and complete in the norm $\|\cdot\|_{W^{1-1/p,p}(\Gamma)}$. When $1 < p < \infty$, the trace operator γ is a Banach space isomorphism from the quotient space $W^{1,p}(\Omega)/W_0^{1,p}(\Omega)$ onto $W^{1-1/p,p}(\Gamma)$ (see [42, §2.5, Theorems 5.5 and 5.7]):

Theorem 2.5. *Let $1 < p < \infty$. If $u \in W^{1,p}(\Omega)$, then $\gamma(u) \in W^{1-1/p,p}(\Gamma)$ and*

$$\|\gamma(u)\|_{W^{1-1/p,p}(\Gamma)} \lesssim_{\Omega,p} \|u\|_{W^{1,p}(\Omega)}.$$

Conversely, for every $w \in W^{1-1/p,p}(\Gamma)$ there exists $u \in W^{1,p}(\Omega)$ such that $\gamma(u) = w$ and $\|u\|_{W^{1,p}(\Omega)} \lesssim_{\Omega,p} \|\gamma(u)\|_{W^{1-1/p,p}(\Gamma)}$.

For convenience we will denote the trace $\gamma(u)$ simply by u . Whenever $u \in C^\infty(\bar{\Omega}, \mathbb{R}^3)$, the normal trace $u \cdot n$ and the tangential trace $u \times n$ are well-defined on the boundary Γ and the generalized Gauss identity and Green’s formula

$$\langle u \cdot n, \varphi \rangle_{\Gamma} = \int_{\Omega} u(x) \cdot \nabla \varphi(x) dx + \int_{\Omega} \varphi(x) \operatorname{div} u(x) dx, \quad \varphi \in W^{1,p'}(\Omega) \quad (2.1)$$

$$\begin{aligned} \langle u \times n, \psi \rangle_{\Gamma} &= \int_{\Omega} \operatorname{curl} u(x) \cdot \psi(x) dx \\ &- \int_{\Omega} u(x) \cdot \operatorname{curl} \psi(x) dx, \quad \psi \in W^{1,p'}(\Omega, \mathbb{R}^3) \end{aligned} \quad (2.2)$$

hold, where $\langle u \cdot n, \varphi \rangle_\Gamma$ and $\langle u \times n, \psi \rangle_\Gamma$ are standard surface integrals (but can also be interpreted in terms of $W^{-1/p,p}(\Gamma)$ – $W^{1-1/p',p'}(\Gamma)$ duality).

Normal and tangential traces are extended to the function spaces defined next: when $1 < p < \infty$, $H^p(\operatorname{div}, \Omega) := \{v \in L^p(\Omega, \mathbb{R}^3) : \operatorname{div} v \in L^p(\Omega)\}$ is endowed with the norm $\|v\|_{H^p(\operatorname{div}, \Omega)} := (\|v\|_{L^p(\Omega)}^p + \|\operatorname{div} v\|_{L^p(\Omega)}^p)^{1/p}$, while $H^p(\operatorname{curl}, \Omega) := \{v \in L^p(\Omega, \mathbb{R}^3) : \operatorname{curl} v \in L^p(\Omega, \mathbb{R}^3)\}$ is given the norm $\|v\|_{H^p(\operatorname{curl}, \Omega)} := (\|v\|_{L^p(\Omega)}^p + \|\operatorname{curl} v\|_{L^p(\Omega)}^p)^{1/p}$.

Theorem 2.6. *Suppose $1 < p < \infty$. Then the normal trace has a unique bounded extension $u \mapsto u \cdot n : H^p(\operatorname{div}, \Omega) \rightarrow W^{-1/p,p}(\Gamma)$ and the generalized Gauss identity (2.1) holds.*

For a proof of Theorem 2.6 see [29, Theorem III.2.2]. In a similar vein, a tangential trace is well-defined whenever $v \in L^p(\Omega, \mathbb{R}^3)$ and $\operatorname{curl} v \in L^p(\Omega, \mathbb{R}^3)$:

Theorem 2.7. *Suppose $1 < p < \infty$. Then the tangential trace has a unique bounded extension $u \mapsto u \times n : H^p(\operatorname{curl}, \Omega) \rightarrow W^{-1/p,p}(\Gamma, \mathbb{R}^3)$ and the generalized Green’s formula (2.2) holds.*

Finally, we mention a characterisation of $W^{1,2}(\Omega, \mathbb{R}^3)$ by Foias and Temam (see e.g. [31, Corollary 3.7]). Here Lipschitz continuity of Γ would not be sufficient (see [3, p. 832]).

Theorem 2.8. *We have $W^{1,2}(\Omega, \mathbb{R}^3) = \{v \in L^2(\Omega, \mathbb{R}^3) : \operatorname{div} v \in L^2(\Omega), \operatorname{curl} v \in L^2(\Omega, \mathbb{R}^3), v \cdot n \in W^{1/2,2}(\Gamma)\}$ and*

$$\|v\|_{W^{1,2}(\Omega)} \lesssim_\Omega \|v\|_{L^2(\Omega)} + \|\operatorname{div} v\|_{L^2(\Omega)} + \|\operatorname{curl} v\|_{L^2(\Omega)} + \|v \cdot n\|_{W^{1/2,2}(\Gamma)}$$

for all $v \in W^{1,2}(\Omega, \mathbb{R}^3)$.

2.3. Bochner spaces. We recall some basic facts on time-dependent mappings in Bochner spaces in a generality needed in this article. We do not discuss the definitions of Bochner measurability and Bochner integrability but refer to [32] for a thorough introduction to Bochner spaces and to [45] for a shorter one with an emphasis on applications in PDE’s.

Whenever $1 \leq p < \infty$ and X is a Banach space, the Bochner space $L^p(0, T; X)$ consists of (classes with respect to equality a.e. $t \in (0, T)$ of) Bochner integrable functions $v : (0, T) \rightarrow X$ satisfying $\int_0^T \|v(\cdot, t)\|_X^p dt < \infty$. For $L^\infty(0, T; X)$ the corresponding condition is $\| \|v(\cdot, t)\|_X \|_{L^\infty(0, T)} < \infty$. If $1 \leq p < \infty$ and X^* is separable, then $(L^p(0, T; X))^* = L^{p'}(0, T; X^*)$ with the duality pairing given by

$$\langle f, v \rangle_{L^{p'}(0, T; X^*) - L^p(0, T; X)} := \int_0^T \langle f(\cdot, t), v(\cdot, t) \rangle_{X^* - X} dt$$

(see [32, Corollary 1.3.22]). Furthermore, then $L^p(0, T; X)$ is separable (see [32, Proposition 1.2.29]) and thus every bounded sequence in $L^{p'}(0, T; X^*)$ has a weak- $*$ convergent subsequence. We also denote by $C_w([0, T]; X)$ the set of mappings $v : [0, T) \rightarrow X$ defined at every $t \in [0, T)$ and satisfying $t_j \rightarrow t$ in $[0, T) \Rightarrow v(\cdot, t_j) \rightharpoonup v(\cdot, t)$ in X .

Whenever $f \in L^1(0, T; X)$, $0 < \delta < T/2$ and $\theta \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}(\theta) \subset (-\delta, \delta)$, we define $f * \theta \in C^\infty(\delta, T - \delta; X)$ by $f * \theta(\cdot, t) := \int_0^T \theta(t - s) f(\cdot, s) ds \in X$. We record a variant of Young’s convolution inequality.

Lemma 2.9. *Suppose $p, q, r \in [1, \infty]$ with $1/p + 1/q = 1 + 1/r$ and $1 \leq s < \infty$. If $v \in L^p(0, T; L^s(\Omega))$ and $\theta \in C_c^\infty(\mathbb{R})$ with $\text{supp}(\theta) \subset (-\delta, \delta)$, then*

$$\|v * \theta\|_{L^r(\delta, T-\delta; L^s(\Omega))} \leq \|v\|_{L^p(0, T; L^s(\Omega))} \|\theta\|_{L^q(-\delta, \delta)}.$$

Proof. By Minkowski’s integral inequality and Young’s convolution inequality,

$$\begin{aligned} & \int_{\delta}^{T-\delta} \left(\int_{\Omega} |v * \theta(x, t)|^s dx \right)^{\frac{r}{s}} dt \\ &= \int_{\delta}^{T-\delta} \left(\int_{\Omega} \left| \int_0^T v(x, \tau) \theta(t - \tau) d\tau \right|^s dx \right)^{\frac{r}{s}} dt \\ &\leq \int_{\delta}^{T-\delta} \left(\int_0^T \left(\int_{\Omega} |v(x, \tau)|^s dx \right)^{\frac{1}{s}} |\theta(t - \tau)| d\tau \right)^r dt \\ &= \int_{\delta}^{T-\delta} (\|x \mapsto v(x, \cdot)\|_{L^s(\Omega)} * |\theta|(t))^r dt \\ &\leq \|v\|_{L^p(0, T; L^s(\Omega))}^r \|\theta\|_{L^q(-\delta, \delta)}^r. \end{aligned}$$

□

We fix, for the rest of this article, an even mollifier $\chi \in C_c^\infty(\mathbb{R})$ with $\text{supp}(\chi) \subset (-1, 1)$ and $\int_{-1}^1 \chi(t) dt = 1$. We denote $\chi^\delta(t) := \delta^{-1} \chi(t/\delta)$ for all $\delta > 0$ and $t \in \mathbb{R}$. For every $f \in L^1(0, T; L^1(\Omega))$ we denote $f_\delta := f * \chi^\delta$. For a proof of the following mollifier approximation lemma see [32, Proposition 1.2.32].

Lemma 2.10. *Let $0 < \epsilon < T/2$ and suppose $1 \leq p, q < \infty$ and $f \in L^p(0, T; L^q(\Omega))$. Then $\|f_\delta - f\|_{L^p(\epsilon, T-\epsilon; L^q(\Omega))} \rightarrow 0$ as $\delta \rightarrow 0$.*

The following interpolation inequalities will also be useful to us.

Lemma 2.11. *For every $v, w \in L^\infty(0, T; L^2(\Omega, \mathbb{R}^3)) \cap L^2(0, T; W^{1,2}(\Omega, \mathbb{R}^3))$ we have*

$$\begin{aligned} \|v\|_{L^4(0, T; L^3(\Omega))} &\lesssim \Omega \|v\|_{L^2(0, T; W^{1,2}(\Omega))}^{1/2} \|v\|_{L^\infty(0, T; L^2(\Omega))}^{1/2}, \\ \|v \otimes w\|_{L^1(0, T; W^{1,3/2}(\Omega))} &\lesssim \Omega \|v\|_{L^2(0, T; W^{1,2}(\Omega))} \|w\|_{L^2(0, T; W^{1,2}(\Omega))}, \\ \|v \otimes w\|_{L^{4/3}(0, T; L^2(\Omega))} &\lesssim \Omega \|v\|_{L^2(0, T; W^{1,2}(\Omega))}^{3/4} \|v\|_{L^\infty(0, T; L^2(\Omega))}^{1/4} \\ &\quad \cdot \|w\|_{L^2(0, T; W^{1,2}(\Omega))}^{3/4} \|w\|_{L^\infty(0, T; L^2(\Omega))}^{1/4}, \end{aligned}$$

where $v \otimes w := [v_i w_j]_{i, j=1}^3$ is the tensor product of v and w .

Proof. The first inequality is a standard interpolation and can be found e.g. at [44, p. 74] (up to a use of the Sobolev embedding $W^{1,2}(\Omega) \subset L^6(\Omega)$). For the second one note that at a.e. $t \in (0, T)$, Hölder’s inequality and the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ yield

$$\| |v| |\nabla w| \|_{L^{3/2}(\Omega)} \leq \|v\|_{L^6(\Omega)} \|\nabla w\|_{L^2(\Omega)} \lesssim \Omega \|v\|_{W^{1,2}(\Omega)} \|w\|_{W^{1,2}(\Omega)}.$$

A similar inequality holds for $|w| |\nabla w|$ and $|v| |w|$, and one then uses the Cauchy–Schwarz inequality on time integrals to finish the proof. Similar reasoning is used to prove the third inequality of the lemma. □

We also recall the Aubin–Lions Lemma which we formulate in a form that suffices for the purposes of this article (see [45, Lemma 7.7]).

Lemma 2.12. *Let X, Y and Z be reflexive Banach spaces such that X embeds compactly into Y and Y embeds into Z . Suppose $1 < p < \infty$ and $1 \leq q \leq \infty$. Then $\{u \in L^p(0, T; X) : \partial_t u \in L^q(0, T; Z)\}$ embeds compactly into $L^p(0, T; Y)$.*

2.4. Leray–Hopf solutions of viscous, resistive MHD equations and the ideal limit. We recall the definition and present an existence theorem on Leray–Hopf solutions of viscous, resistive 3D MHD equations. When $1 < p < \infty$, we denote the relevant function spaces by

$$\begin{aligned}
 C_{c,\sigma}^\infty(\Omega, \mathbb{R}^3) &:= \{\varphi \in C_c^\infty(\Omega, \mathbb{R}^3) : \operatorname{div} \varphi = 0\}, \\
 L_\sigma^p(\Omega, \mathbb{R}^3) &:= \overline{C_{c,\sigma}^\infty(\Omega, \mathbb{R}^3)}^{L^p(\Omega, \mathbb{R}^3)} = \{v \in L^p(\Omega, \mathbb{R}^3) : \operatorname{div} v = 0, v \cdot n|_\Gamma = 0\}, \\
 W_{0,\sigma}^{1,p}(\Omega, \mathbb{R}^3) &:= \overline{C_{c,\sigma}^\infty(\Omega, \mathbb{R}^3)}^{W_0^{1,p}(\Omega, \mathbb{R}^3)} = \{v \in W_0^{1,p}(\Omega, \mathbb{R}^3) : \operatorname{div} v = 0\}, \\
 W_\sigma^{1,p}(\Omega, \mathbb{R}^3) &:= \{v \in W^{1,p}(\Omega, \mathbb{R}^3) : \operatorname{div} v = 0, v \cdot n|_\Gamma = 0\}.
 \end{aligned}$$

(for the two identities see e.g. [29, Theorems III.2.3 and III.4.1]). Leray–Hopf solutions of MHD are defined by the following standard variational formulation.

Definition 2.13. Let $u_0, b_0 \in L_\sigma^2(\Omega, \mathbb{R}^3)$. Suppose that $u \in C_w([0, T]; L_\sigma^2(\Omega, \mathbb{R}^3)) \cap L^2(0, T; W_{0,\sigma}^{1,2}(\Omega))$ and $b \in C_w([0, T]; L_\sigma^2(\Omega, \mathbb{R}^3)) \cap L^2(0, T; W_\sigma^{1,2}(\Omega))$ satisfy $\partial_t u \in L^1(0, T; (W_{0,\sigma}^{1,2}(\Omega, \mathbb{R}^3))^*)$ and $\partial_t b \in L^1(0, T; (W_\sigma^{1,2}(\Omega, \mathbb{R}^3))^*)$, and that

$$\langle \partial_t u, \varphi \rangle + \int_\Omega (u \cdot \nabla u - b \cdot \nabla b) \cdot \varphi + \nu \int_\Omega \nabla u : \nabla \varphi = 0, \tag{2.3}$$

$$\langle \partial_t b, \theta \rangle + \int_\Omega b \times u \cdot \operatorname{curl} \theta + \mu \int_\Omega \operatorname{curl} b \cdot \operatorname{curl} \theta = 0 \tag{2.4}$$

hold at a.e. $t \in [0, T)$ and every $\varphi \in W_{0,\sigma}^{1,2}(\Omega, \mathbb{R}^3)$ and $\theta \in W_\sigma^{1,2}(\Omega, \mathbb{R}^3)$. Suppose furthermore that $u(\cdot, 0) = u_0$ and $b(\cdot, 0) = b_0$ and that u and b satisfy the *energy inequality*

$$\begin{aligned}
 &\frac{1}{2} \int_\Omega (|u(x, t)|^2 + |b(x, t)|^2) dx \\
 &\quad + \int_0^t \int_\Omega (\nu |\nabla u(x, \tau)|^2 + \mu |\operatorname{curl} b(x, \tau)|^2) dx d\tau \\
 &\leq \frac{1}{2} \int_\Omega (|u_0(x)|^2 + |b_0(x)|^2) dx
 \end{aligned}$$

for all $t \in (0, T)$. Then (u, b) is called a *Leray–Hopf solution* of (1.1)–(1.6).

Note that (2.4) captures in a weak sense the condition $(\operatorname{curl} b) \times n|_\Gamma = 0$. Also note that (2.4) and the condition $b(\cdot, 0) = b_0$ imply that

$$\int_0^T \partial_t \eta \int_\Omega b \cdot \theta - \int_0^T \eta \int_\Omega b \times u \cdot \operatorname{curl} \theta - \mu \int_0^t \eta \int_\Omega \operatorname{curl} b \cdot \operatorname{curl} \theta + \eta(0) \int_\Omega b_0 \cdot \theta = 0 \tag{2.5}$$

for all $\eta \in C_c^\infty([0, T])$ and $\theta \in W_\sigma^{1,2}(\Omega, \mathbb{R}^3)$. As mentioned in the introduction, we present a proof of the following theorem in the Appendix.

Theorem 2.14. *Let $u_0, b_0 \in L_\sigma^2(\Omega; \mathbb{R}^3)$. Then there exists a Leray–Hopf solution (u, b) of (1.1)–(1.6).*

Theorems 1.2 and 1.5 do not assume that the ideal (i.e. inviscid, non-resistive) limit, defined below, holds. However, we mention the notion for completeness and also because it falls under the scope of Corollary 1.3. It is a fundamental open problem under what conditions the ideal limit holds in 3D MHD (see [18,52–55,57] for partial results).

Definition 2.15. Suppose viscosities $\nu_j > 0$ and resistivities $\mu_j > 0$ satisfy $\nu_j, \mu_j \rightarrow 0$ and that divergence-free initial data $u_{j,0} \rightarrow u_0$ and $b_{j,0} \rightarrow b_0$ in $L_\sigma^2(\Omega, \mathbb{R}^3)$. Assume that for every $j \in \mathbb{N}$, (u_j, b_j) is a Leray–Hopf solution of (1.1)–(1.6) and that $u, b \in L^\infty(0, T; L_\sigma^2(\Omega, \mathbb{R}^3))$ form a solution of (1.1)–(1.4) with $\mu = \nu = 0$. We say that (u, b) is the *ideal limit* or *inviscid, non-resistive limit* of (u_j, b_j) (in the energy norm) if $\|u_j - u\|_{L^\infty(0,T;L^2(\Omega))} \rightarrow 0$ and $\|b_j - b\|_{L^\infty(0,T;L^2(\Omega))} \rightarrow 0$. We then also say that *the ideal limit holds for (u_j, b_j) and (u, b) .*

3. Vector Potentials and Gauge Dependence of Magnetic Helicity

The aim of this section is to discuss the notion of magnetic helicity in multiply connected domains and to recall the existence of vector potentials satisfying the assumptions of Corollary 1.6.

3.1. Magnetic helicity in multiply connected domains. We first recall the Helmholtz–Weyl decomposition of $L^2(\Omega, \mathbb{R}^3)$ which is, in fact, valid in every domain of \mathbb{R}^n for all $n \geq 2$ (see [28, Theorem III.1.1]).

Theorem 3.1. $L^2(\Omega) = L_\sigma^2(\Omega, \mathbb{R}^3) \oplus \nabla W^{1,2}(\Omega, \mathbb{R}^3)$.

In [27], Foias and Temam performed a further direct decomposition of $L_\sigma^2(\Omega, \mathbb{R}^3)$ into a part that has a vanishing flux across the cuts and an harmonic part (see [27, Proposition 1.1] or [51, Appendix I, Lemma 1.4]). We present the decomposition of Foias and Temam in the notation of [56].

Theorem 3.2. $L_\sigma^2(\Omega, \mathbb{R}^3) = L_\Sigma^2(\Omega, \mathbb{R}^3) \oplus L_H^2(\Omega, \mathbb{R}^3)$, where

$$L_\Sigma^2(\Omega, \mathbb{R}^3) := \left\{ v \in L_\sigma^2(\Omega, \mathbb{R}^3) : \int_{\Sigma_i} v(x) \cdot n(x) \, dS(x) = 0 \text{ for } i = 1, \dots, N \right\},$$

$$L_H^2(\Omega, \mathbb{R}^3) := \{v \in L_\sigma^2(\Omega, \mathbb{R}^3) : \text{curl } v = 0\}.$$

By Theorem 2.8, $L_H^2(\Omega, \mathbb{R}^3) \subset W_\sigma^{1,2}(\Omega, \mathbb{R}^3)$.

Definition 3.3. We denote the projections onto $L_\Sigma^2(\Omega, \mathbb{R}^3)$ and $L_H^2(\Omega, \mathbb{R}^3)$ by $P_\Sigma : L_\sigma^2(\Omega, \mathbb{R}^3) \rightarrow L_\Sigma^2(\Omega, \mathbb{R}^3)$ and $P_H : L_\sigma^2(\Omega, \mathbb{R}^3) \rightarrow L_H^2(\Omega, \mathbb{R}^3)$. For every $v \in L_\sigma^2(\Omega, \mathbb{R}^3)$ we denote $v_\Sigma := P_\Sigma v$ and $v_H := P_H v$.

The vector space $L^2_H(\Omega, \mathbb{R}^3)$ is N -dimensional. For a characterisation of an orthonormal basis $\{h_1, \dots, h_N\}$ of $L^2_H(\Omega, \mathbb{R}^3)$ see [51, Appendix I, Lemma 1.3] or [3, Proposition 3.14]. Theorems 3.1 and 3.2 yield the decomposition

$$L^2(\Omega, \mathbb{R}^3) = L^2_\Sigma(\Omega, \mathbb{R}^3) \oplus \ker(\text{curl}). \tag{3.1}$$

Furthermore, $L^2_\sigma(\Omega, \mathbb{R}^3) \subset \{\text{curl } \psi : \psi \in W^{1,2}(\Omega, \mathbb{R}^3)\}$ (see [51, Appendix I, Proposition 1.3]). We record the following simple observation.

Proposition 3.4. *Suppose $b \in L^2_\sigma(\Omega, \mathbb{R}^3)$. Then the value $\int_\Omega \psi(x) \cdot b(x) dx$ is independent of the solution $\psi \in W^{1,2}(\Omega, \mathbb{R}^3)$ of $\text{curl } \psi = b$ if and only if $b \in L^2_\Sigma(\Omega, \mathbb{R}^3)$.*

Proof. If $b \in L^2_\sigma(\Omega, \mathbb{R}^3)$ and $\int_\Omega \phi(x) \cdot b(x) dx = 0$ for all $\phi \in W^{1,2}(\Omega, \mathbb{R}^3)$ with $\text{curl } \phi = 0$, then in particular $\int_\Omega h_i(x) \cdot b(x) dx = 0$ for all $i \in \{1, \dots, N\}$, giving $b \in L^2_\Sigma(\Omega, \mathbb{R}^3)$. The converse follows immediately from (3.1). \square

Consequently, magnetic helicity is independent of the vector potential for every $b(\cdot, t) \in L^2_\sigma(\Omega, \mathbb{R}^3)$ precisely when $L^2_H(\Omega, \mathbb{R}^3) = \{0\}$. In Proposition 3.5 this helps us to characterise, in multiply connected domains, those magnetic fields whose magnetic helicity is conserved for every vector potential.

Proposition 3.5. *Suppose the mappings $u_j, b_j, u_{j,0}, b_{j,0}, u, b, u_0, b_0$ satisfy the assumptions of Theorem 1.5. Then the following conditions are equivalent.*

- (i) $\int_\Omega \psi(x, t) \cdot b(x, t) dx$ is a.e. constant for every $\psi \in L^\infty(0, T; W^{1,2}(\Omega, \mathbb{R}^3))$ with $\text{curl } \psi = b$.
- (ii) $b_H = 0$.
- (iii) $b_{0,H} = 0$.

If Ω is multiply connected, there exist $u_j, b_j, u_{j,0}, b_{j,0}, u, b, u_0, b_0$ such that (i)–(iii) are not satisfied.

Proof. The equivalence (i) \Leftrightarrow (ii) is a direct corollary of Theorem 1.5 and Proposition 3.4, and the equivalence (ii) \Leftrightarrow (iii) follows immediately from Lemma 4.3. The last claim follows by combining Lemma 4.3 and Theorem 2.14. \square

Proposition 3.5 indicates that in multiply connected domains, magnetic helicity conservation can only hold in the weak ideal limit if some restrictions are imposed on the vector potential.

3.2. Good vector potentials. As stated in Corollary 1.6, a condition that allows magnetic helicity conservation in multiply connected domains is given by

$$\psi^\Sigma \times n|_\Gamma = 0 \quad \text{and} \quad \psi_0^\Sigma \times n|_\Gamma = 0. \tag{3.2}$$

We will, in fact, obtain Theorem 1.5 as a consequence of the fact that (3.2) leads to magnetic helicity conservation. For more information on condition (3.2) see e.g. [3,36,56].

Our next aim is to specify vector potentials that satisfy (3.2). For the L^2_Σ part of the magnetic field we essentially use vector potentials found by Borchers and Sohr in [6, Corollary 2.2]. The boundary condition $\langle \partial_n(\text{div } \psi), 1 \rangle_{\Gamma_i} = 0$, added by Amrouche, Bernardi, Dauge, and Girault in [3], ensures uniqueness. Theorem 3.6 follows from [3, Corollary 3.19] and [3, Theorem 3.20].

Theorem 3.6. *For every $v \in L^2_\Sigma(\Omega, \mathbb{R}^3)$ there exists a unique $T_\Sigma v := \Psi^\Sigma \in W^{1,2}_0(\Omega, \mathbb{R}^3)$ such that*

$$\operatorname{curl} \Psi^\Sigma = v, \quad \operatorname{div}(\Delta \Psi^\Sigma) = 0, \quad \langle \partial_n(\operatorname{div} \Psi^\Sigma), 1 \rangle_{\Gamma_i} = 0 \quad (i = 1, \dots, K).$$

Furthermore, $T_\Sigma : L^2_\Sigma(\Omega, \mathbb{R}^3) \rightarrow W^{1,2}_0(\Omega, \mathbb{R}^3)$ is linear and bounded.

For the space $L^2_H(\Omega, \mathbb{R}^3)$ a natural choice of potentials is a special case of [3, Theorem 3.12] and [3, Corollary 3.16]:

Theorem 3.7. *For every $v \in L^2_H(\Omega, \mathbb{R}^3)$ there exists a unique $T_H v := \psi^H \in W^{1,2}(\Omega, \mathbb{R}^3) \cap L^2_\Sigma(\Omega, \mathbb{R}^3)$ such that $\operatorname{curl}(\psi^H) = v$. Furthermore, $T_H : L^2_H(\Omega, \mathbb{R}^3) \rightarrow W^{1,2}(\Omega, \mathbb{R}^3)$ is linear and bounded.*

We use Theorems 3.6 and 3.7 to record an existence theorem about vector potentials satisfying (3.2).

Corollary 3.8. *For every $v \in L^\infty(0, T; L^2_\sigma(\Omega, \mathbb{R}^3))$, the mappings $\Psi^\Sigma(x, t) := T_\Sigma v_\Sigma(x, t)$ and $\psi^H(x, t) := T_H v_H(x, t)$ belong to $L^\infty(0, T; W^{1,2}(\Omega, \mathbb{R}^3))$ and satisfy $\operatorname{curl} \Psi^\Sigma = v_\Sigma$, $\operatorname{curl} \psi^H = v_H$ and (3.2).*

Given $v \in L^\infty(0, T; L^2_\sigma(\Omega, \mathbb{R}^3))$, the time-dependent mappings $T_\Sigma v_\Sigma$ and $T_H v_H$ are strongly measurable, which follows from the fact that $T_\Sigma \circ P_\Sigma$ and $T_H \circ P_H$ are bounded linear operators from $L^2_\sigma(\Omega, \mathbb{R}^3)$ into $W^{1,2}(\Omega, \mathbb{R}^3)$.

Remark 3.9. Another choice of vector potentials that satisfies (3.2) (a special case of the *Coulomb gauge*) is given in [3, Theorem 3.17] and [3, Corollary 3.19]: for every $v \in L^2_\Sigma(\Omega, \mathbb{R}^3)$ there exists a unique $\phi \in W^{1,2}(\Omega, \mathbb{R}^3)$ with $\operatorname{curl} \phi = v$, $\operatorname{div} \phi = 0$, $\phi \times n = 0$ on Γ and $\langle \phi \cdot n, 1 \rangle_{\Gamma_i} = 0$ for all $i \in \{1, \dots, N\}$ - the condition $\psi^\Sigma \cdot n|_\Gamma = 0$ is thus traded for $\operatorname{div} \phi = 0$.

We will also need a scalar potential for time-dependent curl-free L^p vector fields in simply connected domains.

Lemma 3.10. *Let $1 \leq p \leq \infty$ and $1 < q < \infty$, and suppose $\Omega' \subset \Omega$ is a simply connected domain with smooth boundary. If $v \in L^p(0, T; L^q(\Omega', \mathbb{R}^3))$ is curl-free, then there exists a unique $g \in L^p(0, T; W^{1,q}(\Omega'))$ such that $v = \nabla g$ and $\int_{\Omega'} g(x, t) dx = 0$. Furthermore, $\|g(\cdot, t)\|_{W^{1,q}(\Omega')} \lesssim_{\Omega', q} \|v(\cdot, t)\|_{L^q(\Omega')}$ for a.e. $t \in (0, T)$.*

Proof. For the existence of $g(\cdot, t)$ a.e. $t \in (0, T)$ see e.g. [29, Lemma III.1.1]. Given $t \in (0, T)$ the uniqueness of $g(\cdot, t)$ is clear and the norm estimate follows from the Poincaré inequality. Since $v(\cdot, t) \mapsto g(\cdot, t)$ is linear, g is strongly measurable. \square

4. The proof of Theorems 1.2 and 1.5

Theorem 1.5 is proved in this section, and Theorem 1.2 is obtained as a special case. We begin by motivating our decomposition of vector potentials.

4.1. The decomposition of vector potentials. Given a weak ideal or non-resistive limit (u, b) , our aim is to compute $\int_{\Omega} \psi(x, t) \cdot b(x, t) dx$ at a.e. $t \in (0, T)$ for every vector potential $\psi \in L^\infty(0, T; W^{1,2}(\Omega, \mathbb{R}^3))$ of b . However, we do not assume that (u, b) satisfies the ideal MHD equations and so no neat formula for $\int_{\Omega} \psi(x, t) \cdot b(x, t) dx$ is readily available. We therefore wish to relate $\int_{\Omega} \psi(x, t) \cdot b(x, t) dx$ to $\int_{\Omega} \psi_j(x, t) \cdot b_j(x, t) dx$ and compute the latter for every ψ_j .

A natural idea for computing $\int_{\Omega} \psi_j(x, t) \cdot b_j(x, t) dx$ (which works without major complications in simply connected domains) is to write

$$\int_{\Omega} \psi_j(x, t) \cdot b_j(x, t) dx = \int_0^t \int_{\Omega} [\partial_\tau \psi_j(x, \tau) \cdot b_j(x, \tau) + \psi_j(x, \tau) \cdot \partial_\tau b_j(x, \tau)] dx d\tau + \int_{\Omega} \psi_j(x, 0) \cdot b_j(x, 0) dx$$

and use the induction equation

$$\partial_t b_j + \text{curl}(b_j \times u_j) + \mu_j \text{curl curl } b_j = 0 \tag{4.1}$$

on $\partial_\tau \psi_j$ and $\partial_\tau b_j$. In the multiply connected case, however, (4.1) leads (formally) to $\partial_t \psi_j = -b_j \times u_j - \mu_j \text{curl } b_j + \sum_{i=1}^N d_i(t) h_i + \nabla g$, where $\{h_1, \dots, h_N\}$ is an orthonormal basis of $L^2_H(\Omega, \mathbb{R}^3)$, and the product $\sum_{i=1}^N d_i(t) h_i \cdot b_j$ seems very difficult to control. We therefore maneuver carefully in the proof of Lemma 4.5 to make sure that *we do not multiply $\partial_t \psi_j$ and $b_{j,H}$ at any point of the argument.*

The considerations above prompt us to decompose ψ_j and take advantage of the differences between $b_{j,\Sigma}$ and $b_{j,H}$. Using the notation of Corollary 3.8, we write

$$\psi_j = (\psi_j - \Psi_j^\Sigma - \psi_j^H) + \Psi_j^\Sigma + \psi_j^H. \tag{4.2}$$

In (4.2), $\partial_t \psi_j^H = 0$, while the condition $\Psi_j^\Sigma \times n|_\Gamma = 0$ ensures that many natural integrations by parts do not create unwanted boundary terms. These properties play a key role in the proof of Lemma 4.5. Finally, the 'bad part' $\psi_j - \Psi_j^\Sigma - \psi_j^H$ is curl-free and, consequently, $\int_{\Omega} (\psi_j(x, t) - \Psi_j^\Sigma(x, t) - \psi_j^H(x)) \cdot b_j(x, t) dx$ can be given a simple formula.

Furthermore, while ψ_j need not converge to ψ in any useful sense, the 'good parts' $\Psi_j^\Sigma + \psi_j^H$ of the potentials satisfy

$$\Psi_j^\Sigma + \psi_j^H \rightarrow \Psi^\Sigma + \psi^H \quad \text{in } L^2_{loc}(0, T; L^2(\Omega, \mathbb{R}^3)) \tag{4.3}$$

(see Lemma 4.4). The basic reason is that Ψ_j^Σ and ψ_j^H depend linearly on b_j , which allows us to exploit the weak-* convergence $b_j \xrightarrow{*} b$ in $L^\infty(0, T; L^2_\sigma(\Omega, \mathbb{R}^3))$.

4.2. An overview of the proof. The proof of Theorem 1.5 is reduced to the special case of $\psi_j = \Psi_j^\Sigma + \psi_j^H$ and $\psi = \Psi^\Sigma + \psi^H$ in Lemma 4.2. We therefore introduce a shorthand notation for magnetic helicity in this gauge.

Definition 4.1. Given $v \in L^\infty(0, T; L^2_\sigma(\Omega, \mathbb{R}^3))$ and $\Psi^\Sigma := T_\Sigma v$, $\psi^H := T_H v$ we denote

$$\mathcal{M}(v; t) := \int_{\Omega} (\Psi^\Sigma(x, t) + \psi^H(x, t)) \cdot v(x, t) dx.$$

Our aim is to show that

$$\mathcal{M}(b_j; t) = \mathcal{M}(b_j; 0) - 2\mu_j \int_0^t \int_{\Omega} b_j(x, \tau) \cdot \operatorname{curl} b_j(x, \tau) \, dx \, d\tau \quad (4.4)$$

for every $j \in \mathbb{N}$ and $t \in [0, T]$ and that given $\eta \in C_c^\infty(0, T)$,

$$\int_0^T \eta(t) \mathcal{M}(b; t) \, dt = \lim_{j \rightarrow \infty} \int_0^T \eta(t) \mathcal{M}(b_j; t) \, dt = \lim_{j \rightarrow \infty} \int_0^T \eta(t) \mathcal{M}(b_j; 0) \, dt. \quad (4.5)$$

Once (4.5) is proved, (1.11) follows for $\psi = \Psi^\Sigma + \psi^H$ rather easily (see Lemma 4.7).

The leftmost equality in (4.5) is proved by showing (4.3) and recalling that $b_j \rightharpoonup b$ in $L^2(0, T; L^2(\Omega, \mathbb{R}^3))$. The proof of (4.3) uses the Aubin–Lions Lemma as a main tool and is presented in Sect. 4.5. The rightmost equality of (4.5) is proved in Sect. 4.6 by showing that the double integral on the right-hand side of (4.4) vanishes at the limit $j \rightarrow \infty$.

We finally mention that in the proof of Theorem 1.5 we will on several occasions pass to a subsequence without relabeling it. The limit (1.11) will however hold for the whole sequence $(b_j)_{j=1}^\infty$ as every subsequence will have a subsequence satisfying (1.11).

4.3. Reduction to good vector potentials. The following lemma shows that it suffices to prove the claims of Theorem 1.5 and Corollary 1.6 for the potentials of Definition 4.1. It also indicates to what extent gauge invariance of magnetic helicity fails in multiply connected domains.

Lemma 4.2. *Suppose $\psi = \psi^\Sigma + \psi^H \in L^\infty(0, T; W^{1,2}(\Omega, \mathbb{R}^3))$ is a vector potential of $v \in L^\infty(0, T; L^2_\sigma(\Omega, \mathbb{R}^3))$. Then*

$$\int_{\Omega} \psi(x, t) \cdot v(x, t) \, dx = \mathcal{M}(v; t) - \int_{\Gamma} \psi^\Sigma(x, t) \times n \cdot \psi^H(x) \, dx$$

at a.e. $t \in (0, T)$.

Proof. By using the definition of $\mathcal{M}(v; t)$ and the facts that $\psi^\Sigma - \Psi^\Sigma \in \ker \operatorname{curl} = L^2_\Sigma(\Omega, \mathbb{R}^3)^\perp$ and $\Psi^\Sigma \times n|_\Gamma = 0$ we obtain

$$\begin{aligned} \int_{\Omega} \psi(x, t) \cdot v(x, t) \, dx &= \mathcal{M}(v; t) + \int_{\Omega} (\psi^\Sigma(x, t) - \Psi^\Sigma(x, t)) \cdot v_H(x, t) \, dS(x) \\ &= \mathcal{M}(v; t) + \int_{\Gamma} \psi^H(x) \times n \cdot (\psi^\Sigma(x, t) - \Psi^\Sigma) \, dS(x) \\ &= \mathcal{M}(v; t) - \int_{\Gamma} \psi^\Sigma(x, t) \times n \cdot \psi^H(x) \, dx \end{aligned}$$

at a.e. $t \in (0, T)$. \square

4.4. *Stationarity of the harmonic parts of magnetic fields.* We next show that for Leray–Hopf solutions and their weak ideal limits, the harmonic part of the magnetic field is stationary.

Lemma 4.3. *Under the assumptions of Theorem 1.5, for every $j \in \mathbb{N}$ the harmonic parts $b_{j,H}$ and b_H are of the forms $b_{j,H}(x, t) = b_{j,0,H}(x)$ and $b_H(x, t) = b_{0,H}(x) = \lim_{j \rightarrow \infty} b_{j,0,H}(x)$.*

Proof. Given $j \in \mathbb{N}$ we write $b_{j,H}(x, t) = \sum_{i=1}^N c_i^j(t)h_i(x)$ and set out to prove that c_1^j, \dots, c_N^j are constants independent of t . We fix $i \in \{1, \dots, N\}$ and first show that c_i^j is continuous. Since b_j is weakly L^2 -continuous in time, we get

$$c_i^j(t_k) = \int_{\Omega} b_j(x, t_k) \cdot h_i(x) \, dx \rightarrow \int_{\Omega} b_j(x, t) \cdot h_i(x) \, dx = c_i^j(t)$$

whenever $t_k \rightarrow t$ in $[0, T)$. Now let $\eta \in C_c^\infty(0, T)$ and set $\theta(x, t) := \eta(t)h_i(x)$ in (2.5), getting $\int_0^T \eta'(t)c_i^j(t) \, dt = 0$, which implies that $c_i^j(t) = c_i^j(0)$ for all $t \in [0, T)$. Fixing $i \in \{1, \dots, N\}$ and $\eta \in C_c^\infty(0, T)$ we get

$$\begin{aligned} c_i^j(0) \int_0^T \eta(t) \, dt &= \int_0^T \int_{\Omega} \eta(t)h_i(x) \cdot b_j(x, t) \, dx \, dt \\ &\rightarrow \int_0^T \int_{\Omega} \eta(t)h_i(x) \cdot b(x, t) \, dx \, dt, \end{aligned}$$

which yields the statement on b_H . \square

4.5. *Strong convergence of good vector potentials.* The aim of this section is to prove (4.3) via the Aubin–Lions Lemma. This requires uniform control of the norms $\|\partial_t \psi_j\|_{L^1(0,T;X)}$ in some (reflexive) Banach space $X \supset L^2(\Omega, \mathbb{R}^3)$. Note that since $\partial_t \psi_j^H = 0$, (1.2) yields

$$\operatorname{curl}(\partial_t \psi_j^\Sigma + b_j \times u_j + \mu_j \operatorname{curl} b_j) = 0. \tag{4.6}$$

If $\Omega' \subset \Omega$ is a simply connected subdomain, we can thus write $\partial_t \psi_j^\Sigma + b_j \times u_j + \mu_j \operatorname{curl} b_j = \nabla g$ in Ω' . It is, however, not immediately clear how well-behaved $\partial_t \psi_j^\Sigma$ (and, thus, g) is. In order to circumvent this issue we mollify in time via the functions $t \mapsto \chi^\delta(t)$ mentioned in Sect. 2.3 and write $\partial_t(\psi_j^\Sigma * \chi^\delta) = \psi_j^\Sigma * \partial_t \chi^\delta$.

Lemma 4.4. *The vector potentials $\Psi_j^\Sigma + \psi_j^H$ converge in $L^2_{loc}(0, T; L^2(\Omega, \mathbb{R}^3))$ to the vector potential $\Psi^\Sigma + \psi^H$ of b .*

Proof. Lemma 4.3 implies that ψ_j^H converges to ψ^H in $L^2(0, T; L^2(\Omega, \mathbb{R}^3))$. The more elaborate part is the strong convergence of the potentials Ψ_j^Σ .

We fix a sequence of numbers $\epsilon_j \in (0, T/2)$. We then choose another sequence of numbers $\delta_j \in (0, \epsilon_j)$ such that $\|\Psi_j^\Sigma * \chi^{\delta_j} - \Psi_j^\Sigma\|_{L^2(\epsilon_j, T-\epsilon_j)} < 1/j$ for all $j \in \mathbb{N}$, so that it suffices to prove the convergence of the sequence $(\Psi_j^\Sigma * \chi^{\delta_j})_{j=1}^\infty$ in

$L^2_{loc}(0, T; L^2(\Omega, \mathbb{R}^3))$. We fix $\epsilon > 0$ and a non-empty simply connected, smooth sub-domain $\Omega' \subset \Omega$ and aim to show that

$$\sup_{j \in \mathbb{N}} (\|\Psi_j^\Sigma * \chi^{\delta_j}\|_{L^2(\epsilon, T-\epsilon; W^{1,2}(\Omega, \mathbb{R}^3))} + \|\partial_t \Psi_j^\Sigma * \chi^{\delta_j}\|_{L^2(\epsilon, T-\epsilon; (W^{1,4}_{0,\sigma}(\Omega', \mathbb{R}^3))^*)}) < \infty; \tag{4.7}$$

the Aubin–Lions Lemma then implies norm convergence of a subsequence of $(\Psi_j^\Sigma)_{j=1}^\infty$ to some $\tilde{\psi} \in L^2(\epsilon, T - \epsilon; L^2(\Omega))$. On the other hand, for a further subsequence, $\Psi_j^\Sigma = T_\Sigma b_{j,\Sigma} \rightharpoonup T_\Sigma b_\Sigma$ in $L^2(\epsilon, T - \epsilon; W^{1,2}(\Omega, \mathbb{R}^3))$, and thus $\tilde{\psi} = T_\Sigma b_\Sigma =: \Psi^\Sigma$.

For (4.7) we fix $j \in \mathbb{N}$ and use the fact that by Lemma 3.6, Theorem 2.14 and Lemma 2.9,

$$\begin{aligned} \|\Psi_j^\Sigma * \chi^{\delta_j}\|_{L^2(\epsilon, T-\epsilon; W^{1,2}(\Omega, \mathbb{R}^3))} &\lesssim \|b_{j,\Sigma} * \chi^{\delta_j}\|_{L^2(\epsilon, T-\epsilon; L^2(\Omega))} \leq \|b_{j,\Sigma}\|_{L^2(0, T; L^2(\Omega))} \\ &\leq \|b_j\|_{L^2(0, T; L^2(\Omega))} \leq \sqrt{T(\|b_{j,0}\|_{L^2(\Omega)}^2 + \|u_{j,0}\|_{L^2(\Omega)}^2)}. \end{aligned}$$

In order to control the norm of $\partial_t(\Psi_j^\Sigma * \chi^{\delta_j})$ we note that $\partial_t(\Psi_j * \chi^{\delta_j}) = \Psi_j^\Sigma * \partial_t \chi^{\delta_j} \in L^\infty(\epsilon, T - \epsilon; W^{1,2}_0(\Omega'))$, and so Lemma 3.10 and (4.6) yield

$$\Psi_j^\Sigma * \partial_t \chi^{\delta_j} + (b_j \times u_j) * \chi^{\delta_j} - \mu_j \operatorname{curl} b_j * \chi^{\delta_j} = \nabla g_j, \tag{4.8}$$

where $g_j \in L^\infty(\epsilon, T - \epsilon; W^{1,2}(\Omega'))$. We estimate, at every $t \in (\epsilon, T - \epsilon)$,

$$\begin{aligned} \left| \int_{\Omega'} \partial_t [\Psi_j^\Sigma * \chi^{\delta_j}](x, t) \cdot v(x) \, dx \right| &\leq \|b_j \times u_j * \chi^{\delta_j}(\cdot, t)\|_{L^1(\Omega)} \|v\|_{L^\infty(\Omega)} \\ &\quad + \mu_j \|\operatorname{curl} b_j * \chi^{\delta_j}(\cdot, t)\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \end{aligned}$$

for all $v \in W^{1,4}_{0,\sigma}(\Omega', \mathbb{R}^3)$ so that, by Lemma 2.9,

$$\begin{aligned} \|\partial_t \Psi_j^\Sigma * \chi^{\delta_j}\|_{L^2(\epsilon, T-\epsilon; (W^{1,4}_{0,\sigma}(\Omega'))^*)} &\lesssim_\Omega \|b_j \times u_j * \chi^{\delta_j}\|_{L^2(\epsilon, T-\epsilon; L^1(\Omega))} \\ &\quad + \mu_j \|\operatorname{curl} b_j * \chi^{\delta_j}\|_{L^2(\epsilon, T-\epsilon; L^2(\Omega))} \\ &\leq \|b_j \times u_j\|_{L^2(0, T; L^1(\Omega))} + \mu_j \|\operatorname{curl} b_j\|_{L^2(0, T; L^2(\Omega))} \end{aligned}$$

which yields (4.7). \square

4.6. Completion of the proof. The proof of Theorem 1.5 will be finished in the following two lemmas by showing (4.4) and controlling the size of the integral term in (4.4). A third lemma then proves Corollary 1.6.

Lemma 4.5. *For every $j \in \mathbb{N}$ and every $t \in [0, T)$ we have*

$$\mathcal{M}(b_j; t) = \mathcal{M}(b_j; 0) - 2\mu_j \int_0^t \int_\Omega b_j(x, \tau) \cdot \operatorname{curl} b_j(x, \tau) \, dx \, d\tau.$$

Proof. We intend to show that

$$\partial_t \mathcal{M}(b_j; t) = -2\mu_j \int_{\Omega} b_j(x, t) \cdot \operatorname{curl} b_j(x, t) \, dx \tag{4.9}$$

in the sense of distributions; the claim then follows since the Cauchy–Schwarz inequality and the energy inequality give $\partial_t \mathcal{M}(b_j; \cdot) \in L^1(0, T)$.

Let $\eta \in C_c^\infty(0, T)$ and note that $\int_0^T \eta(t) \mathcal{M}(b_j; t) \, dt = \lim_{\delta \rightarrow 0} \int_0^T \eta(t) \mathcal{M}(b_{j,\delta}; t) \, dt$ by Lemma 2.10. Fix $\delta \in (0, T/2)$ such that $\operatorname{supp}(\eta) \subset [2\delta, T - 2\delta]$. Then, integrating by parts several times and using the facts that $\Psi_{j,\delta}^\Sigma \times n|_\Gamma = 0$ and $\partial_t \Psi_j^H = \partial_t b_{j,H} = 0$ we get

$$\begin{aligned} \int_0^T \partial_t \eta(t) \mathcal{M}(b_{j,\delta}; t) \, dt &= \int_0^T \partial_t \eta(t) \int_{\Omega} (\Psi_j^\Sigma + \psi_j^H)_\delta(x, t) \cdot (b_{j,\Sigma} + b_{j,H})_\delta(x, t) \, dx \, dt \\ &= -2 \int_0^T \eta(t) \int_{\Omega} \partial_t \Psi_{j,\delta}^\Sigma(x, t) \cdot b_{j,\Sigma,\delta}(x, t) \, dx \, dt \\ &\quad + 2 \int_0^T \partial_t \eta_\delta(t) \int_{\Omega} \psi_j^H(x) \cdot b_{j,\Sigma}(x, t) \, dx \, dt =: I_1 + I_2. \end{aligned}$$

For I_1 we note that $\partial_t \Psi_{j,\delta}^\Sigma + (b_j \times u_j)_\delta + \mu_j \operatorname{curl} b_{j,\delta} \in L^\infty(\delta, T - \delta; \ker(\operatorname{curl})) = L^\infty(\delta, T - \delta; (L^2_\Sigma(\Omega, \mathbb{R}^3))^\perp)$ so that

$$I_1 = 2 \int_0^T \eta(t) \int_{\Omega} [(b_j \times u_j)_\delta(x, t) + \mu_j \operatorname{curl} b_{j,\delta}(x, t)] \cdot (b_{j,\delta}^\Sigma(x, t)) \, dx \, dt.$$

For I_2 we note that since $\partial_t(\psi_j^H \cdot b_{j,H}) = 0$, we get

$$I_2 = 2 \int_0^T \partial_t \eta_\delta(t) \int_{\Omega} \psi_j^H(x) \cdot b_j(x, t) \, dx \, dt.$$

By setting $\theta(x, t) := 2\eta_\delta(t)\psi_j^H(x)$ in (2.5) we obtain

$$\begin{aligned} I_2 &= 2 \int_0^T \eta_\delta(t) \int_{\Omega} \operatorname{curl} \psi_j^H(x) \cdot [b_j \times u_j(x, t) + \mu_j \operatorname{curl} b_j(x, t)] \, dx \, dt \\ &= 2 \int_0^T \eta(t) \int_{\Omega} b_j^H(x) \cdot [(b_j \times u_j)_\delta(x, t) + \mu_j \operatorname{curl} b_{j,\delta}(x, t)] \, dx \, dt. \end{aligned}$$

Collecting the identities, taking the limit $\delta \rightarrow 0$ (via Lemma 2.11) and using the pointwise identity $b_j \cdot b_j \times u_j = 0$ we conclude that

$$\int_0^T \partial_t \eta(t) \mathcal{M}(b_j; t) \, dt = 2\mu_j \int_0^T \eta(t) \int_{\Omega} b_j(x, t) \cdot \operatorname{curl} b_j(x, t) \, dx,$$

which yields (4.9). \square

The following estimate, which goes back to [4], completes the proof of Theorem 1.5.

Lemma 4.6. *For every $j \in \mathbb{N}$,*

$$\mu_j \int_0^T \int_{\Omega} |b_j(x, t) \cdot \operatorname{curl} b_j(x, t)| \, dx \, dt \lesssim_T \sqrt{\mu_j} (\|u_{j,0}\|_{L^2}^2 + \|b_{j,0}\|_{L^2}^2).$$

Proof. By Young’s inequality and the energy inequality,

$$\begin{aligned} & \mu_j \int_0^T \int_{\Omega} |b_j(x, t) \cdot \operatorname{curl} b_j(x, t)| \, dx \, dt \\ & \leq \frac{\sqrt{\mu_j}}{2} \int_0^T \int_{\Omega} (|b_j(x, t)|^2 + \mu_j |\operatorname{curl} b_j(x, t)|^2) \, dx \, dt \\ & \lesssim_T \sqrt{\mu_j} (\|u_{j,0}\|_{L^2}^2 + \|b_{j,0}\|_{L^2}^2). \end{aligned}$$

□

A simple lemma gives the rightmost equality in (1.13).

Lemma 4.7. $\lim_{j \rightarrow \infty} \int_{\Omega} (\Psi_{j,0}^{\Sigma}(x) + \psi_{j,0}^H(x)) \cdot b_{j,0}(x) \, dx = \int_{\Omega} (\Psi_{j,0}^{\Sigma}(x) + \psi_{j,0}^H(x)) \cdot b_0(x) \, dx.$

Proof. By assumption, $b_{j,0} \rightharpoonup b_0$ in $L^2_{\sigma}(\Omega, \mathbb{R}^3)$, and therefore $\Psi_{j,0}^{\Sigma} + \psi_{j,0}^H = T_{\Sigma} b_{j,0, \Sigma} + T_H b_{j,0, H} \rightharpoonup T_{\Sigma} b_{0, \Sigma} + T_H b_{0, H} = \Psi_0^{\Sigma} + \psi_0^H$ in $W^{1,2}(\Omega, \mathbb{R}^3)$. The Rellich–Kondrachov Theorem then yields $\|(\Psi_{j,0}^{\Sigma} + \psi_{j,0}^H) - (\Psi_0^{\Sigma} + \psi_0^H)\|_{L^2(\Omega)} \rightarrow 0$, which implies the claim. □

5. A Two-Dimensional Analogue

Magnetic helicity has a two-dimensional counterpart, the mean-square magnetic potential. It is defined as the L^2 energy of the canonical stream function of b , and it is conserved in time by smooth solutions of ideal 2D MHD. In Sect. 5.1 we define the mean-square magnetic potential in multiply connected domains and formulate Theorem 5.4 which says that it is also conserved in the weak ideal limit. As a byproduct, we prove that if a weak solution of 2D ideal MHD lies in the energy space, then it conserves mean-square magnetic potential in time. The proof of Theorem 5.4 is presented in Sect. 5.2. As main tools, apart from ones already used in 3D, we use C. Fefferman’s \mathcal{H}^1 –BMO duality theorem from [23] and the Hardy space theory of compensated compactness quantities of Coifman, Lions, Meyer and Semmes from [14].

5.1. Mean-square magnetic potential and statement of the theorem. In two dimensions, the viscous, resistive MHD equations are given by

$$\partial_t u + (u \cdot \nabla)u - (b \cdot \nabla)b - \nu \Delta u + \nabla \Pi = 0, \tag{5.1}$$

$$\partial_t b - \nabla^{\perp}(b \times u) - \mu \nabla^{\perp}(\operatorname{curl} b) = 0, \tag{5.2}$$

$$\operatorname{div} u = \operatorname{div} b = 0, \tag{5.3}$$

$$u(\cdot, 0) = u_0, \quad b(\cdot, 0) = b_0, \tag{5.4}$$

where $\nabla^{\perp} = (-\partial_2, \partial_1)$ and $\operatorname{curl} = \nabla^{\perp} \cdot$. We now record our assumptions on the domain; we weaken the regularity condition that we placed on the boundary in three dimensions. Assumption 5.1 is strong enough to ensure the existence of a canonical stream function for every vector field in $L^2_{\sigma}(\Omega, \mathbb{R}^2)$ (see Theorem 5.2).

Assumption 5.1. The set $\Omega \subset \mathbb{R}^2$ is open and bounded. Furthermore, Ω is connected and its boundary Γ is Lipschitz-continuous and has a finite number of connected components denoted by $\Gamma_1, \dots, \Gamma_K$.

The boundary conditions corresponding to (1.5)–(1.6) are

$$u|_{\Gamma} = 0, \tag{5.5}$$

$$b \cdot n|_{\Gamma} = 0 \quad \text{and} \quad \text{curl } b|_{\Gamma} = 0. \tag{5.6}$$

Equations (5.1)–(5.6) are understood in analogy to the 3D case, but (2.4) needs to be replaced by the formula

$$\langle \partial_t b, \theta \rangle + \int_{\Omega} b \times u \text{curl } \theta + \mu \int_{\Omega} \text{curl } b \text{curl } \theta = 0. \tag{5.7}$$

We enumerate $\Gamma_1, \dots, \Gamma_K$ in such a way that Γ_1 is the boundary of the unbounded component of $\mathbb{R}^2 \setminus \overline{\Omega}$. Following [31, p. 40] we denote

$$\Phi := \{\psi \in W^{1,2}(\Omega) : \psi|_{\Gamma_1} = 0, \psi|_{\Gamma_i} \text{ is constant for } 2 \leq i \leq K\};$$

note that if Γ is connected, then $\Phi = W_0^{1,2}(\Omega)$. The following theorem gives a canonical choice of stream functions (see [31, Corollary I.3.1]).

Theorem 5.2. *The mapping $-\nabla^{\perp} : \Phi \rightarrow L^2_{\sigma}(\Omega, \mathbb{R}^2)$ is an isomorphism.*

When $v \in L^2_{\sigma}(\Omega, \mathbb{R}^2)$, we call $(-\nabla^{\perp})^{-1}v \in \Phi$ the *stream function of v* . Leray–Hopf solutions are defined in direct analogy to Definition 2.13.

Definition 5.3. Suppose (u, b) is a Leray–Hopf solution of (5.1)–(5.6) and $\psi \in C_w([0, T]; \Phi)$ is the stream function of b . For every $t \in [0, T]$, $\int_{\Omega} |\psi(x, t)|^2 dx$ is called the *mean-square magnetic potential of b at time t* .

We formulate an analogue of Theorem 1.5 for the mean-square magnetic potential, denoting the stream functions of the initial datas $b_{j,0}$ and b_0 by $\psi_{j,0}$ and ψ_0 . The weak ideal limit and weak non-resistive limit are defined in direct analogy to Definition 1.1.

Theorem 5.4. *Suppose $\Omega \subset \mathbb{R}^2$ satisfies Assumption 5.1, and assume that $u, b \in L^{\infty}(0, T; L^2_{\sigma}(\Omega, \mathbb{R}^3))$ are a weak ideal limit or weak non-resistive limit of Leray–Hopf solutions (u_j, b_j) , $j \in \mathbb{N}$. Then $b \in C_w([0, T]; L^2_{\sigma}(\Omega, \mathbb{R}^2))$, $\partial_t b - \nabla^{\perp}(b \times u) = 0$ with $b(\cdot, 0) = b_0$ and*

$$\int_{\Omega} |\psi(x, t)|^2 dx = \int_{\Omega} |\psi_0(x)|^2 dx = \lim_{j \rightarrow \infty} \int_{\Omega} |\psi_{j,0}(x)|^2 dx \tag{5.8}$$

for all $t \in [0, T]$.

Note that Theorem 5.4 is stronger than Theorem 1.5 in the sense that the induction equation $\partial_t b - \nabla^{\perp}(b \times u) = 0$ holds in the weak ideal limit. Theorem 5.4 is proved in the following subsection.

5.2. *Proof of Theorem 5.4.* Our first task is to prove that the induction equation $\partial_t b - \nabla^\perp(b \times u) = 0$ holds and $b(\cdot, 0) = b_0$. We begin the proof by showing a 2D analogue of Lemma 4.4.

Lemma 5.5. $\psi_j \rightarrow \psi$ in $L^2(0, T; L^2(\Omega))$.

Proof. Since $-\nabla^\perp \psi_j = b_j \rightharpoonup b = -\nabla^\perp \psi$ in $L^2(0, T; L^2_\sigma(\Omega, \mathbb{R}^2))$, we have $\psi^j \rightharpoonup \psi$ in $L^2(0, T; \Phi)$ by Theorem 5.2. Hence, it suffices, by the Aubin–Lions Lemma, to show that

$$\sup_{j \in \mathbb{N}} \|\partial_t \psi_j\|_{L^1(0, T; W^{-1,2}(\Omega))} < \infty. \tag{5.9}$$

We write $\partial_t \psi_j = -b_j \times u_j - \mu_j \operatorname{curl} b_j$ and estimate the terms separately. First, we set

$$B_j(x, t) := \begin{cases} b_j(x, t), & x \in \Omega, \\ 0, & x \notin \Omega, \end{cases} \quad U_j(x, t) := \begin{cases} u_j(x, t), & x \in \Omega, \\ 0, & x \notin \Omega \end{cases} \tag{5.10}$$

and note that $\operatorname{div} B_j = \operatorname{div} U_j = 0$ in $\mathbb{R}^2 \times (0, T)$. Fix $t \in (0, T)$ and $\varphi \in C_c^\infty(\Omega)$, and denote $\Phi(x) = \varphi(x)$ for $x \in \Omega$ and $\Phi(x) = 0$ for $x \notin \Omega$. Fefferman’s \mathcal{H}^1 -BMO duality theorem and the div-curl estimate of Coifman, Lions, Meyer and Semmes give

$$\begin{aligned} \int_\Omega b_j(x, t) \times u_j(x, t) \varphi(x) \, dx &= \int_{\mathbb{R}^2} B_j(x, t) \times U_j(x, t) \Phi(x) \, dx \\ &\lesssim \|B_j(\cdot, t) \times U_j(\cdot, t)\|_{\mathcal{H}^1} \|\Phi\|_{\text{BMO}} \\ &\lesssim \|B_j(\cdot, t)\|_{L^2} \|U_j(\cdot, t)\|_{L^2} \|\nabla \Phi\|_{L^2}, \end{aligned}$$

yielding, by the Cauchy–Schwartz inequality,

$$\sup_{j \in \mathbb{N}} \|b_j \times u_j\|_{L^1(0, T; W^{-1,2}(\Omega))} \lesssim \sup_{j \in \mathbb{N}} \|b_j\|_{L^2(0, T; L^2(\Omega))} \|u_j\|_{L^2(0, T; L^2(\Omega))} < \infty.$$

Furthermore, trivially, $\sup_{j \in \mathbb{N}} \|\mu_j \operatorname{curl} b_j\|_{L^1(0, T; W^{-1,2}(\Omega))} < \infty$, and so (5.9) holds. \square

We next show that the limit mappings u and b satisfy the ideal induction equation. Given $j \in \mathbb{N}$, the mappings u_j and b_j satisfy (5.1)–(5.6) and using standard arguments (see e.g. [28, Lemma 2.4]), (5.7) and the initial value condition $b_j(\cdot, 0) = b_{j,0}$ yield

$$\int_0^T \int_\Omega (b_j \cdot \partial_t \phi - b_j \times u_j \operatorname{curl} \phi - \mu \operatorname{curl} b_j \operatorname{curl} \phi) \, dx \, dt + \int_\Omega b_{j,0} \cdot \phi(\cdot, 0) \tag{5.11}$$

for every $\phi \in C_c^\infty(\Omega \times [0, T], \mathbb{R}^2)$ with $\operatorname{div} \phi = 0$.

Lemma 5.6. $\partial_t b - \nabla^\perp(b \times u) = 0$ with initial value $b(\cdot, 0) = b_0$.

Proof. Fix $\phi \in C_c^\infty(\Omega \times [0, T], \mathbb{R}^2)$ with $\operatorname{div} \phi = 0$. By using the formula $b_j \times u_j = \nabla \psi_j \cdot u_j$, Theorem 3.1 and Lemma 5.5,

$$\begin{aligned} & \int_0^T \int_\Omega b_j(x, t) \times u_j(x, t) \operatorname{curl} \phi(x, t) \, dx \, dt \\ &= \int_0^T \int_\Omega \nabla[\operatorname{curl} \phi(x, t) \psi_j(x, t)] \cdot u_j(x, t) \, dx \, dt \\ &\quad - \int_0^T \int_\Omega \psi_j(x, t) \nabla \operatorname{curl} \phi(x, t) \cdot u_j(x, t) \, dx \, dt \\ &\rightarrow - \int_0^T \int_\Omega \psi(x, t) \nabla \operatorname{curl} \phi(x, t) \cdot u(x, t) \, dx \, dt \\ &= \int_0^T \int_\Omega b(x, t) \times u(x, t) \operatorname{curl} \phi(x, t) \, dx \, dt. \end{aligned}$$

The claim now follows immediately by inspection of (5.11), since the energy inequality yields $\mu_j \int_0^T \int_\Omega \operatorname{curl} b_j(x, t) \operatorname{curl} \phi(x, t) \, dx \, dt \rightarrow 0$. \square

As in 3D, by adapting [28, Lemmas 2.1 and 2.2] we may assume that $b \in C_w([0, T]; L_\sigma^2(\Omega, \mathbb{R}^2))$. Thus the stream function ψ belongs to $C_w([0, T]; \Phi) \subset C([0, T]; L^2(\Omega))$.

Theorem 5.4 will be proved once we show (5.8). The right equality in (5.8) follows from the assumption $b_{j,0} \rightharpoonup b_0$ and the Rellich–Kondrachov Theorem. We next prove the left equality—in fact, we also prove that every weak solution of ideal MHD in the energy space conserves magnetic helicity in time.

Lemma 5.7. *Suppose $u \in L^\infty(0, T; L_\sigma^2(\Omega, \mathbb{R}^2))$ and $b \in C_w([0, T]; L_\sigma^2(\Omega, \mathbb{R}^2))$ satisfy $\partial_t b - \nabla^\perp(b \times u) = 0$ with initial value $b(\cdot, 0) = b_0 \in L_\sigma^2(\Omega, \mathbb{R}^2)$. Then b conserves mean square magnetic potential in time.*

Proof. Since $\psi \in C([0, T]; L^2(\Omega))$, it suffices to show that for every $\eta \in C_c^\infty(0, T)$ we have $\int_0^T \partial_t \eta(t) \int_\Omega |\psi(x, t)|^2 \, dx \, dt = 0$. Fix such an η and choose $\epsilon > 0$ such that $\operatorname{supp}(\eta) \subset [\epsilon, T - \epsilon]$. Whenever $0 < \delta < \epsilon$, we mollify in time and write $\psi_\delta := \psi * \chi_\delta$. By Lemma 2.10, $\int_0^T \partial_t \eta(t) \int_\Omega |\psi(x, t)|^2 \, dx \, dt = \lim_{\delta \rightarrow 0} \int_0^T \partial_t \eta(t) \int_\Omega |\psi_\delta(x, t)|^2 \, dx \, dt$.

Let now $0 < \delta < \epsilon$. The induction equation $\partial_t b - \nabla^\perp(b \times u) = 0$ and the assumptions about the boundary values of u, b and ψ imply that $\partial_t \psi + b \times u = 0$, and thus $(b \times u)_\delta = -\psi * \partial_t \chi_\delta \in L^\infty(\epsilon, T - \epsilon; W^{1,2}(\Omega))$, giving

$$\int_0^T \partial_t \eta(t) \int_\Omega |\psi_\delta(x, t)|^2 \, dx \, dt = 2 \int_0^T \eta(t) \int_\Omega \psi_\delta(x, t) (b \times u)_\delta(x, t) \, dx \, dt.$$

As in (5.10), we denote the zero extensions of b and u outside Ω by B and U . Likewise, for every $t \in [0, T]$, we denote by $\Psi(\cdot, t) \in W^{1,2}(\mathbb{R}^2)$ the unique compactly supported solution of $-\nabla^\perp \Psi(\cdot, t) = B(\cdot, t)$. Thus $\Psi \in L^\infty(0, T; W^{1,2}(\mathbb{R}^2)) \subset L^2(0, T; \operatorname{VMO}(\mathbb{R}^2))$ and $B \times U \in L^\infty(0, T; \mathcal{H}^1(\mathbb{R}^2)) \subset (L^2(0, T; \operatorname{VMO}(\mathbb{R}^2)))^*$. This allows us to write, using Lemma 2.10,

$$\begin{aligned} \int_0^T \eta(t) \int_\Omega \psi_\delta(x, t) (b \times u)_\delta(x, t) \, dx \, dt &= \int_0^T \eta(t) \int_{\mathbb{R}^2} \Psi_\delta(x, t) (B \times U)_\delta(x, t) \, dx \, dt \\ &\rightarrow \int_0^T \eta(t) \langle \Psi(\cdot, t), B \times U(\cdot, t) \rangle_{\operatorname{VMO}-\mathcal{H}^1} \, dt. \end{aligned}$$

We finally mollify Ψ and B in *space* and use the Dominated Convergence Theorem in time to conclude that

$$\begin{aligned} & \int_0^T \eta(t) \langle \Psi(\cdot, t), B \times U(\cdot, t) \rangle_{\text{VMO}-\mathcal{H}^1} dt \\ &= \lim_{\epsilon \rightarrow 0} \int_0^T \eta(t) \int_{\mathbb{R}^2} \Psi_\epsilon(x, t) B_\epsilon(x, t) \times U(x, t) \, dx \, dt \\ &= \frac{1}{2} \lim_{\epsilon \rightarrow 0} \int_0^T \eta(t) \int_{\mathbb{R}^2} \nabla |\Psi_\epsilon(x, t)|^2 \cdot U(x, t) \, dx \, dt = 0. \end{aligned}$$

□

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Appendix A. The Existence of Leray–Hopf Solutions in Multiply Connected Domains

We give a proof of the existence of Leray–Hopf solutions of (1.1)–(1.6), referring to the literature on some of the steps that are familiar from Navier–Stokes equations. A proof for simply connected domains is sketched in [48] and presented in more detail in [30]. As we cover multiply connected domains, more technicalities are needed although we follow the general scheme of the proof given in [30]. We reformulate Theorem 2.14 for the convenience of the reader.

Theorem A.1. *Suppose Ω satisfies Assumption 2.1 and let $u_0, b_0 \in L^2_\sigma(\Omega, \mathbb{R}^3)$. Then there exists a Leray–Hopf solution (u, b) of (1.1)–(1.6).*

The basic strategy of the proof, via finite-dimensional *Galerkin approximations*, is classical, but we discuss the main ideas. The solution is built via orthonormal bases $\{v_j\}_{j \in \mathbb{N}}$ and $\{w_j\}_{j \in \mathbb{N}}$ of $L^2_\sigma(\Omega, \mathbb{R}^3)$ satisfying the sought boundary conditions, that is, $v_j \in W^{1,2}_{0,\sigma}(\Omega, \mathbb{R}^3)$ and $w_j \in W^{1,2}(\Omega, \mathbb{R}^3)$ with $(\text{curl } w_j \times n)|_\Gamma = 0$.

Definition A.2. Suppose $u_0, b_0 \in L^2_\sigma(\Omega, \mathbb{R}^3)$ and let $n \in \mathbb{N}$. Mappings of the forms

$$u_n(x, t) = \sum_{j=1}^n c_{nj}(t)v_j(x), \quad b_n(x, t) = \sum_{j=1}^n d_{nj}(t)w_j(x), \tag{A.1}$$

where $c_{nj}, d_{nj} \in C^1([0, T])$, satisfy the *n*th order *Galerkin approximation* of (1.1)–(1.6) if

$$\begin{aligned} & \frac{d}{dt} (u_n, v_j)_{L^2} + \nu (\nabla u_n, \nabla v_j)_{L^2} + \langle (u_n \cdot \nabla)u_n - (b_n \cdot \nabla)b_n, v_j \rangle_{(W^{1,2}_{0,\sigma})^* - W^{1,2}_{0,\sigma}} = 0, \\ & \frac{d}{dt} (b_n, w_j)_{L^2} + \mu (\text{curl } b_n, \text{curl } w_j)_{L^2} + \langle \text{curl}(b_n \times u_n), w_j \rangle_{(W^{1,2}_\sigma)^* - W^{1,2}_\sigma} = 0, \\ & u_n(\cdot, 0) = P_n u_0, \quad b_n(\cdot, 0) = Q_n b_0 \end{aligned}$$

for all $j = 1, \dots, n$.

For every $n \in \mathbb{N}$, standard theory of ordinary differential equations gives a unique solution of the Galerkin approximation satisfying the energy equality

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} (|u_n(x, t)|^2 + |b_n(x, t)|^2) dx \\ & + \int_0^t \int_{\Omega} (v |\nabla u_n(x, \tau)|^2 + \mu |\operatorname{curl} b_n(x, \tau)|^2) dx d\tau \\ & = \frac{1}{2} \int_{\Omega} (|P_n u_0(x)|^2 + |Q_n b_0(x)|^2) dx \end{aligned} \tag{A.2}$$

for all $t \in (0, T)$ (see Lemma A.8). With some work, the energy equality allows us to subtract a subsequence with $u_n \rightharpoonup u$ in $L^2(0, T; W_{0,\sigma}^{1,2}(\Omega, \mathbb{R}^3))$ and $b_n \rightharpoonup b$ in $L^2(0, T; W_{\sigma}^{1,2}(\Omega, \mathbb{R}^3))$. Our goal is to show that (u, b) is a Leray–Hopf solution with initial data (u_0, b_0) .

For every $n \in \mathbb{N}$ we denote by P_n and Q_n the projections of $L_{\sigma}^2(\Omega, \mathbb{R}^3)$ onto $\operatorname{span}\{v_1, \dots, v_n\}$ and $\operatorname{span}\{w_1, \dots, w_n\}$. Note that $P_n : W_{0,\sigma}^{1,2}(\Omega, \mathbb{R}^3) \rightarrow W_{0,\sigma}^{1,2}(\Omega, \mathbb{R}^3)$ and $Q_n : W_{\sigma}^{1,2}(\Omega, \mathbb{R}^3) \rightarrow W_{\sigma}^{1,2}(\Omega, \mathbb{R}^3)$ are also bounded operators, and we denote their (Banach space) adjoints by $P_n^* : (W_{0,\sigma}^{1,2}(\Omega, \mathbb{R}^3))^* \rightarrow (W_{0,\sigma}^{1,2}(\Omega, \mathbb{R}^3))^*$ and $Q_n^* : (W_{\sigma}^{1,2}(\Omega, \mathbb{R}^3))^* \rightarrow (W_{\sigma}^{1,2}(\Omega, \mathbb{R}^3))^*$. We also define the Stokes operator and a corresponding operator for magnetic fields,

$$\Lambda_1 : W_{0,\sigma}^{1,2}(\Omega, \mathbb{R}^3) \rightarrow (W_{0,\sigma}^{1,2}(\Omega, \mathbb{R}^3))^* \quad \text{and} \quad \Lambda_2 : W_{\sigma}^{1,2}(\Omega, \mathbb{R}^3) \rightarrow (W_{\sigma}^{1,2}(\Omega, \mathbb{R}^3))^*,$$

by

$$\begin{aligned} \langle \Lambda_1 u, v \rangle_{(W_{0,\sigma}^{1,2})^* - W_{0,\sigma}^{1,2}} & := \int_{\Omega} \nabla u : \nabla v, \\ \langle \Lambda_2 b, w \rangle_{(W_{\sigma}^{1,2})^* - W_{\sigma}^{1,2}} & := \int_{\Omega} \operatorname{curl} b \cdot \operatorname{curl} w. \end{aligned}$$

We write the Galerkin approximation in the condensed form

$$\partial_t u_n - v P_n^* \Lambda_1 u_n + P_n^* [(u_n \cdot \nabla) u_n - (b_n \cdot \nabla) b_n] = 0, \tag{A.3}$$

$$\partial_t b_n - \mu Q_n^* \Lambda_2 b_n + Q_n^* [\operatorname{curl}(b_n \times u_n)] = 0, \tag{A.4}$$

$$u_n(\cdot, 0) = P_n u_0, \quad b_n(\cdot, 0) = Q_n b_0. \tag{A.5}$$

In order for the weak limit (u, b) to satisfy the MHD equations (1.1)–(1.6) we need to gain enough compactness in the nonlinear terms $P_n^* [(u_n \cdot \nabla) \cdot u_n - b_n \times (\operatorname{curl} b_n)]$ and $Q_n^* [\operatorname{curl}(b_n \times u_n)]$. This is eventually achieved by using the Aubin Lions Lemma to get $u_n \rightarrow u$ and $b_n \rightarrow b$ in $L^2(0, T; L_{\sigma}^2(\Omega, \mathbb{R}^3))$. In order to satisfy the assumptions of the Aubin–Lions Lemma we wish to choose suitable bases $\{v_j\}_{j \in \mathbb{N}}$ and $\{w_j\}_{j \in \mathbb{N}}$ (see Sect. A.1) that ensure the uniform norm control

$$\sup_{n \in \mathbb{N}} (\|P_n^*\|_{(W_{0,\sigma}^{1,2}(\Omega, \mathbb{R}^3))^* \rightarrow (W_{0,\sigma}^{1,2}(\Omega, \mathbb{R}^3))^*} + \|Q_n^*\|_{(W_{\sigma}^{1,2}(\Omega, \mathbb{R}^3))^* \rightarrow (W_{\sigma}^{1,2}(\Omega, \mathbb{R}^3))^*}) < \infty. \tag{A.6}$$

As is customary, we select v_j to be eigenfunctions of Λ_1 , while a basis of $L_{\Sigma}^2(\Omega, \mathbb{R}^3)$ consists of eigenfunctions of Λ_2 . Since we deal with multiply connected domains, $\{w_j\}_{j \in \mathbb{N}}$ also needs to include a basis of $L_H^2(\Omega, \mathbb{R}^3)$, and some care is needed in the ensuing arguments. The proof is completed in Sect. A.3.

A.1. The choice of bases. This subsection is devoted to the choice of the orthonormal bases of $L^2_\sigma(\Omega, \mathbb{R}^3)$. The first basis $\{v_j\}_{j \in \mathbb{N}}$ consists of eigenfunctions of the Stokes operator and its existence is classical; we refer to [51, p. 39]. We endow $W^{1,2}_{0,\sigma}(\Omega, \mathbb{R}^3)$ with the Hilbert norm $\|\cdot\|_{W^{1,2}_{0,\sigma}} := \|\nabla \cdot\|_{L^2}$.

Lemma A.3. $L^2_\sigma(\Omega, \mathbb{R}^3)$ has an orthonormal basis $\{v_j\}_{j \in \mathbb{N}}$ with the following properties: for every $j \in \mathbb{N}$ there exists $\lambda_j > 0$ such that $v_j \in W^{1,2}_{0,\sigma}(\Omega, \mathbb{R}^3)$ satisfies

$$(v_j, \phi)_{W^{1,2}_{0,\sigma}} = \lambda_j (v_j, \phi)_{L^2}$$

for all $\phi \in W^{1,2}_{0,\sigma}(\Omega, \mathbb{R}^3)$. In particular, $\{v_j/\sqrt{\lambda_j}\}_{j \in \mathbb{N}}$ is an orthonormal system in $W^{1,2}_{0,\sigma}(\Omega, \mathbb{R}^3)$.

The analysis of the second basis is simplified by using the following lemma which is essentially a special case of [3, Corollary 3.16].

Lemma A.4. On $W^{1,2}_\sigma(\Omega, \mathbb{R}^3)$, the norm $\|\cdot\|_{W^{1,2}_\sigma}$ induced by the inner product

$$(v, w)_{W^{1,2}_\sigma} := \int_\Omega \operatorname{curl} v(x) \cdot \operatorname{curl} w(x) \, dx + \sum_{i=1}^N \gamma_i \langle v \cdot n, 1 \rangle_{\Sigma_i} \langle w \cdot n, 1 \rangle_{\Sigma_i}$$

(where $\gamma_i > 0$ is chosen such that $\|h_i\|_{W^{1,2}_\sigma} = 1$ for all $i \in \{1, \dots, N\}$) is equivalent to the norm inherited from $W^{1,2}(\Omega, \mathbb{R}^3)$.

Lemma A.4 has the following consequence ([19, Lemme II.6]).

Lemma A.5. The vector spaces $W^{1,2}_\Sigma(\Omega, \mathbb{R}^3) := W^{1,2}_\sigma(\Omega, \mathbb{R}^3) \cap L^2_\Sigma(\Omega, \mathbb{R}^3)$ and $W^{1,2}_H(\Omega, \mathbb{R}^3) := W^{1,2}_\sigma(\Omega, \mathbb{R}^3) \cap L^2_H(\Omega, \mathbb{R}^3)$ satisfy

$$W^{1,2}_\sigma(\Omega, \mathbb{R}^3) = W^{1,2}_\Sigma(\Omega, \mathbb{R}^3) \oplus W^{1,2}_H(\Omega, \mathbb{R}^3). \tag{A.7}$$

Furthermore, $W^{1,2}_\Sigma(\Omega, \mathbb{R}^3)$ is dense in $L^2_\Sigma(\Omega, \mathbb{R}^3)$.

Proof. In the proof of (A.7) the only non-trivial condition to check is that when $w \in W^{1,2}_\sigma(\Omega, \mathbb{R}^3)$, we have $w_\Sigma := P_\Sigma w \in W^{1,2}_\Sigma(\Omega, \mathbb{R}^3)$ and $w_H := P_H w \in W^{1,2}_H(\Omega, \mathbb{R}^3)$. Note that Theorem 2.8 gives $w_H \in W^{1,2}_H(\Omega, \mathbb{R}^3)$, which immediately implies $w_\Sigma \in W^{1,2}_\Sigma(\Omega, \mathbb{R}^3)$. Furthermore, the projection $P_\Sigma : L^2_\sigma(\Omega, \mathbb{R}^3) \rightarrow L^2_\Sigma(\Omega, \mathbb{R}^3)$ is also a bounded operator from $W^{1,2}_\sigma(\Omega, \mathbb{R}^3)$ into $W^{1,2}_\Sigma(\Omega, \mathbb{R}^3)$.

Let now $f \in L^2_\Sigma(\Omega, \mathbb{R}^3)$ and choose mappings $\psi_j \in C^\infty_{c,\sigma}(\Omega, \mathbb{R}^3) \subset W^{1,2}_\sigma(\Omega, \mathbb{R}^3)$ such that $\|\psi_j - f\|_{L^2} \rightarrow 0$. Then $P_\Sigma \psi_j \in W^{1,2}_\Sigma(\Omega, \mathbb{R}^3)$ for all $j \in \mathbb{N}$ and $P_\Sigma \psi_j \rightarrow P_\Sigma f = f$ in $L^2(\Omega, \mathbb{R}^3)$. \square

We use Lemma A.5 to find the basis of $L^2_\sigma(\Omega, \mathbb{R}^3)$ that is used to construct the magnetic field in Theorem A.1. In the case of simply connected domains this is done by analysing the magnetostatic problem instead of the stationary Stokes problem (see [30, pp. 67–69]). In multiply connected domains the situation is a bit more complicated because $W^{1,2}_H(\Omega, \mathbb{R}^3)$ is non-trivial.

Lemma A.6. $L^2_\sigma(\Omega, \mathbb{R}^3)$ has an orthonormal basis $\{w_j\}_{j \in \mathbb{N}}$ with the following properties: $\{w_1, \dots, w_N\} = \{h_1, \dots, h_N\}$ and for every $j \in \mathbb{N}$ there exists $\tilde{\lambda}_j > 0$ such that $w_j \in W^{1,2}_\sigma(\Omega, \mathbb{R}^3)$ satisfies

$$(w_j, \psi)_{W^{1,2}_\sigma} = \tilde{\lambda}_j (w_j, \psi)_{L^2} \tag{A.8}$$

for all $\psi \in W^{1,2}_\sigma(\Omega, \mathbb{R}^3)$. In particular, $\{w_j/\sqrt{\tilde{\lambda}_j}\}_{j \in \mathbb{N}}$ is an orthonormal system in $W^{1,2}_\sigma(\Omega, \mathbb{R}^3)$.

Proof. Given $j \in \{1, \dots, N\}$ we first check that (A.8) holds for $w_j = h_j$ with $\tilde{\lambda}_j = 1$. Let $\psi \in W^{1,2}_\sigma(\Omega, \mathbb{R}^3)$. Writing $\psi = \sum_{i=1}^N (\psi, h_i)_{L^2} h_i + \psi_\Sigma$ and using Lemma A.5, (A.8) follows immediately.

Next we set out to find $w_j \in W^{1,2}_\Sigma(\Omega, \mathbb{R}^3)$ for every $j > N$. Given $f \in L^2_\Sigma(\Omega, \mathbb{R}^3)$ we define a quadratic functional $K : W^{1,2}_\Sigma(\Omega, \mathbb{R}^3) \rightarrow \mathbb{R}$ by

$$K(C) := \frac{1}{2} \int_\Omega |\operatorname{curl} C(x)|^2 dx - \int_\Omega f(x) \cdot C(x) dx.$$

The quadratic part of K is coercive and K is strictly convex, and therefore K has a unique minimizer w in $W^{1,2}_\Sigma(\Omega, \mathbb{R}^3)$. Thus

$$\int_\Omega \operatorname{curl} w(x) \cdot \operatorname{curl} \psi(x) dx = \int_\Omega f(x) \cdot \psi(x) dx \tag{A.9}$$

for all $\psi \in W^{1,2}_\Sigma(\Omega, \mathbb{R}^3)$.

We define a bounded linear operator $\mathcal{A}_2 : L^2_\Sigma(\Omega, \mathbb{R}^3) \rightarrow L^2_\Sigma(\Omega, \mathbb{R}^3)$ by $\mathcal{A}_2 f := w$. Our aim is to choose $w_j, j > N$, as eigenfunctions of \mathcal{A}_2 . Since

$$\|w\|_{W^{1,2}(\Omega)}^2 \lesssim_\Omega \|\operatorname{curl} w\|_{L^2(\Omega)}^2 = \int_\Omega f \cdot w \leq \|f\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)}$$

for all $f \in L^2_\Sigma(\Omega, \mathbb{R}^3)$, the Rellich–Kondrachov Theorem implies that \mathcal{A}_2 is compact. In addition, $\int_\Omega \mathcal{A}_2 f(x) \cdot g(x) dx = \int_\Omega \operatorname{curl} \mathcal{A}_2 f(x) \cdot \operatorname{curl} \mathcal{A}_2 g(x) dx = \int_\Omega f(x) \cdot \mathcal{A}_2 g(x) dx$ for all $f, g \in L^2_\Sigma(\Omega, \mathbb{R}^3)$ so that \mathcal{A}_2 is self-adjoint. Furthermore, \mathcal{A}_2 is a positive operator. Indeed, given $f \in L^2_\Sigma(\Omega, \mathbb{R}^3) \setminus \{0\}$ we have $\int_\Omega \mathcal{A}_2 f(x) \cdot f(x) dx = \|\operatorname{curl} \mathcal{A}_2 f\|_{L^2(\Omega)}^2$. Now (A.9) and the assumption $f \neq 0$ imply that $\operatorname{curl} \mathcal{A}_2 f \neq 0$: since $W^{1,2}_\Sigma(\Omega, \mathbb{R}^3)$ is dense in $L^2_\Sigma(\Omega, \mathbb{R}^3)$ by Lemma A.5, we may choose $\psi \in W^{1,2}_\Sigma(\Omega, \mathbb{R}^3)$ such that $\int_\Omega \operatorname{curl} \mathcal{A}_2 f(x) \cdot \operatorname{curl} \psi(x) dx = \int_\Omega f(x) \cdot \psi(x) dx \neq 0$. Hence, $\int_\Omega \mathcal{A}_2 f(x) \cdot f(x) dx > 0$.

The Spectral Theorem for compact self-adjoint operators now yields an orthonormal basis $\{w_j\}_{j \in \mathbb{N}}$ of $L^2_\Sigma(\Omega, \mathbb{R}^3)$ and corresponding strictly positive eigenvalues $\mu_j \rightarrow 0$. We denote $\tilde{\lambda}_j := 1/\mu_j \rightarrow \infty$. Equality (A.8) implies that the mappings $w_j/\sqrt{\tilde{\lambda}_j}$ form an orthonormal system in $W^{1,2}_\sigma(\Omega, \mathbb{R}^3)$. \square

When $u \in W_{0,\sigma}^{1,2}(\Omega, \mathbb{R}^3)$, $b \in W_{\sigma}^{1,2}(\Omega, \mathbb{R}^3)$ and $n \in \mathbb{N}$, Lemmas A.3 and A.6 allow us to write $P_n u$ and $Q_n b$ as

$$P_n u = \sum_{j=1}^n \left(u, \frac{v_j}{\sqrt{\lambda_j}} \right)_{W_{0,\sigma}^{1,2}} \frac{v_j}{\sqrt{\lambda_j}}, \quad Q_n b = \sum_{j=1}^n \left(b, \frac{w_j}{\sqrt{\tilde{\lambda}_j}} \right)_{W_{\sigma}^{1,2}} \frac{w_j}{\sqrt{\tilde{\lambda}_j}}.$$

This immediately implies the following result, which in turn yields the norm bound in (A.6).

Proposition A.7. *Both of the linear operators $P_n : W_{0,\sigma}^{1,2}(\Omega, \mathbb{R}^3) \rightarrow W_{0,\sigma}^{1,2}(\Omega, \mathbb{R}^3)$ and $Q_n : W_{\sigma}^{1,2}(\Omega, \mathbb{R}^3) \rightarrow W_{\sigma}^{1,2}(\Omega, \mathbb{R}^3)$ are self-adjoint and bounded uniformly in n .*

In the next subsection we give a solution of the Galerkin approximation equations.

A.2. The Galerkin approximation. In order to smoothen the exposition we work with the bases constructed in the previous subsection, although the following lemma holds for any orthonormal bases $\{v_j\}_{j \in \mathbb{N}}$ and $\{w_j\}_{j \in \mathbb{N}}$ of $L_{\sigma}^2(\Omega, \mathbb{R}^3)$ with $v_j \in W_{0,\sigma}^{1,2}(\Omega, \mathbb{R}^3)$ and $w_j \in W_{\sigma}^{1,2}(\Omega, \mathbb{R}^3)$.

Lemma A.8. *For every $n \in \mathbb{N}$, the Galerkin approximation has a solution of the form (A.1) with the energy equality (A.2) holding for all $t \in [0, T)$.*

Proof. When u_n and b_n are of the form (A.1), Lemmas A.3 and A.6 imply that the Galerkin equations read as

$$\dot{c}_{nj}(t) - \nu \lambda_j c_{nj}(t) + \sum_{k,l=1}^n c_{nk}(t) c_{nl}(t) \alpha_{jkl} - \sum_{k,l=1}^n d_{nk}(t) d_{nl}(t) \beta_{jkl} = 0, \quad (\text{A.10})$$

$$\dot{d}_{nj}(t) - \mu \tilde{\lambda}_j \chi_{j > N} d_{nj}(t) + \sum_{k,l=1}^n c_{nk}(t) d_{nl}(t) \gamma_{jkl} - \sum_{k,l=1}^n d_{nk}(t) c_{nl}(t) \delta_{jkl} = 0, \quad (\text{A.11})$$

$$c_{nj}(0) = \int_{\Omega} u_0(x) \cdot v_j(x) \, dx, \quad d_{nj}(0) = \int_{\Omega} b_0(x) \cdot w_j(x) \, dx \quad (\text{A.12})$$

for $j = 1, \dots, n$, where

$$\begin{aligned} \alpha_{jkl} &:= \int_{\Omega} (v_k(x) \cdot \nabla) v_l(x) \cdot v_j(x) \, dx, & \beta_{jkl} &:= \int_{\Omega} (w_k(x) \cdot \nabla) w_l(x) \cdot v_j(x) \, dx \\ \gamma_{jkl} &:= \int_{\Omega} (w_k(x) \cdot \nabla) v_l(x) \cdot w_j(x) \, dx, & \delta_{jkl} &:= \int_{\Omega} (v_k(x) \cdot \nabla) w_l(x) \cdot w_j(x) \, dx. \end{aligned}$$

Note that (A.10)–(A.12) is an initial value problem for a system of $2n$ ODE’s on the $2n$ functions c_{nj}, d_{nj} , and by standard theory of ODE’s there exists $T_n > 0$ and a solution $c_{n1}, \dots, d_{nn} \in C^{\infty}([0, T_n))$. Note also that

$$\alpha_{jkl} = -\alpha_{lkj}, \quad \beta_{jkl} = -\beta_{lkj}, \quad \delta_{jkl} = -\delta_{lkj} = 0. \quad (\text{A.13})$$

The energy equality can be written as

$$\begin{aligned} & \sum_{j=1}^n c_{nj}(t)^2 + \sum_{j=1}^n d_{nj}(t)^2 + 2 \sum_{j=1}^n \int_0^t (\nu \lambda_j c_{nj}(\tau)^2 + \mu \tilde{\lambda}_j \chi_{j>N} d_{nj}(\tau)^2) d\tau \\ &= \sum_{j=1}^n c_{nj}(0)^2 + \sum_{j=1}^n d_{nj}(0)^2 \end{aligned}$$

and is proved by multiplying (A.10) by $c_{nj}(t)$ and (A.11) by $d_{nj}(t)$, summing in j , integrating in time and using (A.13). The energy equality allows us to continue the solution of (A.10)–(A.12) to the whole interval $[0, T]$. \square

A.3. *Passing to the limit.* The Leray–Hopf solution (u, b) of (1.1)–(1.6) will be obtained as a strong L^2 limit of (u_n, b_n) by using the Aubin–Lions Lemma. To that end we prove norm bounds on (u_n, b_n) .

Lemma A.9. *There exists $C > 0$ such that*

$$\begin{aligned} & \|u_n\|_{L^\infty(0,T;L^2(\Omega))} + \|u_n\|_{L^2(0,T;W^{1,2}(\Omega))} + \|\partial_t u_n\|_{L^{4/3}(0,T;(W_{0,\sigma}^{1,2}(\Omega))^*)} \leq C, \\ & \|b_n\|_{L^\infty(0,T;L^2(\Omega))} + \|b_n\|_{L^2(0,T;W^{1,2}(\Omega))} + \|\partial_t b_n\|_{L^{4/3}(0,T;(W_\sigma^{1,2}(\Omega))^*)} \leq C \end{aligned}$$

for all $n \in \mathbb{N}$.

Proof. First, since u_n and b_n satisfy the energy equality for every $n \in \mathbb{N}$, $\|P_n u_0 - u_0\|_{L^2} \rightarrow 0$ and $\|Q_n b_0 - b_0\|_{L^2} \rightarrow 0$, it follows that $\sup_{n \in \mathbb{N}} (\|u_n\|_{L^\infty(0,T;L^2(\Omega))} + \|b_n\|_{L^\infty(0,T;L^2(\Omega))}) < \infty$. By another use of the energy equality and Lemma A.4, $|\nabla u_n|$ and $|\nabla b_n, \Sigma|$ are uniformly bounded in $L^2(0, T; L^2(\Omega))$. Furthermore,

$$\begin{aligned} \|\nabla b_{n,H}\|_{L^2(0,T;L^2(\Omega))} &= \left\| \sum_{i=1}^{\min n,N} d_{nj} \nabla w_j \right\|_{L^2(0,T;L^2(\Omega))} \\ &\lesssim_\Omega \left\| \sum_{i=1}^{\min n,N} d_{nj} w_j \right\|_{L^2(0,T;L^2(\Omega))} \\ &= \|b_{n,H}\|_{L^2(0,T;L^2(\Omega))} \end{aligned}$$

for all $n \in \mathbb{N}$, and thus $\sup_{n \in \mathbb{N}} (\|u_n\|_{L^2(0,T;W^{1,2}(\Omega))} + \|b_n\|_{L^2(0,T;W^{1,2}(\Omega))}) < \infty$.

We now deal with $\partial_t b_n, \partial_t u_n$ being similar but slightly simpler. At a.e. $t \in [0, T]$ and for all $\theta \in W_\sigma^{1,2}(\Omega, \mathbb{R}^3)$ we write

$$\langle \partial_t b_n, \theta \rangle_{(W_\sigma^{1,2}(\Omega))^* - W_\sigma^{1,2}(\Omega)} + \langle \mu \Lambda_2 b_n + \text{curl}(b_n \times u_n), Q_n \theta \rangle_{(W_\sigma^{1,2}(\Omega))^* - W_\sigma^{1,2}(\Omega)} = 0.$$

Proposition A.7 gives

$$\begin{aligned} \langle \mu \Lambda_2 b_n, Q_n \theta \rangle_{(W_\sigma^{1,2}(\Omega))^* - W_\sigma^{1,2}(\Omega)} &= \mu (\text{curl } b_n(\cdot, t), \text{curl } Q_n \theta)_{L^2} \\ &\lesssim \mu \|b_n(\cdot, t)\|_{W_\sigma^{1,2}(\Omega)} \|\theta\|_{W_\sigma^{1,2}(\Omega)}. \end{aligned}$$

By using Proposition A.7 again, given $n \in \mathbb{N}$ and $\theta \in W_{\sigma}^{1,2}(\Omega, \mathbb{R}^3)$ we get

$$\begin{aligned} |(\operatorname{curl}(b_n \times u_n), Q_n \theta)_{(W_{\sigma}^{1,2}(\Omega))^* - W_{\sigma}^{1,2}(\Omega)}| &\leq \|b_n(\cdot, t)\|_{L^6} \|u_n(\cdot, t)\|_{L^3} \|\operatorname{curl} Q_n \theta\|_{L^2} \\ &\lesssim_{\Omega} \|\nabla b_n(\cdot, t)\|_{L^2} \|u_n(\cdot, t)\|_{L^3} \|\theta\|_{W_{\sigma}^{1,2}}, \end{aligned}$$

so that, by using the previous inequality and Hölder’s inequality with exponents 3/2 and 3 in t ,

$$\begin{aligned} \|\operatorname{curl}(u_n \times b_n)\|_{L^{4/3}(0,T;(W_{0,\sigma}^{1,2}(\Omega))^*)} &\lesssim_{\Omega} \|\nabla b_n\|_{L^2(\Omega)} \|u_n\|_{L^3(\Omega)} \|L^{4/3}(0,T) \\ &\leq \|\nabla b_n\|_{L^2(0,T;L^2(\Omega))} \|u_n\|_{L^4(0,T;L^3(\Omega))} \end{aligned}$$

which, when combined with Lemma 2.11, completes the proof. \square

The Aubin–Lions Lemma and interpolation give various convergence properties.

Lemma A.10. *There exist $u \in L^{\infty}(0, T; L^2_{\sigma}(\Omega, \mathbb{R}^3)) \cap L^2(0, T; W_{0,\sigma}^{1,2}(\Omega, \mathbb{R}^3))$ and $b \in L^{\infty}(0, T; L^2_{\sigma}(\Omega, \mathbb{R}^3)) \cap L^2(0, T; W_{\sigma}^{1,2}(\Omega, \mathbb{R}^3))$ such that, up to a subsequence, the following convergences hold:*

- (i) $u_n \rightarrow u$ and $b_n \rightarrow b$ in $L^2(0, T; L^2_{\sigma}(\Omega, \mathbb{R}^3))$,
- (ii) $u_n \rightharpoonup u$ in $L^2(0, T; W_{0,\sigma}^{1,2}(\Omega, \mathbb{R}^3))$ and $b_n \rightharpoonup b$ in $L^2(0, T; W_{\sigma}^{1,2}(\Omega, \mathbb{R}^3))$,
- (iii) $\partial_t u_n \rightharpoonup \partial_t u$ in $L^{4/3}(0, T; (W_{0,\sigma}^{1,2}(\Omega, \mathbb{R}^3))^*)$ and furthermore $\partial_t b_n \rightharpoonup \partial_t b$ in $L^{4/3}(0, T; (W_{\sigma}^{1,2}(\Omega, \mathbb{R}^3))^*)$,
- (iv) $u_n \otimes u_n \rightharpoonup u \otimes u$ and $b_n \otimes b_n \rightharpoonup b \otimes b$ in $L^{4/3}(0, T; L^2(\Omega, \mathbb{R}^{3 \times 3}))$, $b_n \times u_n \rightharpoonup b \times u$ in $L^{4/3}(0, T; L^2(\Omega, \mathbb{R}^3))$.

Proof. While (i) and (ii) follow immediately from the Aubin–Lions Lemma and Lemma A.9, claims (iii)–(iv) follow from (i) and Lemma 2.11. The claim $u, b \in L^{\infty}(0, T; L^2_{\sigma}(\Omega, \mathbb{R}^3))$ follows from the fact that up to a subsequence, $u_n \overset{*}{\rightharpoonup} u$ and $b_n \overset{*}{\rightharpoonup} b$ in $L^{\infty}(0, T; L^2_{\sigma}(\Omega, \mathbb{R}^3))$. \square

We show that (u, b) solves the equations (1.1)–(5.3) and (1.5)–(1.6), and we refer to [28] for the proof of the claims that $u, b \in C_w([0, T]; L^2_{\sigma}(\Omega, \mathbb{R}^3))$ and that $u(\cdot, 0) = u_0$ and $b(\cdot, 0) = b_0$. The energy inequality is then obtained as a consequence.

Lemma A.11. *The mappings u and b mentioned in Lemma A.10 form a Leray–Hopf solution of (1.1)–(1.6).*

Proof. We first show that u and b satisfy (2.3) and (2.4) a.t. a.e. $t \in [0, T]$ for every $\varphi \in W_{0,\sigma}^{1,2}(\Omega, \mathbb{R}^3)$ and $\theta \in W_{\sigma}^{1,2}(\Omega, \mathbb{R}^3)$. Note that whenever $\eta \in C_c^{\infty}([0, T])$ and $k \in \mathbb{N}$, Lemmas A.8 and A.10 give

$$\begin{aligned} 0 &= -\eta(0)(P_n u_0, v_k)_{L^2} - \int_0^T \eta'(t)(u_n, v_k)_{L^2} dt \\ &\quad - \int_0^T \eta(t)(u_n \otimes u_n - b_n \otimes b_n, \nabla v_k)_{L^2} dt + v \int_0^T \eta(t)(\nabla u_n, \nabla v_k)_{L^2} dt \\ &\rightarrow -\eta(0)(u_0, v_k)_{L^2} - \int_0^T \eta'(t)(u, v_k)_{L^2} dt \\ &\quad - \int_0^T \eta(t)(u \otimes u - b \otimes b, \nabla v_k)_{L^2} dt + v \int_0^T \eta(t)(\nabla u, \nabla v_k)_{L^2} dt. \end{aligned}$$

Given any $\varphi \in W_{0,\sigma}^{1,2}(\Omega, \mathbb{R}^3)$ we can replace v_k above by $\varphi_k := P_k \varphi$ by taking linear combinations. Now $\|P_k \varphi - \varphi\|_{L^2(\Omega, \mathbb{R}^3)} \rightarrow 0$ and $\sup_{k \in \mathbb{N}} \|P_k \varphi\|_{W_{0,\sigma}^{1,2}(\Omega, \mathbb{R}^3)} < \infty$ imply that $P_k \varphi \rightarrow \varphi$ in $W_{0,\sigma}^{1,2}(\Omega, \mathbb{R}^3)$. We let $k \rightarrow \infty$ to obtain

$$\begin{aligned} 0 &= -\eta(0)(u_0, \varphi)_{L^2} - \int_0^T \eta'(t)(u, \varphi)_{L^2} dt \\ &\quad - \int_0^T \eta(t)(u \otimes u - b \otimes b, \nabla \varphi)_{L^2} dt + \nu \int_0^T \eta(t)(\nabla u, \nabla \varphi)_{L^2} dt, \end{aligned}$$

which in particular gives (2.3) at a.e. $t \in [0, T)$. Similarly, if $\theta \in W_\sigma^{1,2}(\Omega, \mathbb{R}^3)$, equation (2.4) holds at a.e. $t \in [0, T)$.

The claims that $u, b \in C_w([0, T); L_\sigma^2(\Omega, \mathbb{R}^3))$ and that $u(\cdot, 0) = u_0$ and $b(\cdot, 0) = b_0$ can be proved by slightly modifying [28, Lemmas 2.1–2.2]. Since we have $u, b \in C_w([0, T); L_\sigma^2(\Omega, \mathbb{R}^3))$, it suffices to show the energy inequality at a.e. $t \in [0, T)$. Since $u_n \rightarrow u$ and $b_n \rightarrow b$ in $L^2(0, T; L^2(\Omega, \mathbb{R}^3))$, passing to a subsequence we get $u_n(\cdot, t) \rightarrow u(\cdot, t)$ and $b_n(\cdot, t) \rightarrow b(\cdot, t)$ in $L^2(\Omega, \mathbb{R}^3)$ at a.e. $t \in [0, T)$. At those times t the energy inequality for u and b now follows from the energy equality of u_n and b_n . \square

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