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Existence of Corotating and Counter-Rotating Vortex Pairs for Active Scalar Equations

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Abstract: In this paper, we study the existence of corotating and counter-rotating pairs of simply connected patches for Euler equations and the $(\mathrm{SQG})_\alpha$ equations with $\alpha \in (0, 1)$. From the numerical experiments implemented for Euler equations in Deem and Zabusky (Phys Rev Lett 40(13):859–862, 1978), Pierrehumbert (J Fluid Mech 99:129–144, 1980), Saffman and Szeto (Phys Fluids 23(12):2339–2342, 1980) it is conjectured the existence of a curve of steady vortex pairs passing through the point vortex pairs. There are some analytical proofs based on variational principle (Keady in J Aust Math Soc Ser B 26:487–502, 1985; Turkington in Nonlinear Anal Theory Methods Appl 9(4):351–369, 1985); however, they do not give enough information about the pairs, such as the uniqueness or the topological structure of each single vortex. We intend in this paper to give direct proofs confirming the numerical experiments and extend these results for the $(\mathrm{SQG})_\alpha$ equation when $\alpha \in (0, 1)$. The proofs rely on the contour dynamics equations combined with a desingularization of the point vortex pairs and the application of the implicit function theorem.

Contents

1. Introduction

The present work deals with the dynamics of vortex pairs for some nonlinear transport equations arising in fluid dynamics. The equations that we shall consider are the generalized surface quasi-geostrophic equations which describe the evolution of the potential temperature θ through the system,

$$\begin{cases} \partial_t\theta + v\cdot\nabla\theta = 0, \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^2,\\ v = -\nabla^\perp(-\Delta)^{-1+\frac{\alpha}{2}}\theta,\\ \theta_{|t=0} = \theta_0. \end{cases} \tag{1}$$

Here v refers to the velocity field, $\nabla^\perp = (-\partial_2, \partial_1)$ and α is a real parameter taken in $[0, 2)$. The operator $(-\Delta)^{-1+\frac{\alpha}{2}}$ is of convolution type and defined as follows

$$(-\Delta)^{-1+\frac{\alpha}{2}}\theta(x) = \int_{\mathbb{R}^2} K_\alpha(x-y)\theta(y)dy$$

with

$$K_\alpha(x) = \begin{cases} -\frac{1}{2\pi}\log|x|, & \text{if } \alpha = 0\\ \frac{C_\alpha}{2\pi}\frac{1}{|x|^\alpha}, & \text{if } \alpha\in(0,2) \end{cases} \tag{2}$$

and $C_\alpha = \frac{\Gamma(\alpha/2)}{2^{1-\alpha}\Gamma(\frac{2-\alpha}{2})}$ where Γ stands for the gamma function. Note that this model was proposed by Córdoba et al. in [11] as an interpolation between Euler equations and the surface quasi-geostrophic model (SQG) corresponding to $\alpha = 0$ and $\alpha = 1$, respectively. We mention that the SQG equation is used in [21,25] to describe the atmosphere circulation near the tropopause and to track the ocean dynamics in the upper layers [30]. The mathematical analogy with the classical three-dimensional incompressible Euler equations was pointed out in [10].

In the last few years there has been a growing interest in the mathematical study of these active scalar equations. Local well-posedness of classical solutions has been discussed in various function spaces. For instance, this was implemented in the framework of Sobolev spaces [8], however, the global existence is still an open problem except for Euler equations. The second restriction with the $(\text{SQG})_\alpha$ equation concerns the construction of Yudovich solutions—known to exist globally in time for Euler equations [43]—which remains unsolved even locally in time. The main difficulty is due to the velocity, which is in general singular and scales below the Lipschitz class. Nonetheless, one can say more about this issue for some special class of concentrated vortices. More precisely, when the initial data is a single vortex patch, that is, $\theta_0(x) = \chi_D$ is the characteristic function of a bounded simply connected smooth domain D, there is a unique local solution in the patch form $\theta(t) = \chi_{D_t}$. In this case, the boundary motion of the domain D_t is described by the contour dynamics formulation. Indeed, the Lagrangian parametrization $\gamma_t : \mathbb{T}\to\partial D_t$ obeys the following integro-differential equations

$$\partial_t\gamma_t(w) = \int_{\mathbb{T}} K_\alpha\big(\gamma_t(w)-\gamma_t(\xi)\big)\gamma_t'(\xi)d\xi.$$

For more details, see [9,19,37]. The global persistence of the boundary regularity is established for Euler equations by Chemin [9]; we refer also to the paper of Bertozzi and Constantin [2] for another proof. However for $\alpha > 0$ only local persistence result is known and numerical experiments carried out in [11] reveal a singularity formation in finite time. Let us mention that the contour dynamics equation remains locally well-posed when the domain of the initial data is composed of multiple patches with different magnitudes in each component.

In this paper we shall focus on steady single and multiple patches moving without changing shape, called relative equilibria or V-states according to the terminology of Deem and Zabusky. Their dynamics is seemingly simple flow configurations described by rotating or translating motion but it is immensely rich and exhibits complex behaviors. There is abundant literature dealing with numerical and analytical structures for the isolated rotating patches and the first example goes back to Kirchhoff [28], who proved for Euler equations that an ellipse of semi-axes a and b rotates uniformly with the angular velocity $\Omega = ab/(a+b)^2$. About one century later, Deem and Zabusky [12] provided strong numerical evidence for the existence of rotating patches with to be m-fold symmetry for the integers $m \in \{3, 4, 5\}$. Recall that a domain is said to be m-fold symmetric if it is invariant by the action of the dihedral group D_m. A few years later, Burbea gave in [3] an analytical proof and showed for any integer $m \geq 2$ the existence of a curve of V-states with m-fold symmetry bifurcating from Rankine vortex at the angular velocity $\frac{m-1}{2m}$. The proof relies on the use of complex analysis tools combined with the bifurcation theory. The regularity of the boundary close to Rankine vortices has been discussed very recently by the authors and Verdera in [22] where it was proved that the boundary is C^∞ and convex. It seems that the boundary is actually analytic according to the recent result of Castro, Córdoba and Gómez-Serrano [7]. We also refer to the paper [42] where it is proved that corners with right angles is the only plausible scenario for the limiting V-states. It is worth pointing out that Burbea's approach has been successfully implemented for the $(\mathrm{SQG})_\alpha$ equations in [7,20] but with much more delicate computations. Similarly to the case $\alpha = 0$, we find countable family of bifurcating curves at some known angular velocities related to gamma function. In the same context, it turns out that for Euler equations a second bifurcation of countable branches from the ellipses occurs but the shapes have in fact less symmetry and being at most two-folds. This was first observed numerically in [26,31] and analytical proofs were recently discussed in [6,23]. Another valuable investigation has been devoted to the existence of doubly connected V-states where the rotating patches have only one hole. In this case the boundary is comprised of two Jordan curves obeying to two coupled singular nonlinear equations and thereby the dynamics acquires more richness and significant behaviors. The existence of such structures was first accomplished for Euler equations in [14] by using bifurcation tools in the spirit of Burbea's approach. Roughly speaking, for higher symmetry m we get two branches of $m-$fold V-states bifurcating from the annulus $\{b < |z| < 1\}$ and numerical experiments about the limiting V-states reveal different plausible configurations depending on the size of the parameter b. Later, this result has been extended for the $(\mathrm{SQG})_\alpha$ equations in [13] for $\alpha \in (0, 1)$, which surprisingly exhibit various completely new behaviors compared to Euler equations. For example we find rotating patches with negative and positive angular velocities for any $\alpha \in (0, 1)$. It is worth mentioning that the bifurcation in the preceding cases is obtained under the transversality assumption of Crandall–Rabinowitz corresponding to simple nonlinear eigenvalues. However the bifurcation in the degenerate case where there is crossing eigenvalues is more complicated and has been recently solved in [24].

The main task of this paper is to deal with non connected V-states where the bifurcation arguments discussed above are out of use. To be more precise, we shall be concerned with vortex pairs moving without deformation. This is a fundamental and rich subject in vortex dynamics and they serve for instance to model trailing vortices behind the wings of aircraft in steady horizontal flight or to describe the interaction between isolated vortex and a solid wall. We point out that the literature is very abundant and it is by no means an easy task to collect and recall all the results done in this field. Therefore we shall restrict the discussion to the cases of counter-rotating and corotating vortices and recall some results that fit with our main goal. In the first case, the most common studied configuration is two symmetric vortex pairs with opposite circulations moving steadily with constant speed in a fixed direction. Notice that an explicit example is given by a pair of point vortices with opposite circulations which translates steadily with the speed $U_{sing} = \frac{\gamma}{2\pi d}$, where d is the distance separating the point vortices and γ is the magnitude, see for instance [29]. Another nontrivial explicit example of touching counter-rotating vortex pair was discovered by Lamb [29], where the vortex is not uniformly distributed but has a smooth compactly supported profile related to Bessel functions of the first kind. Later, Deem and Zabusky [12] and Pierrehumbert [36] provided numerically a class of translating vortex pairs of symmetric patches and they conjectured the existence of a curve of translating symmetric pair of simply connected patches emerging from two point vortices and ending with two touching patches at right angle. We mention that Keady [27] used a variational principle in order to explore the existence part and give asymptotic estimates for some significant functionals such as the excess kinetic energy and the speed of the pairs. The basic idea is to maximize the excess kinetic energy supplemented with some additional constraints and to show the existence of a maximizer taking the form of a pair of vortex patches in the spirit of the paper of Turkington [41]. However, this approach does not give sufficient information on the structure of the pairs. For example the uniqueness of the maximizer is left open and the topology of the patches is not well-explored, and it is not clear from the proof whether or not each single patch is simply connected as it is suggested numerically. Concerning the corotating vortex pair, which consists of two symmetric patches with the same circulations and rotating about the centroid of the system with constant angular velocity, it was investigated numerically by Saffman and Szeto in [39]. They showed that when far apart, the vortices are almost circular and when the distance between them decreases they become more deformed until they touch. We remark that a pair of point vortices far away at a distance d and with the same magnitude γ rotates steadily with the angular velocity $\Omega_{sing} = \frac{\gamma}{\pi d^2}$. By using variational principle, Turkington gave in [41] an analytic proof of the existence of corotating vortex pairs but this general approach does not give enough precision on the topological structure of each vortex patch, similarly to the translating case commented before. Note that in the same direction Dritschel [16] calculated numerically V-states of vortex pairs with different shapes and discussed their linear stability. Very recently, Denisov established in [15] for a modified Euler equations the existence of corotating simply connected vortex patches and analyzed the contact point of the limiting V-states. To end this short discussion, we want to emphasize once again that the subject of vortex pairs has been intensively studied during the past and it is difficult to track, know and recall here everything written about it. So, we have only selected some basic results and the reader can find more details not only in this subject but also in some other connected topics in [1,4,5,17,18,31,32,34,35,38,40] and the references therein.

In the current paper we intend to give direct proofs for the existence of corotating and counter-rotating vortex pairs using the contour dynamics equations. We shall also extend these results to the $(\mathrm{SQG})_\alpha$ equations for $\alpha \in (0,1)$. Now we shall fix some notations before stating our main result. Let $0 < \varepsilon < 1$, $d > 2$ and take a small simply connected domain D_1 containing the origin and contained in the open ball $D(0,2)$ centered at the origin and with radius 2. Define

$$\theta_{0,\varepsilon} = \frac{1}{\varepsilon^2}\chi_{D_1^\varepsilon} + \delta\frac{1}{\varepsilon^2}\chi_{D_2^\varepsilon}, \quad D_1^\varepsilon = \varepsilon D_1, \quad D_2^\varepsilon = -D_1^\varepsilon + 2d, \tag{3}$$

where the number δ is taken in $\{\pm 1\}$. As we can readily observe, this initial data is composed of symmetric pair of simply connected patches with equal or opposite circulations. The main result of the paper is the following.

Main Theorem. *Let $\alpha \in [0,1)$, there exists $\varepsilon_0 > 0$ such that the following results hold true.*

(i) *Case $\delta = 1$. For any $\varepsilon \in (0, \varepsilon_0]$ there exists a strictly convex domain D_1^ε at least of class C^1 such that $\theta_{0,\varepsilon}$ in (3) generates a corotating vortex pair for (1).*
(ii) *Case $\delta = -1$. For any $\varepsilon \in (0, \varepsilon_0]$ there exists a strictly convex domain D_1^ε of class C^1 such that $\theta_{0,\varepsilon}$ generates a counter-rotating vortex pair for (1).*

Before giving the basic ideas of the proofs some remarks are in order.

Remark 1. The domain D_1^ε is a small perturbation of the disc $D(0,\varepsilon)$, centered at zero and of radius ε. Moreover, it can be described by the conformal parametrization $\phi_\varepsilon : \mathbb{T} \to \partial D_1^\varepsilon$ which belongs for $0 < \alpha < 1$ to $C^{2-\alpha}(\mathbb{T})$ and for $\alpha = 0$ to $C^{1+\beta}$ for any $\beta \in (0,1)$, and satisfies

$$\phi_\varepsilon(w) = \varepsilon w + \varepsilon^{2+\alpha} f_\varepsilon(w) \quad \text{with} \quad \|f_\varepsilon\|_{C^{2-\alpha}} \le 1.$$

Therefore the boundary of each V-state is at least C^1. Note that with slight modifications we can adapt the proofs and show that the domain D_1^ε belongs to $C^{n+\beta}$ for any fixed $n \in \mathbb{N}$. Of course, the size of ε_0 depends on the parameter n and cannot be uniform; it shrinks to zero as n grows to infinity. However, we expect the boundary to be analytic, meaning that the conformal mapping possesses a holomorphic extension in $D(0,r)^c$ for some $0 < r < 1$. The ideas developed in the recent paper [6] might be useful to confirm such expectation.

Remark 2. In the setting of the vortex patches the global existence with smooth boundaries is not known for $\alpha \in (0,2)$. In [7,13,20] we exhibit the first nontrivial examples of simply connected and doubly connected V-states which are periodic in time. We find here another class of global solutions which are the vortex pairs.

Remark 3. The proof is valid for $\alpha \in [0,1)$ but we expect that the result remains true for $\alpha \in [1,2)$. We believe that the use of the spaces introduced in [7] could be helpful for solving these cases.

Remark 4. As we shall see later, we can unify the formalism leading to the existence of corotating patches with the point vortex model. The latter one is obtained when $\varepsilon = 0$ in which case we find the classical result which says that two point vortices at distance $2d$ and with the same magnitude rotate uniformly about their center with the angular velocity $\Omega_{sing}^\alpha = \frac{\alpha C_\alpha}{\pi(2d)^{2+\alpha}}$. However, when they have opposite signs they exhibit a uniform translating motion with the speed $U_{sing}^\alpha = \frac{\alpha C_\alpha}{\pi(2d)^{1+\alpha}}$.

Next we shall sketch the basic ideas used to prove the main result. We will just restrict the discussion to the corotating pairs for Euler equations since the proofs for the remaining cases follow the same lines but with much more involved computations. The proof relies on the desingularization of point vortex pairs combined with the implicit function theorem. We first formulate the equations governing the corotating vortex pairs using the complex variable, and we shall see later in Sect. 3 more details. However, we think that at this stage it is convenient to sketch the arguments needed to get the contour dynamics equations of the boundary. Let D_1 and D_2 be two disjoint simply connected domains and a, b be two non vanishing real numbers. Then the initial datum

$$\theta_0 = a\chi_{D_1} + b\chi_{D_2}$$

gives rise to a rotating vortex pair about the point $(d, 0)$ and with the angular velocity Ω if the solution $\theta(t)$ of (1) takes the form

$$\theta(t, z-d) = \theta_0\big(e^{it\Omega}(z-d)\big).$$

Therefore, inserting this expression into the equation (1) we find

$$\Big(v_0(z) - \Omega(z-d)\Big)\cdot\nabla\theta_0(z) = 0$$

with v_0 the velocity associated to θ_0. Using the patch structure, the preceding equation reduces to

$$\Big(v_0(z) - \Omega(z-d)\Big)\cdot\boldsymbol{n}(z) = 0, \quad \forall\, z \in \partial D_1 \cup \partial D_2$$

where $\boldsymbol{n}(z)$ is a normal vector to the boundary. Combining Biot–Savart law with Green–Stokes formula we find

$$\overline{v_0(z)} = \frac{a}{4\pi}\int_{\partial D_1}\frac{\overline{\xi}-\overline{z}}{\xi - z}\,d\xi + \frac{b}{4\pi}\int_{\partial D_2}\frac{\overline{\xi}-\overline{z}}{\xi - z}\,d\xi, \quad \forall\, z \in \mathbb{C}.$$

Consequently, using the special structure (3) combined with some elementary transformations in the complex plane we deduce that the two equations governing the boundary are actually equivalent to the following equation

$$\mathrm{Re}\Big\{\Big(2\Omega\big(\varepsilon\overline{z} - d\big) + I_\varepsilon(z)\Big)\,\boldsymbol{\tau}(z)\Big\} = 0, \quad \forall\, z \in \partial D_1$$

with $\boldsymbol{\tau}(z)$ being the unit tangent vector to the boundary ∂D_1 positively oriented and

$$I_\varepsilon(z) = \frac{1}{2i\pi\varepsilon}\int_{\partial D_1}\frac{\overline{\xi}-\overline{z}}{\xi - z}d\xi - \frac{1}{2i\pi}\int_{\partial D_1^\varepsilon}\frac{\overline{\xi}}{\varepsilon\xi + \varepsilon z - 2d}d\xi.$$

The basic idea of the proof is to extend the functional defining the vortex pairs beyond $\varepsilon = 0$ corresponding to point vortex pairs and afterwards to apply the implicit function theorem. As we can see, the first integral term in $I_\varepsilon(z)$ is singular and to remove the singularity we should seek for domains which are slight perturbation of the unit disc with a small amplitude of order ε. In other words, we look for a conformal parametrization of D_1 in the form

$$\forall\, w \in \mathbb{T}, \quad \phi_\varepsilon(w) = w + \varepsilon f(w)$$

where the Fourier expansion of f takes the form

$$f(w) = \sum_{n\geq 1} a_n w^{-n}, \quad a_n \in \mathbb{R} \quad \text{and} \quad \|f\|_{C^{1+\beta}} \leq 1$$

for some $\beta \in (0, 1)$. Note that the singularity in ε is in fact removable owing to the symmetry of the disc. Indeed, following standard computations we get the expansion

$$I_\varepsilon(\phi_\varepsilon(w)) = -\frac{1}{\varepsilon}\overline{w} + J_\varepsilon(\phi_\varepsilon(w)) \tag{4}$$

where J_ε belongs to the space C^β and can be extended for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ with $\varepsilon_0 > 0$. Setting

$$G(\varepsilon, \Omega, f(w)) \equiv \operatorname{Im}\left\{\Big(2\Omega\big(\varepsilon\phi_\varepsilon(w) - d\big) + I_\varepsilon\big(\phi_\varepsilon(w)\big)\Big)\, w\, \phi_\varepsilon'(w)\right\},$$

then the equation of the vortex pairs is simply given by

$$\forall\, w \in \mathbb{T}, \quad G(\varepsilon, \Omega, f(w)) = 0.$$

It follows from the expansion (4) that we can get rid of the singularity in ε and this is the first step towards the application of the implicit function theorem. Before giving further details we should first fix the function spaces. Let

$$X^0 = \Big\{f \in C^{1+\beta}(\mathbb{T}),\ f(w) = \sum_{n\geq 1} a_n w^{-n}\Big\},$$

and

$$Y^0 = \Big\{f \in C^\beta(\mathbb{T}),\ f = \sum_{n\geq 1} a_n e_n,\ a_n \in \mathbb{R}\Big\}, \quad \widehat{Y^0} = \big\{f \in Y^0, a_1 = 0\big\},$$
$$e_n(w) \equiv \operatorname{Im}(w^n).$$

According to Proposition 1 the function $G : (-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R} \times B_1^0 \to Y^0$ is well-defined and it is of class C^1, where B_1^0 is the open unit ball of X^0. Moreover

$$\partial_f G(0, \Omega, 0)h(w) = -\operatorname{Im}(h'(w)).$$

However, this operator is not invertible from X^0 to Y^0 but it does from X^0 to $\widehat{Y^0}$. The next step is to choose carefully Ω such that the image of the nonlinear functional G is contained in the vector space $\widehat{Y^0}$. This will be done carefully in Sect. 4.2 and leads eventually to a new nonlinear constraint of the type $\Omega = \Omega(\varepsilon, f)$. Consequently the equation of the vortex pairs becomes

$$F(\varepsilon, f(w)) \equiv G(\varepsilon, \Omega(\varepsilon, f), f) = 0.$$

Note that with this formulation the point vortex configuration corresponds to $F(0, 0) = 0$ in which case $\Omega = \Omega_{sing}^0 = \frac{1}{4\pi d^2}$. In addition, from the platitude of Ω we deduce that the linearized operator remains the same, that is,

$$\partial_f F(0, 0) = \partial_f G(0, \Omega_{sing}^0, 0)$$

which is invertible from X^0 to $\widehat{Y^0}$. Therefore and at this stage one can use the implicit function theorem which implies the local existence of a unique curve of solutions $\varepsilon \mapsto \phi_\varepsilon$ passing through (0, 0) and remark that each point of this curve is a nontrivial corotating vortex pair of symmetric simply connected patches.

The remaining of the paper is organized as follows. In Sect. 2 we shall gather some tools dealing with the function spaces and give some results on Newton and Riesz potentials. In Sect. 3 we shall write down the equations governing the corotating and translating vortex pairs of symmetric patches for both Euler and $(\text{SQG})_\alpha$ equations. Sections 4 and 5 are dedicated to the proofs of the Main Theorem.

Notation We need to fix some notations that will be frequently used along this paper. We denote by C any positive constant that may change from line to line. We denote by $\mathbb{D}$ the unit disc and its boundary, the unit circle, is denoted by $\mathbb{T}$. Let $f : \mathbb{T} \to \mathbb{C}$ be a continuous function, we define its mean value by,

$$\fint_{\mathbb{T}} f(\tau)d\tau \equiv \frac{1}{2i\pi}\int_{\mathbb{T}} f(\tau)d\tau,$$

where $d\tau$ stands for the complex integration. Finally, for $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we use the notation $(x)_n$ to denote the Pochhammer symbol defined by,

$$(x)_n = \begin{cases} 1 & n = 0 \\ x(x+1)\cdots(x+n-1) & n \geq 1. \end{cases}$$

2. Preliminaries and Background

In this section we shall briefly recall the classical Hölder spaces on the periodic case and state some classical facts on the continuity of fractional integrals over these spaces. It is convenient to think of $2\pi-$periodic function $f : \mathbb{R} \to \mathbb{C}$ as a function of the complex variable $w = e^{i\eta}$ rather than a function of the real variable η. To be more precise, let $f : \mathbb{T} \to \mathbb{R}^2$, be a continuous function, then it can be assimilated to a $2\pi-$periodic function $g : \mathbb{R} \to \mathbb{R}$ via the relation

$$f(w) = g(\eta), \quad w = e^{i\eta}.$$

Hence when f is smooth enough we get

$$f'(w) \equiv \frac{df}{dw} = -ie^{-i\eta}g'(\eta).$$

Because d/dw and $d/d\eta$ differ only by a smooth factor with modulus one we shall in the sequel work with d/dw instead of $d/d\eta$ which appears to be more convenient in the computations. Now we shall introduce Hölder spaces on the unit circle $\mathbb{T}$.

Definition 1. Let $0 < \beta < 1$. We denote by $C^\beta(\mathbb{T})$ the space of continuous functions f such that

$$\|f\|_{C^\beta(\mathbb{T})} \equiv \|f\|_{L^\infty(\mathbb{T})} + \sup_{x \neq y \in \mathbb{T}} \frac{|f(x) - f(y)|}{|x-y|^\beta} < \infty.$$

For any integer n the space $C^{n+\beta}(\mathbb{T})$ stands for the set of functions f of class C^n whose $n-$th order derivatives are Hölder continuous with exponent β. This space is equipped with the usual norm,

$$\|f\|_{C^{n+\beta}(\mathbb{T})} \equiv \|f\|_{L^\infty(\mathbb{T})} + \left\|\frac{d^n f}{dw^n}\right\|_{C^\beta(\mathbb{T})}.$$

Recall that the Lipschitz (semi)-norm is defined as follows.

$$\|f\|_{\mathrm{Lip}(\mathbb{T})} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

Now we list some classical properties that will be used later in several sections.

(i) For $n \in \mathbb{N}$, $\beta \in]0, 1[$ the space $C^{n+\beta}(\mathbb{T})$ is an algebra.
(ii) For $K \in L^1(\mathbb{T})$ and $f \in C^{n+\beta}(\mathbb{T})$ we have the convolution law,

$$\|K * f\|_{C^{n+\beta}(\mathbb{T})} \leq \|K\|_{L^1(\mathbb{T})} \|f\|_{C^{n+\beta}(\mathbb{T})}.$$

The next result is used frequently and it deals with fractional integrals of the following type,

$$\mathcal{T}(f)(w) = \int_{\mathbb{T}} K(w, \tau)\, f(\tau) d\tau, \tag{5}$$

with $K : \mathbb{T} \times \mathbb{T} \to \mathbb{C}$ being a singular kernel satisfying some properties. The problem on the smoothness of this operator will appear naturally when we shall deal with the regularity of the nonlinear functional defining steady vortex pairs. The result that we shall discuss with respect to this subject is classical and whose proof can be found for instance in [20,33].

Lemma 1. *Let* $0 \leq \alpha < 1$ *and consider a function* $K : \mathbb{T} \times \mathbb{T} \to \mathbb{C}$ *with the following properties. There exits* $C_0 > 0$ *such that,*

(i) *K is measurable on* $\mathbb{T} \times \mathbb{T} \backslash \{(w, w),\ w \in \mathbb{T}\}$ *and*

$$\left|K(w, \tau)\right| \leq \frac{C_0}{|w - \tau|^\alpha}, \quad \forall\, w \neq \tau \in \mathbb{T}.$$

(ii) *For each* $\tau \in \mathbb{T}$, $w \mapsto K(w, \tau)$ *is differentiable in* $\mathbb{T} \backslash \{\tau\}$ *and*

$$\left|\partial_w K(w, \tau)\right| \leq \frac{C_0}{|w - \tau|^{1+\alpha}}, \quad \forall\, w \neq \tau \in \mathbb{T}.$$

Then

(A) *The operator* $\mathcal{T}$ *defined by (5) is continuous from* $L^\infty(\mathbb{T})$ *to* $C^{1-\alpha}(\mathbb{T})$. *More precisely, there exists a constant* C_α *depending only on* α *such that*

$$\|\mathcal{T}(f)\|_{1-\alpha} \leq C_\alpha C_0 \|f\|_{L^\infty}.$$

(B) *For* $\alpha = 0$ *the operator* $\mathcal{T}$ *is continuous from* $L^\infty(\mathbb{T})$ *to* $C^\beta(\mathbb{T})$ *for any* $0 < \beta < 1$. *That is, there exists a constant* C_β *depending only on* β *such that*

$$\|\mathcal{T}(f)\|_\beta \leq C_\beta C_0 \|f\|_{L^\infty}.$$

As a by-product we obtain a result that will be frequently used through this paper.

Corollary 1. *Let* $0 < \alpha < 1$, $\phi : \mathbb{T} \to \phi(\mathbb{T})$ *be a bi-Lipschitz function with real Fourier coefficients and define the operator*

$$\mathcal{T}_\phi : f \mapsto \fint_{\mathbb{T}} \frac{f(\tau)}{|\phi(w) - \phi(\tau)|^\alpha} d\tau, \quad w \in \mathbb{T}.$$

Then $\mathcal{T}_\phi : L^\infty(\mathbb{T}) \to C^{1-\alpha}(\mathbb{T})$ *is continuous with the estimate*

$$\|\mathcal{T}_\phi(f)\|_{C^{1-\alpha}(\mathbb{T})} \leq C\Big(\|\phi^{-1}\|^\alpha_{\mathrm{Lip}(\mathbb{T})} + \|\phi\|^2_{\mathrm{Lip}(\mathbb{T})} \|\phi^{-1}\|^{1+\alpha}_{\mathrm{Lip}(\mathbb{T})}\Big) \|f\|_{L^\infty(\mathbb{T})},$$

where C *is a positive constant depending only on* α.

3. Steady Vortex Pairs Models

The aim of this section is to derive the equations governing co-rotating and translating symmetric pairs of patches. In the first step, we shall write down the equations for the rotating pairs for Euler and $(SQG)_\alpha$ equations. In the second step, we shall be concerned with the counter-rotating vortex pairs sometimes called translating pairs. Notice that we prefer to use the conformal parametrization because it is more convenient in the computations especially through its holomorphic structure.

3.1. Corotating vortex pairs. Let D_1 be a bounded simply connected domain containing the origin and contained in the ball $B(0,2)$. For $\varepsilon\in]0,1[$ and $d>2$ we define the domains

$$D_1^\varepsilon=\varepsilon D_1\quad\text{and}\quad D_2^\varepsilon=-D_1^\varepsilon+2d.$$

Set

$$\theta_{0,\varepsilon}=\frac{1}{\varepsilon^2}\chi_{D_1^\varepsilon}+\frac{1}{\varepsilon^2}\chi_{D_2^\varepsilon}$$

and assume that this gives rise to a rotating pairs of patches about the centroid of the system $(d,0)$ and with an angular velocity Ω. According to ([14], p. 1896) this condition holds true if and only if

$$\mathrm{Re}\big(-i\,\Omega\,\big(\overline{z}-d\big)\,\boldsymbol{n}(z)\big)=\mathrm{Re}\big(\overline{v(z)}\,\boldsymbol{n}\big),\quad\forall\,z\in\partial D_1^\varepsilon\cup\partial D_2^\varepsilon,\tag{6}$$

where $\boldsymbol{n}(z)$ is the exterior unit normal vector to the boundary of $D_1^\varepsilon\cup D_2^\varepsilon$ at the point z. Next we shall discuss separately Euler equations and the case $\alpha\in(0,1)$ due to the difference structures of their Green functions.

3.1.1. Euler equations It is well-known that the velocity can be recovered for the vorticity according to Biot–Savart law,

$$\overline{v(z)}=-\frac{i}{2\pi\,\varepsilon^2}\int_{D_1^\varepsilon}\frac{dA(\zeta)}{z-\zeta}-\frac{i}{2\pi\,\varepsilon^2}\int_{D_2^\varepsilon}\frac{dA(\zeta)}{z-\zeta},\quad\forall\,z\in\mathbb{C}.$$

From Green–Stokes formula we record that

$$-\frac{1}{\pi}\int_D\frac{dA(\zeta)}{z-\zeta}=\fint_{\partial D}\frac{\overline{\xi}-\overline{z}}{\xi-z}d\xi,\quad\forall\,z\in\mathbb{C}.$$

Therefore

$$\mathrm{Re}\Big\{\Big(2\Omega\big(\overline{z}-d\big)+I(z)\Big)\,\boldsymbol{\tau}\Big\}=0,\quad\forall\,z\in\partial D_1^\varepsilon\cup\partial D_2^\varepsilon,\tag{7}$$

with $\boldsymbol{\tau}$ being the unit tangent vector to $\partial D_1^\varepsilon\cup\partial D_2^\varepsilon$ positively oriented and

$$I(z)=\frac{1}{\varepsilon^2}\fint_{\partial D_1^\varepsilon}\frac{\overline{\xi}-\overline{z}}{\xi-z}d\xi+\frac{1}{\varepsilon^2}\fint_{\partial D_2^\varepsilon}\frac{\overline{\xi}-\overline{z}}{\xi-z}d\xi.$$

Changing in the last integral ξ to $-\xi + 2d$, which sends ∂D_2^ε to ∂D_1^ε, we get

$$I(z) = \frac{1}{\varepsilon^2} \fint_{\partial D_1^\varepsilon} \frac{\overline{\xi} - \overline{z}}{\xi - z} d\xi - \frac{1}{\varepsilon^2} \fint_{\partial D_1^\varepsilon} \frac{\overline{\xi} + \overline{z} - 2d}{\xi + z - 2d} d\xi.$$

We can check that if the Eq. (7) is satisfied for all $z \in \partial D_1^\varepsilon$, then it will be surely satisfied for all $z \in \partial D_2^\varepsilon$. This follows easily from the identity

$$I(-z + 2d) = -I(z).$$

Now observe that when $z \in \partial D_1^\varepsilon$ then $-z + 2d \notin \overline{D_1^\varepsilon}$ and thus residue theorem allows to get

$$I(z) = \frac{1}{\varepsilon^2} \fint_{\partial D_1^\varepsilon} \frac{\overline{\xi} - \overline{z}}{\xi - z} d\xi - \frac{1}{\varepsilon^2} \fint_{\partial D_1^\varepsilon} \frac{\overline{\xi}}{\xi + z - 2d} d\xi.$$

Denote $\Gamma_1 = \partial D_1$ then by the change of variables $\xi \mapsto \varepsilon\xi$ and $z \mapsto \varepsilon z$ the Eq. (7) becomes

$$\mathrm{Re}\Big\{\Big(2\Omega\big(\varepsilon\overline{z} - d\big) + I_\varepsilon(z)\Big)\,\boldsymbol{\tau}\Big\} = 0, \quad \forall\, z \in \Gamma_1.$$

with

$$\begin{aligned} I_\varepsilon(z) &\equiv I(\varepsilon z) \\ &= \frac{1}{\varepsilon} \fint_{\Gamma_1} \frac{\overline{\xi} - \overline{z}}{\xi - z} d\xi - \fint_{\Gamma_1} \frac{\overline{\xi}}{\varepsilon\xi + \varepsilon z - 2d} d\xi \\ &\equiv I_\varepsilon^1(z) - I_\varepsilon^2(z). \end{aligned}$$

We shall search for domains D_1 which are small perturbations of the unit disc with an amplitude of order ε. More precisely, we shall in the conformal parametrization $\phi : \mathbb{T} \to \partial D_1$ look for a solution in the form

$$\phi(w) = w + \varepsilon f(w), \quad \text{with} \;\; f(w) = \sum_{n \geq 1} \frac{a_n}{w^n}, \quad a_n \in \mathbb{R}.$$

We remark that the assumption $a_n \in \mathbb{R}$ means that the domain D_1 is symmetric with respect to the real axis. Setting $z = \phi(w)$, then for $w \in \mathbb{T}$ a tangent vector to the boundary at the point z is given by

$$\boldsymbol{\tau} = i\, w\, \phi'(w) = i\, w\big(1 + \varepsilon f'(w)\big).$$

Thus the steady vortex pairs equation becomes

$$\mathrm{Im}\Big\{\Big(2\Omega\Big[\varepsilon\overline{w} + \varepsilon^2 f(\overline{w}) - d\Big] + I_\varepsilon(\phi(w))\Big)\, w\big(1 + \varepsilon f'(w)\big)\Big\} = 0, \quad \forall\, w \in \mathbb{T}. \quad (8)$$

Notice that we have used that f has real Fourier coefficients and thus $\overline{f(w)} = f(\overline{w})$. By using the notation $A = \tau - w$ and $B = f(\tau) - f(w)$ we can write for all $w \in \mathbb{T}$

$$\begin{aligned} I_\varepsilon^1(\phi(w)) &= \frac{1}{\varepsilon} \fint_{\mathbb{T}} \frac{\overline{\tau} - \overline{w} + \varepsilon\big(f(\overline{\tau}) - f(\overline{w})\big)}{\tau - w + \varepsilon(f(\tau) - f(w))} \big(1 + \varepsilon f'(\tau)\big) d\tau \\ &= \fint_{\mathbb{T}} \frac{\overline{A} + \varepsilon\overline{B}}{A + \varepsilon B} f'(\tau) d\tau + \fint_{\mathbb{T}} \frac{A\overline{B} - \overline{A}B}{A(A + \varepsilon B)} d\tau + \frac{1}{\varepsilon} \fint_{\mathbb{T}} \frac{\overline{A}}{A} d\tau \\ &= \fint_{\mathbb{T}} \frac{\overline{A} + \varepsilon\overline{B}}{A + \varepsilon B} f'(\tau) d\tau + \fint_{\mathbb{T}} \frac{A\overline{B} - \overline{A}B}{A(A + \varepsilon B)} d\tau - \frac{1}{\varepsilon}\overline{w}, \end{aligned}$$

where we have used the obvious formula

$$\fint_{\mathbb{T}} \frac{\overline{A}}{A} d\tau = -\overline{w} \fint_{\mathbb{T}} \frac{d\tau}{\tau} \\ = -\overline{w}.$$

This leads to a significant cancellation and the singular term will disappear from the full nonlinearity due in particular to the symmetry of the disc,

$$\begin{aligned} &\mathrm{Im}\Big\{ I_\varepsilon^1(\phi(w))\, w\big(1+\varepsilon f'(w)\big)\Big\} \\ &= \mathrm{Im}\Big\{ \Big(\fint_{\mathbb{T}} \frac{\overline{A}+\varepsilon \overline{B}}{A+\varepsilon B} f'(\tau) d\tau + \fint_{\mathbb{T}} \frac{A\overline{B}-\overline{A}B}{A(A+\varepsilon B)} d\tau \Big) w\big[1+\varepsilon f'(w)\big] \Big\} \\ &\quad - \mathrm{Im}(f'(w)), \quad \forall\, w \in \mathbb{T}. \end{aligned}$$

For the second term $I_\varepsilon^2(\phi(w)$ it takes the form

$$I_\varepsilon^2(\phi(w) = \fint_{\mathbb{T}} \frac{\overline{(\tau} + \varepsilon f(\overline{\tau}))(1+\varepsilon f'(\tau))}{\varepsilon(\tau+w)+\varepsilon^2\big(f(\tau)+f(w)\big)-2d} d\tau.$$

Hence the steady vortex pairs equation is equivalent to

$$-2G^0(\varepsilon, \Omega, f) \equiv \mathrm{Im}(F^0(\varepsilon, \Omega, f)) = 0 \tag{9}$$

with

$$\begin{aligned} F^0(\varepsilon, \Omega, f(w)) &= 2\Omega\Big(\varepsilon\overline{w} + \varepsilon^2 f(\overline{w}) - d\Big) w\big(1+\varepsilon f'(w)\big) - f'(w) \\ &\quad + \Big(\fint_{\mathbb{T}} \frac{\overline{A}+\varepsilon\overline{B}}{A+\varepsilon B} f'(\tau) d\tau + \fint_{\mathbb{T}} \frac{A\overline{B}-\overline{A}B}{A(A+\varepsilon B)} d\tau \Big) w\big(1+\varepsilon f'(w)\big) \\ &\quad - \Bigg(\fint_{\mathbb{T}} \frac{\big(\overline{\tau}+\varepsilon f(\overline{\tau})\big)\big(1+\varepsilon f'(\tau)\big)}{\varepsilon(\tau+w)+\varepsilon^2\big(f(\tau)+f(w)\big)-2d} d\tau \Bigg) w\big(1+\varepsilon f'(w)\big) \\ &\equiv F_1(\varepsilon, \Omega, f(w)) + F_2(\varepsilon, f(w)) + F_3(\varepsilon, f(w)). \end{aligned}$$

We point out that we have added a factor -2 in the definition of G^0 given by (9) in order to unify the notation with the function G^α that we shall introduce in next section for the $(\mathrm{SQG})_\alpha$.

3.1.2. $(SQG)_\alpha$ equations. First we remark that the Eq. (6) can be written in the form,

$$\Omega\, \mathrm{Re}\big\{(z-d)\,\overline{\boldsymbol{\tau}}\big\} = \mathrm{Im}\big\{v(z)\,\overline{\boldsymbol{\tau}}\big\}, \quad \forall\, z \in \partial D_1^\varepsilon \cup \partial D_2^\varepsilon, \tag{10}$$

where as before $\boldsymbol{\tau}$ denotes a tangent vector to the boundary at the point z. This equation is equivalent to

$$\mathrm{Re}\Big\{ \Big(\Omega\big(z-d\big) + i v(z) \Big)\, \overline{\boldsymbol{\tau}} \Big\} = 0, \quad \forall\, z \in \partial D_1^\varepsilon \cup \partial D_2^\varepsilon.$$

The velocity can be recovered from the boundary as follows, see for instance [20],

$$v(z) = \frac{C_\alpha}{2\pi\,\varepsilon^2} \int_{\partial D_1^\varepsilon} \frac{1}{|z-\xi|^\alpha} d\xi + \frac{C_\alpha}{2\pi\,\varepsilon^2} \int_{\partial D_2^\varepsilon} \frac{1}{|z-\xi|^\alpha} d\xi, \quad \forall\, z \in \mathbb{C}.$$

Using in the last integral the change of variables $\xi \mapsto -\xi + 2d$, we deduce that

$$v(z) = \frac{C_\alpha}{2\pi\,\varepsilon^2}\int_{\partial D_1^\varepsilon}\frac{1}{|z-\xi|^\alpha}d\xi - \frac{C_\alpha}{2\pi\,\varepsilon^2}\int_{\partial D_1^\varepsilon}\frac{1}{|z+\xi-2d|^\alpha}d\xi, \quad \forall\, z\in\mathbb{C}. \quad (11)$$

We point out that by a symmetry argument if the equation (10) is satisfied for all $z\in\partial D_1^\varepsilon$ then it will be also satisfied for all $z\in\partial D_2^\varepsilon$. This follows from the identity

$$v(-z+2d) = -v(z).$$

As $D_1^\varepsilon = \varepsilon D_1$ then using a change of variable the equation becomes

$$\mathrm{Re}\Big\{\Big(\Omega\big(\varepsilon z - d\big) + I_\varepsilon(z)\Big)\,\overline{\boldsymbol{\tau}}\Big\} = 0, \quad \forall\, z\in\partial D_1, \quad (12)$$

with

$$I_\varepsilon(z) = -\frac{C_\alpha}{\varepsilon^{1+\alpha}}\fint_{\partial D_1}\frac{1}{|z-\xi|^\alpha}d\xi + \frac{C_\alpha}{\varepsilon}\fint_{\partial D_1}\frac{1}{|\varepsilon z+\varepsilon\xi-2d|^\alpha}d\xi.$$

We shall look for the domains D_1 which are small perturbation of the unit disc with an amplitude of order $\varepsilon^{1+\alpha}$. More precisely, we shall in the conformal parametrization $\phi:\mathbb{T}\to\partial D_1$ look for a solution in the form

$$\begin{aligned}\phi(w) &= w + \varepsilon^{1+\alpha} f(w)\\ &= w + \varepsilon^{1+\alpha}\sum_{n\geq 1}\frac{a_n}{w^n}, \quad a_n\in\mathbb{R}.\end{aligned}$$

For $w\in\mathbb{T}$ the conjugate of a tangent vector is given by $\overline{\boldsymbol{\tau}} = -i\,\overline{w}\,\overline{\phi'(w)}$ and therefore for any $w\in\mathbb{T}$,

$$\begin{aligned}G^\alpha(\varepsilon,\Omega,f(w)) &\equiv \mathrm{Im}\Big\{\Big(\Omega\Big[\varepsilon w + \varepsilon^{2+\alpha} f(w) - d\Big] + I(\varepsilon,f(w))\Big)\,\overline{w}\,\Big(1+\varepsilon^{1+\alpha}\overline{f'(w)}\Big)\Big\}\\ &= \mathrm{Im}\Big(F^\alpha(\varepsilon,\Omega,f(w))\Big) = 0, \end{aligned} \quad (13)$$

with

$$\begin{aligned}I(\varepsilon,f(w)) &= -\frac{C_\alpha}{\varepsilon^{1+\alpha}}\fint_{\mathbb{T}}\frac{\phi'(\tau)d\tau}{|\phi(w)-\phi(\tau)|^\alpha} + \frac{C_\alpha}{\varepsilon}\fint_{\mathbb{T}}\frac{\phi'(\tau)d\tau}{|\varepsilon\phi(w)+\varepsilon\phi(\tau)-2d|^\alpha}\\ &\equiv -I_1(\varepsilon,f(w)) + I_2(\varepsilon,f(w)). \end{aligned} \quad (14)$$

We shall split G into three terms

$$G^\alpha = G_1 - G_2 + G_3 \quad (15)$$

with

$$\begin{aligned}G_1(\varepsilon,\Omega,f(w)) &= \mathrm{Im}\Big\{\Omega\Big[\varepsilon w + \varepsilon^{2+\alpha} f(w) - d\Big]\,\overline{w}\,\Big(1+\varepsilon^{1+\alpha}\overline{f'(w)}\Big)\Big\},\\ G_2(\varepsilon,f(w)) &= \mathrm{Im}\Big\{I_1(\varepsilon,f(w))\,\overline{w}\,\Big(1+\varepsilon^{1+\alpha}\overline{f'(w)}\Big)\Big\},\end{aligned}$$

and

$$G_3(\varepsilon,f(w)) = \mathrm{Im}\Big\{I_2(\varepsilon,f(w))\,\overline{w}\,\Big(1+\varepsilon^{1+\alpha}\overline{f'(w)}\Big)\Big\}.$$

3.2. Counter-rotating vortex pairs. As for the corotating pairs we shall distinguish between Euler equations and the case $0 < \alpha < 1$. As before let D_1 be a bounded domain containing the origin and contained in the ball $B(0, 2)$. For $\varepsilon \in]0, 1[$ and $d > 2$ we define

$$D_1^\varepsilon = \varepsilon D_1 \quad \text{and} \quad D_2^\varepsilon = -D_1^\varepsilon + 2d.$$

Set

$$\theta_0 = \frac{1}{\varepsilon^2}\chi_{D_1^\varepsilon} - \frac{1}{\varepsilon^2}\chi_{D_2^\varepsilon}$$

and assume that θ_0 travels steadily in the (Oy) direction with uniform velocity U. Then in the moving frame the pair of the patches is stationary and consequently the analogous of the equation (6) is

$$\text{Re}\Big\{\Big(\overline{v(z)} + iU\Big)\boldsymbol{n}\Big\} = 0, \quad \forall\, z \in \partial D_1^\varepsilon \cup \partial D_2^\varepsilon. \tag{16}$$

3.2.1. Euler equations. One has from (16)

$$\text{Re}\left\{(2U + I(z))\ \boldsymbol{\tau}\right\} = 0, \quad \forall\, z \in \partial D_1^\varepsilon \cup \partial D_2^\varepsilon, \tag{17}$$

with

$$I(z) = \frac{1}{\varepsilon^2}\fint_{\partial D_1^\varepsilon}\frac{\overline{\xi} - \overline{z}}{\xi - z}d\xi - \frac{1}{\varepsilon^2}\fint_{\partial D_2^\varepsilon}\frac{\overline{\xi} - \overline{z}}{\xi - z}d\xi.$$

In the last integral changing ξ to $-\xi + 2d$ which sends ∂D_2^ε to ∂D_1^ε we get

$$I(z) = \frac{1}{\varepsilon^2}\fint_{\partial D_1^\varepsilon}\frac{\overline{\xi} - \overline{z}}{\xi - z}d\xi + \frac{1}{\varepsilon^2}\fint_{\partial D_1^\varepsilon}\frac{\overline{\xi} + \overline{z} - 2d}{\xi + z - 2d}d\xi.$$

As for the corotating case, using the identity

$$I(-z + 2d) = I(z)$$

one can check that if the equation (17) is satisfied for all $z \in \partial D_1^\varepsilon$ then it is also satisfied for all $z \in \partial D_2^\varepsilon$.

Now observe that when $z \in \partial D_1^\varepsilon$ then $-z + 2d \notin \overline{D_1^\varepsilon}$ and using residue theorem we obtain

$$I(z) = \frac{1}{\varepsilon^2}\fint_{\partial D_1^\varepsilon}\frac{\overline{\xi} - \overline{z}}{\xi - z}d\xi + \frac{1}{\varepsilon^2}\fint_{\partial D_1^\varepsilon}\frac{\overline{\xi}}{\xi + z - 2d}d\xi.$$

Let $\Gamma_1 = \partial D_1$ then by change of variables $\xi \to \varepsilon\xi$ and $z \to \varepsilon z$. The equation (17) becomes

$$\text{Re}\Big\{\Big(2U + I_\varepsilon(z)\Big)\boldsymbol{\tau}\Big\} = 0, \quad \forall\, z \in \Gamma_1,$$

with

$$\begin{aligned} I_\varepsilon(z) &= I(\varepsilon z) \\ &= \frac{1}{\varepsilon}\fint_{\Gamma_1}\frac{\overline{\xi} - \overline{z}}{\xi - z}d\xi + \fint_{\Gamma_1}\frac{\overline{\xi}}{\varepsilon\xi + \varepsilon z - 2d}d\xi \\ &\equiv I_\varepsilon^1(z) + I_\varepsilon^2(z). \end{aligned}$$

we shall now use the conformal parametrization of the boundary Γ_1,

$$\phi(w) = w + \varepsilon f(w), \quad \text{with } \ f(w) = \sum_{n\geq 1} \frac{a_n}{w^n}, a_n \in \mathbb{R}.$$

Setting $z = \phi(w)$ and $\xi = \phi(\tau)$, then for $w \in \mathbb{T}$ a tangent vector at the point $\phi(w)$ is given by

$$\boldsymbol{\tau} = i w\, \phi'(w) = i w\big(1 + \varepsilon f'(w)\big).$$

The V-states equation becomes

$$\mathrm{Im}\Big\{\Big(2U + I_\varepsilon(\phi(w))\Big)\, w\big(1 + \varepsilon f'(w)\big)\Big\} = 0, \quad \forall \ w \in \mathbb{T}.$$

As in the rotating case, with the notation $A = \tau - w$ and $B = f(\tau) - f(w)$ we get for $w \in \mathbb{T}$

$$I_\varepsilon^1(\phi(w)) = \fint_{\mathbb{T}} \frac{\overline{A} + \varepsilon\overline{B}}{A + \varepsilon B} f'(\tau) d\tau + \fint_{\mathbb{T}} \frac{A\overline{B} - \overline{A}B}{A(A + \varepsilon B)} d\tau - \frac{1}{\varepsilon}\overline{w}.$$

This yields

$$\begin{aligned}
&\mathrm{Im}\Big\{I_\varepsilon^1(\phi(w))\, w\big(1 + \varepsilon f'(w)\big)\Big\} \\
&= \mathrm{Im}\Big\{\Big(\fint_{\mathbb{T}} \frac{\overline{A} + \varepsilon\overline{B}}{A + \varepsilon B} f'(\tau) d\tau + \fint_{\mathbb{T}} \frac{A\overline{B} - \overline{A}B}{A(A + \varepsilon B)} d\tau\Big) w\big(1 + \varepsilon f'(w)\big)\Big\} \\
&\quad - \mathrm{Im}(f'(w)), \quad \forall\, w \in \mathbb{T}.
\end{aligned}$$

The second term $I_\varepsilon^2(\phi(w)$ takes the form

$$I_\varepsilon^2(\phi(w)) = \fint_{\mathbb{T}} \frac{(\overline{\tau} + \varepsilon\overline{f(\tau)})(1 + \varepsilon f'(\tau))}{\varepsilon(\tau + w) + \varepsilon^2\big(f(\tau) + f(w)\big) - 2d} d\tau.$$

Hence the V-states equation becomes

$$-2G^0(U, \varepsilon, f) \equiv \mathrm{Im}(F^0(U, \varepsilon, f) = 0 \tag{18}$$

with

$$\begin{aligned}
F^0(U, \varepsilon, f(w)) &= 2U w\big(1 + \varepsilon f'(w)\big) - f'(w) \\
&\quad + \Big(\fint_{\mathbb{T}} \frac{\overline{A} + \varepsilon\overline{B}}{A + \varepsilon B} f'(\tau) d\tau + \fint_{\mathbb{T}} \frac{A\overline{B} - \overline{A}B}{A(A + \varepsilon B)} d\tau\Big) w\big(1 + \varepsilon f'(w)\big) \\
&\quad + \Big(\fint_{\mathbb{T}} \frac{\overline{\tau} + \varepsilon f(\overline{\tau})}{\varepsilon(\tau + w) + \varepsilon^2\big(f(\tau) + f(w)\big) - 2d}\big(1 + \varepsilon f'(\tau)\big) d\tau\Big) \\
&\quad \times w\big(1 + \varepsilon f'(w)\big) \\
&\equiv F_1(U, \varepsilon, f(w)) + F_2(\varepsilon, f(w)) + F_3(\varepsilon, f(w)).
\end{aligned}$$

For the same reason as in the rotating case we add the factor -2 in (18) in order to unify the expression with the function G^α that will appear later for the $(\mathrm{SQG})_\alpha$.

3.2.2. Case $\alpha \in (0,1)$. The Eq. (16) can be written in the form

$$\mathrm{Re}\Big\{\big(v(z)-iU\big)\,\overline{\boldsymbol{n}}\Big\}=0,\quad \forall\, z\in\partial D_1^\varepsilon\cup\partial D_2^\varepsilon.$$

The velocity associated to this model is

$$v(z)=\frac{C_\alpha}{2\pi\,\varepsilon^2}\int_{\partial D_1^\varepsilon}\frac{1}{|z-\xi|^\alpha}d\xi-\frac{C_\alpha}{2\pi\,\varepsilon^2}\int_{\partial D_2^\varepsilon}\frac{1}{|z-\xi|^\alpha}d\xi,\quad \forall\, z\in\mathbb{C}.$$

Changing ξ to $-\xi+2d$ in the last integral we get

$$v(z)=\frac{C_\alpha}{2\pi\,\varepsilon^2}\int_{\partial D_1^\varepsilon}\frac{1}{|z-\xi|^\alpha}d\xi+\frac{C_\alpha}{2\pi\,\varepsilon^2}\int_{\partial D_1^\varepsilon}\frac{1}{|z+\xi-2d|^\alpha}d\xi,\quad \forall\, z\in\mathbb{C}. \tag{19}$$

Therefore the V-states equation be can be written in the form

$$\mathrm{Re}\Big\{\Big(-U+I_\varepsilon(z)\Big)\,\overline{\boldsymbol{\tau}}\Big\}=0,\quad \forall\, z\in\partial D_1 \tag{20}$$

with

$$I_\varepsilon(z)\equiv\frac{C_\alpha}{\varepsilon^{1+\alpha}}\fint_{\partial D_1}\frac{1}{|z-\xi|^\alpha}d\xi+\frac{C_\alpha}{\varepsilon}\fint_{\partial D_1}\frac{1}{|\varepsilon z+\varepsilon\xi-2d|^\alpha}d\xi.$$

Using the conformal parametrization,

$$\begin{aligned}\phi(w)&=w+\varepsilon^{1+\alpha}f(w)\\&\equiv w+\varepsilon^{1+\alpha}\sum_{n\geq1}\frac{a_n}{w^n}.\end{aligned}$$

For $w\in\mathbb{T}$ the conjugate of a tangent vector is given by

$$\overline{z'}=-i\overline{w}\,\overline{\phi'(w)}.$$

Therefore for any $w\in\mathbb{T}$,

$$G^\alpha(\varepsilon,\Omega,f(w))\equiv\mathrm{Im}\Big\{\Big(-U+I\big(\varepsilon,f(w)\big)\Big)\,\overline{w}\,\Big(1+\varepsilon^{1+\alpha}\overline{f'(w)}\Big)\Big\}=0, \tag{21}$$

with

$$\begin{aligned}I(\varepsilon,f(w))&=\frac{C_\alpha}{\varepsilon^{1+\alpha}}\fint_{\mathbb{T}}\frac{\phi'(\tau)d\tau}{|\phi(w)-\phi(\tau)|^\alpha}+\frac{C_\alpha}{\varepsilon}\fint_{\mathbb{T}}\frac{\phi'(\tau)d\tau}{|\varepsilon\phi(w)+\varepsilon\phi(\tau)-2d|^\alpha}\\&\equiv I_1(\varepsilon,f(w))+I_2(\varepsilon,f(w)).\end{aligned} \tag{22}$$

We shall split, as before, G^α into three terms

$$G^\alpha=G_1+G_2+G_3 \tag{23}$$

with

$$\begin{aligned}G_1(\varepsilon,\Omega,f(w))&=-U\,\mathrm{Im}\Big\{\overline{w}\,\Big(1+\varepsilon^{1+\alpha}\overline{f'(w)}\Big)\Big\},\\G_2(\varepsilon,f(w))&=\mathrm{Im}\Big\{I_1(\varepsilon,f(w))\,\overline{w}\,\Big(1+\varepsilon^{1+\alpha}\overline{f'(w)}\Big)\Big\}\end{aligned}$$

and

$$G_3(\varepsilon,f(w))=\mathrm{Im}\Big\{I_2(\varepsilon,f(w))\,\overline{w}\,\Big(1+\varepsilon^{1+\alpha}\overline{f'(w)}\Big)\Big\}.$$

4. Existence of Corotating Vortex Pairs

In this section we will prove the existence of rotating pairs of patches for the $(\mathrm{SQG})_\alpha$ model with $\alpha \in [0, 1)$. Recall that the equations governing the boundaries of the vortices were formulated in the Sect. 3.1. The first goal is to discuss the regularity of the functionals defining the V-states and to compute the associated linear operator at the trivial solutions corresponding to the point vortex pair. In the Sect. 4.2 we shall see how the angular velocity is uniquely determined through the geometry of the domain. In this setting Ω plays the role of the Lagrangian multiplier such that the first Fourier coefficient of the nonlinear functional vanishes. Finally, in the Sect. 4.3, we shall see that the existence of the vortex pairs is a simple consequence of the implicit function theorem in suitable Banach spaces and the convexity of each single patch is done in a standard way through a perturbative argument applied for the curvature.

4.1. Extension and regularity of the functional G^α. The main idea to prove the existence of rotating vortex pairs is to apply the implicit function theorem to the Eqs. (9) and (13). To this end we have to check that the functions G^α defined in (9) and (13) satisfy some regularity conditions. The spaces which are relevant in this study and used throughout the paper are described below. Take $\alpha \in [0, 1)$ and $\beta \in (0, 1)$ an arbitrary given number, then we define the spaces

$$X^\alpha \equiv \begin{cases} \left\{ f \in C^{1+\beta}(\mathbb{T}),\ f(w) = \sum_{n\geq 1} a_n w^{-n},\ a_n \in \mathbb{R} \right\}, & \text{if } \alpha = 0 \\ \left\{ f \in C^{2-\alpha}(\mathbb{T}),\ f(w) = \sum_{n\geq 1} a_n w^{-n},\ a_n \in \mathbb{R} \right\}, & \text{if } \alpha \in (0,1) \end{cases}$$

and

$$Y^\alpha \equiv \begin{cases} \left\{ f \in C^{\beta}(\mathbb{T}),\ f(w) = \sum_{n\geq 1} a_n e_n,\ a_n \in \mathbb{R} \right\}, & \text{if } \alpha = 0 \\ \left\{ f \in C^{1-\alpha}(\mathbb{T}),\ f(w) = \sum_{n\geq 1} a_n e_n,\ a_n \in \mathbb{R} \right\}, & \text{if } \alpha \in (0,1) \end{cases}$$

with the notation $e_n(w) = \mathrm{Im}(w^n)$. We shall also consider the subspaces

$$\widehat{Y}^\alpha = \left\{ f \in Y^\alpha,\ a_1 = 0 \right\}.$$

For $r > 0$ we denote by B_r^α the open ball of X^α centered at zero and of radius r. The next result deals with some properties of the function G^α. Before giving the main statement of this section it is convenient to introduce the notation

$$\widehat{C}_\alpha \equiv \alpha C_\alpha = 2^\alpha \frac{\Gamma(\frac{2+\alpha}{2})}{\Gamma(\frac{2-\alpha}{2})}, \quad \text{for } \alpha \in [0, 1).$$

Proposition 1. *Let $\alpha \in [0, 1)$, then the function G^α can be extended from $(-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R} \times B_1^\alpha$ to Y^α as a C^1 function. Moreover, for any $\Omega \in \mathbb{R}$ the operator $\partial_f G^\alpha(0, \Omega, 0) : X^\alpha \to \widehat{Y}^\alpha$ is an isomorphism. More precisely, for $h = \sum_{n\geq 1} a_n w^{-n} \in X^\alpha$, we get*

$$\partial_f G^\alpha(0, \Omega, 0)h(w) = \sum_{n\geq 1} a_n \widehat{\gamma}_n e_{n+1}$$

with

$$\widehat{\gamma}_n = \frac{\widehat{C}_\alpha \Gamma(1-\alpha)}{4\Gamma^2(1-\frac{\alpha}{2})} \left(\frac{2(1+n)}{1-\frac{\alpha}{2}} - \frac{\left(1+\frac{\alpha}{2}\right)_n}{\left(1-\frac{\alpha}{2}\right)_n} - \frac{\left(1+\frac{\alpha}{2}\right)_{n+1}}{\left(1-\frac{\alpha}{2}\right)_{n+1}} \right).$$

Remark 5. We can easily check from the proofs that two initial point vortex $\pi\delta_{(0,0)}$ and $\pi\delta_{(2d,0)}$ rotate uniformly about $(d,0)$ with the angular velocity

$$\Omega_{sing}^{\alpha} \equiv \frac{\widehat{C}_\alpha}{(2d)^{2+\alpha}}\cdot \tag{24}$$

Remark 6. By adapting the proof below we can check that the preceding proposition remains true if we change in the definition of X^α and Y^α the parameter α by $\alpha - n$ for any $n \in \mathbb{N}^\star$. As a consequence, the boundaries of the V-states belong to the Hölderian class $C^{n-\alpha}$ for any $n \in \mathbb{N}$.

Proof. The proof will be divided in two pieces. In the first one we shall discuss the case $\alpha = 0$ and the second one will be devoted to $\alpha \in (0,1)$.

Part I: Euler case According to the definition (9) we have the decomposition

$$\begin{aligned} -2G^0 &\equiv G_1 + G_2 + G_3 \\ &= \operatorname{Im}(F_1) + \operatorname{Im}(F_2) + \operatorname{Im}(F_3). \end{aligned}$$

We will first proceed with the regularity of the first term, that is,

$$G_1(\varepsilon,\Omega,f) = \operatorname{Im}\Big\{2\Omega\Big(\varepsilon\overline{w} + \varepsilon^2\overline{f(w)} - d\Big)w\big(1+\varepsilon f'(w)\big) - f'(w)\Big\}.$$

Clearly this function can be defined from the set $(-\frac{1}{2},\frac{1}{2}) \times \mathbb{R} \times B_1^0$ to Y^0 because the function in the brackets is in $C^\beta(\mathbb{T})$, and is obtained as sums and products of functions with real coefficients. In order to prove its differentiability we have to compute their partial derivatives,

$$\partial_\varepsilon G_1(\varepsilon,\Omega,f) = \operatorname{Im}\Big\{2\Omega(\overline{w} + 2\varepsilon\overline{f(w)})w(1+\varepsilon f'(w)) + 2\Omega(\varepsilon\overline{w} + \varepsilon^2\overline{f(w)} - d)wf'(w)\Big\},$$

and clearly this is a continuous function from $(-\frac{1}{2},\frac{1}{2}) \times \mathbb{R} \times B_1^0$ to Y^0.

Taking now the derivative in Ω we get

$$\partial_\Omega G_1(\varepsilon,\Omega,f) = \operatorname{Im}\Big\{2\Big(\varepsilon\overline{w} + \varepsilon^2\overline{f(w)} - d\Big)w\big(1+\varepsilon f'(w)\big)\Big\},$$

which is continuous from $(-\frac{1}{2},\frac{1}{2}) \times \mathbb{R} \times B_1^0$ to Y^0. Let us note that G_1 is a polynomial also in f and f' and consequently the derivative is polynomial in f and f'. Thus, it is necessary a continuous function from $(-\frac{1}{2},\frac{1}{2}) \times \mathbb{R} \times B_1^0$ to Y^0. It is an easy computation to check that

$$\partial_f G_1(0,\Omega,0)(h) = -\operatorname{Im}\{h'(w)\}.$$

Let's take now

$$\begin{aligned} G_2(\varepsilon,f) &= \operatorname{Im}\Big\{\Big(\fint_{\mathbb{T}} \frac{\overline{A} + \varepsilon\overline{B}}{A+\varepsilon B} f'(\tau)d\tau + \fint_{\mathbb{T}} \frac{A\overline{B} - \overline{A}B}{A(A+\varepsilon B)}d\tau\Big)w\big(1+\varepsilon f'(w)\big)\Big\} \\ &= \operatorname{Im}\Big\{\big(G_{21} + G_{22}\big)w\big(1+\varepsilon f'(w)\big)\Big\}. \end{aligned}$$

To prove that $G_2(\varepsilon, f)$ is a function from $(-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R} \times B_1^0$ to Y^0 it is enough to verify that the functions $G_{21}(\varepsilon, f)$ and $G_{22}(\varepsilon, f)$ satisfy the same property. Observe that the function

$$G_{21}(\varepsilon, f) = \fint_{\mathbb{T}} \frac{\overline{\tau} - \overline{w} + \varepsilon(f(\overline{\tau}) - f(\overline{w}))}{\tau - w + \varepsilon(f(\tau) - f(w))} f'(\tau) d\tau$$

is given by an integral operator. Since f is in $C^{1+\beta}(\mathbb{T})$, we will have that G_{21} belongs to the space $C^{\beta}(\mathbb{T})$ provided the kernel

$$K(\tau, w) = \frac{\overline{\tau} - \overline{w} + \varepsilon(f(\overline{\tau}) - f(\overline{w}))}{\tau - w + \varepsilon(f(\tau) - f(w))}$$

satisfies the hypotheses of Lemma 1 for $\alpha = 0$. It is obvious that

$$\sup_{\tau \neq w} |K(\tau, w)| \leq 1,$$

and moreover

$$\begin{aligned} |\partial_w K(\tau, w)| &= \Bigg| \frac{(1 + \varepsilon f'(w)\big((\overline{\tau} - \overline{w}) + \varepsilon(f(\overline{\tau}) - f(\overline{w})\big)}{((\tau - w) + \varepsilon(f(\tau) - f(w)))^2} \\ &\quad + \frac{1}{w^2} \frac{1 + \varepsilon f'(\overline{w})}{(\tau - w) + \varepsilon(f(\tau) - f(w))} \Bigg| \\ &\leq \frac{M^2 + M}{|\tau - w|}, \end{aligned}$$

where $M = \frac{1+\varepsilon \|f\|_{C^{1+\alpha}(\mathbb{T})}}{1-\varepsilon \|f\|_{C^{1+\alpha}(\mathbb{T})}}$. Now to check that this function has real coefficients we have to show that $\overline{G_{21}(\varepsilon, f)(w)} = G_{21}(\varepsilon, f)(\overline{w})$. Using the change of variable $\eta = \overline{\tau}$, it is an easy computation to see that

$$\begin{aligned} \overline{G_{21}(\varepsilon, f)(w)} &= -\fint_{\mathbb{T}} \frac{\tau - w + \varepsilon(f(\tau) - f(w))}{\overline{\tau} - \overline{w} + \varepsilon(f(\overline{\tau}) - f(\overline{w}))} f'(\overline{\tau}) d\overline{\tau} \\ &= \fint_{\mathbb{T}} \frac{\overline{\eta - \overline{w}} + \varepsilon\overline{(f(\eta) - f(\overline{w}))}}{\eta - \overline{w} + \varepsilon(f(\eta) - f(\overline{w}))} f'(\eta) d\eta \\ &= G_{21}(\varepsilon, f)(\overline{w}). \end{aligned}$$

On the other hand the function

$$G_{22}(\varepsilon, f) = \fint_{\mathbb{T}} \frac{(\tau - w)\big(f(\overline{\tau}) - f(\overline{w})\big) - (\overline{\tau} - \overline{w})(f(\tau) - f(w))}{(\tau - w)\big((\tau - w) + \varepsilon(f(\tau) - f(w)\big)} d\tau$$

will be in the space $C^{\beta}(\mathbb{T})$ if the kernel

$$K(\tau, w) = \frac{(\tau - w)\big(f(\overline{\tau}) - f(\overline{w})\big) - (\overline{\tau} - \overline{w})(f(\tau) - f(w))}{(\tau - w)\big((\tau - w) + \varepsilon(f(\tau) - f(w)\big)}$$

satisfies the hypotheses of Lemma 1 for $\alpha = 0$. As before, it is straightforward that

$$\sup_{\tau \neq w} |K(\tau, w)| \leq \frac{2\|f\|_{C^{1+\alpha}}}{1 - \varepsilon \|f\|_{C^{1+\alpha}(\mathbb{T})}}$$

and

$$|\partial_w K(\tau, w) \le \frac{C}{|\tau - w|},$$

where the constant C depends on ε and $\|f\|_{C^{1+\beta}(\mathbb{T})}$. To check that the function G_{22} has real coefficients one can repeat the same procedure used before for the function G_{21}.

Now we will verify that the function G_2 is of class C^1 from $(-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R} \times B_1^0$ to Y^0. To do so, we will check the continuity of the partial derivatives of G_{21} and G_{22}. Simple computations prove that

$$\partial_\varepsilon G_{21} = \fint_{\mathbb{T}} \frac{f(\overline{\tau}) - f(\overline{w})}{\tau - w + \varepsilon(f(\tau) - f(w))} f'(\tau) d\tau \\ - \fint_{\mathbb{T}} \frac{\overline{\tau} - \overline{w} + \varepsilon\big(f(\overline{\tau}) - f(\overline{w})\big)}{(\tau - w + \varepsilon(f(\tau) - f(w))^2} (f(\tau) - f(w)) f'(\tau) d\tau$$

and

$$\partial_\varepsilon G_{22} = -2i \fint_{\mathbb{T}} \frac{\operatorname{Im}\Big\{(\tau - w)\big(f(\overline{\tau}) - f(\overline{w})\big)\Big\}}{(\tau - w)\big(\tau - w + \varepsilon(f(\tau) - f(w))\big)^2} (f(\tau) - f(w)) d\tau.$$

The existence and the continuity of this partial derivative can be obtained by proving that the kernels that appear in the integral operators satisfy the conditions of Lemma 1. Take $h \in X$ we will compute the Gâteaux derivative in the direction h of the function G_2. For it we only need to calculate the Gâteaux derivatives of the functions G_{21} and G_{22}.

$$\partial_f G_{21}(\varepsilon, f)h(w) \equiv \fint_{\mathbb{T}} \frac{(h(\overline{\tau}) - h(\overline{w}))}{\tau - w + \varepsilon(f(\tau) - f(w))} f'(\tau) d\tau \\ + \fint_{\mathbb{T}} \frac{\overline{\tau} - \overline{w} + \varepsilon(f(\overline{\tau}) - f(\overline{w}))}{\tau - w + \varepsilon(f(\tau) - f(w))} h'(\tau) d\tau \\ - \varepsilon \fint_{\mathbb{T}} \frac{\overline{\tau} - \overline{w} + \varepsilon(f(\overline{\tau}) - f(\overline{w}))}{\big(\tau - w + \varepsilon(f(\tau) - f(w))\big)^2} \big(h(\tau) - h(w)\big) f'(\tau) d\tau.$$

Moreover, one can easily check that

$$\partial_f G_{22}(\varepsilon, f)h(w) = 2i \fint_{\mathbb{T}} \frac{\operatorname{Im}\big\{(\tau - w)(h(\overline{\tau}) - h(\overline{w}))\big\}}{(\tau - w)(\tau - w + \varepsilon(f(\tau) - f(w))} d\tau \\ - 2i\varepsilon \fint_{\mathbb{T}} \frac{\operatorname{Im}\big\{(\tau - w)(f(\overline{\tau}) - f(\overline{w}))\big\}}{(\tau - w)\big((\tau - w + \varepsilon(f(\tau) - f(w))\big)^2} (h(\tau) - h(w)) d\tau.$$

Again Lemma 1 applied to the kernels that appear in the Gâteaux derivatives of the functions G_{21} and G_{22} will give the existence and the continuity of the functions $\partial_f G_{21}$ and $\partial_f G_{22}$. On the other hand,

$$\partial_f G_2(0, 0)(h) = \operatorname{Im}\Big\{\Big(\partial_f G_{21}(0, 0)(h) - \partial_f G_{22}(0, 0)(h)\Big) w\Big\}.$$

In addition, using the residue theorem we get

$$\partial_f G_{21}(0, 0)(h) = \fint_{\mathbb{T}} \frac{\overline{\tau} - \overline{w}}{\tau - w} h'(\tau) d\tau = 0$$

and

$$\partial_f G_{22}(0,0)(h) = 2i \fint_{\mathbb{T}} \frac{\operatorname{Im}\{(\tau - w)(h(\overline{\tau}) - h(\overline{w}))\}}{(\tau - w)^2} d\tau = 0.$$

Consequently $\partial_f G_2(0,0)(h) = 0$. Let's now study the last function in (9)

$$\begin{aligned} G_3(\varepsilon, f) &= -\operatorname{Im}\Big\{\Big(\fint_{\mathbb{T}} \frac{\overline{\tau} + \varepsilon f(\overline{\tau})}{\varepsilon(\tau + w) + \varepsilon^2(f(\tau) + f(w)) - 2d}(1 + \varepsilon f'(\tau))d\tau\Big) w(1 + \varepsilon f'(w))\Big\} \\ &= -\operatorname{Im}\Big\{G_{31}(\varepsilon, f) w(1 + \varepsilon f'(w)\Big\}. \end{aligned}$$

So, the regularity of the function G_3 is equivalent to the regularity of the function G_{31}. Now this function is given by an integral operator with kernel

$$K(\tau, w) = \frac{\overline{\tau} + \varepsilon f(\overline{\tau})}{\varepsilon(\tau + w) + \varepsilon^2(f(\tau) + f(w)) - 2d}.$$

It is clear that $|K(\tau, w)| \le C$ and moreover

$$|\partial_w K(\tau, w)| = \Big| \frac{(\overline{\tau} + \varepsilon f(\overline{\tau}))\big(\varepsilon + \varepsilon^2 f'(w)\big)}{\big(\varepsilon(\tau + w) + \varepsilon^2(f(\tau) + f(w)) - 2d\big)^2} \Big| \le C.$$

Since $1 + \varepsilon f'(\tau)$ is in $C^\beta(\mathbb{T})$ then applying once again Lemma 1 to the above kernel we get that G_{31} is a function in $C^\beta(\mathbb{T})$. To prove that G_{31} has real coefficients one only has to repeat the arguments given in the case of the function G_{21}. Now, to check that the function $(\varepsilon, f) \mapsto G_{31}(\varepsilon, f)$ is C^1 we have to compute its partial derivatives

$$\begin{aligned} \partial_\varepsilon G_{31} &= \fint_{\mathbb{T}} \frac{f(\overline{\tau})(1 + \varepsilon f'(\tau))}{\varepsilon(\tau + w) + \varepsilon^2(f(\tau) + f(w)) - 2d} d\tau \\ &\quad + \fint_{\mathbb{T}} \frac{(\overline{\tau} + \varepsilon f(\overline{\tau}))f'(\tau)}{\varepsilon(\tau + w) + \varepsilon^2(f(\tau) + f(w)) - 2d} d\tau \\ &\quad - \fint_{\mathbb{T}} \frac{(\overline{\tau} + \varepsilon f(\overline{\tau}))(\tau + w + 2\varepsilon(f(\tau) + f(w))}{\big(\varepsilon(\tau + w) + \varepsilon^2(f(\tau) + f(w)) - 2d\big)^2}(1 + \varepsilon f'(\tau))d\tau. \end{aligned}$$

Easy computations, using Lemma 1, prove that these operators are continuous from $(-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R} \times B_1^0$ to $C^\beta(\mathbb{T})$. Since they are functions with real coefficients we can conclude that $\partial_\varepsilon G_3$ is continuous from $(-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R} \times B_1^0$ to Y^0. On the other hand, we can compute the Gâteaux derivative of G_{31} in a given direction $h \in X$

$$\begin{aligned} \partial_f G_{31}(\varepsilon, f)(h) &= \varepsilon \fint_{\mathbb{T}} \frac{h(\overline{\tau})(1 + \varepsilon f'(\tau))}{\varepsilon(\tau + w) + \varepsilon^2(f(\tau) + f(w)) - d} d\tau \\ &\quad + \varepsilon \fint_{\mathbb{T}} \frac{(\overline{\tau} + \varepsilon f(\overline{\tau}))h'(\tau)}{\varepsilon(\tau + w) + \varepsilon^2(f(\tau) + f(w)) - d} d\tau \\ &\quad - \varepsilon^2 \fint_{\mathbb{T}} \frac{(\overline{\tau} + \varepsilon f(\overline{\tau}))(h(\tau) + h(w))}{\big(\varepsilon(\tau + w) + \varepsilon^2(f(\tau) + f(w)) - d\big)^2}(1 + \varepsilon f'(\tau))d\tau. \end{aligned}$$

Again, by straightforward computations one can verify that the integral operators defined by these partial derivatives are continuous and so we obtain that $\partial_f G_3$ is continuous from $(-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R} \times B_1^0$ to Y^0. Moreover we find $\partial_f G_{31}(0,0)(h) = 0$, and consequently

$$\partial_f G_{31}(0,0)(h) = 0.$$

Since by definition $-2G^0 = G_1 + G_2 + G_3$ then we deduce that $\partial_f G^0(0, \Omega, 0)(h) = \frac{1}{2}\mathrm{Im}\{h'\}$, which is clearly an isomorphism from X^0 to $\widehat{Y}^0$. Finally we note that when $\varepsilon = 0$ one should get the two point vortices. Indeed, we can easily check that

$$G^0(0, \Omega, 0) = \mathrm{Im}\Big\{\big(\Omega d - \frac{1}{4d}\big)w\Big\}$$

and therefore $G^0(0, \Omega, 0) = 0$ if and only if

$$\Omega = \Omega^0_{sing} = \frac{1}{4d^2}.$$

Part II: case $\alpha \in (0, 1)$ Now we shall move to the proof of the statement when $\alpha \in (0, 1)$. Recall that the functional defining the rotating pairs is given in (15). We shall start with the proof of the regularity for the function G_1 which is the easiest one. The suitable extension of this function, still denoted G_1, is given by

$$G_1(\varepsilon, \Omega, f(w)) = \Omega\,\mathrm{Im}\Big\{\Big(\varepsilon w + \varepsilon^2|\varepsilon|^\alpha f(w) - d\Big)\,\overline{w}\,\Big(1 + \varepsilon|\varepsilon|^\alpha\overline{f'(w)}\Big)\Big\}. \quad (25)$$

G_1 is well-defined from $(-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R} \times B_1^\alpha$ to Y^α because, using the algebra structure of Hölder spaces with positive regularities, the function between the brackets belongs to $C^{1-\alpha}(\mathbb{T})$ and has real Fourier coefficients. To prove that this functional is C^1 it suffices to check that the partial derivatives exist and are continuous. It is clear that

$$\partial_\varepsilon G_1(\varepsilon, \Omega, f(w)) = \Omega\,\mathrm{Im}\Big\{\Big(w + (2\varepsilon|\varepsilon|^\alpha + \alpha\,\mathrm{sign}(\varepsilon)|\varepsilon|^{1+\alpha})f(w)\Big)\,\overline{w}\,\Big(1 + \varepsilon|\varepsilon|^\alpha\overline{f'(w)}\Big)\Big\}$$
$$+\,\Omega(|\varepsilon|^\alpha + \alpha\,\mathrm{sign}(\varepsilon)\varepsilon|\varepsilon|^{\alpha-1})\mathrm{Im}\Big\{\Big(\varepsilon w + \varepsilon^2|\varepsilon|^\alpha f(w) - d\Big)\,\overline{w}\,\overline{f'(w)}\Big\}$$

This function is polynomial in f and f' and therefore it is continuous from $(-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R} \times B_1^\alpha$ to Y^α. On the other hand, the partial derivative with respect to Ω it is given by

$$\partial_\Omega G_1(\varepsilon, \Omega, f(w)) = \mathrm{Im}\Big\{\Big(\varepsilon w + \varepsilon^2|\varepsilon|^\alpha f(w) - d\Big)\,\overline{w}\,\Big(1 + \varepsilon|\varepsilon|^\alpha\overline{f'(w)}\Big)\Big\}$$

and this is obviously continuous from $(-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R} \times B_1^\alpha$ to Y^α. Note also that G_1 is polynomial with respect to f and f' and consequently $\partial_f G_1$ exists and is continuous. This concludes the fact that G_1 is C^1. It is easy to check that for any direction $h \in X^\alpha$

$$\partial_f G_1(0, \Omega, 0)(h) = 0. \quad (26)$$

For the remaining functionals the situation is much more complicated. As we shall see the reasoning is very classical and we will give just some significant details. We first start with the term G_3. To find the suitable extension note that the ansatz of the solution is very crucial and allows to get rid of the singularity in ε. Recall that

$$G_3(\varepsilon, f(w)) = \mathrm{Im}\Big\{I_2(\varepsilon, f(w))L(\varepsilon, f(w))\Big\} \quad \text{with} \quad L(\varepsilon, f(w)) = \overline{w}\,\Big(1 + \varepsilon|\varepsilon|^\alpha\overline{f'(w)}\Big) \quad (27)$$

where we have extended the tangent vector to L and as previously,

$$L : (-\frac{1}{2}, \frac{1}{2}) \times B_1^\alpha \to Y^\alpha$$

is well-defined and is of class C^1. Therefore it suffices to prove that I_2 can be extended from $(-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R} \times B_1^\alpha$ to Y^α as a C^1 function. The key point is Taylor formula:

$$\frac{1}{|A+B|^\alpha} = \frac{1}{|A|^\alpha} - \alpha \int_0^1 \frac{\mathrm{Re}(A\overline{B}) + t|B|^2}{|A+tB|^{2+\alpha}} dt \tag{28}$$

which is true for any complex numbers A, B such that $B| < |A|$. As an application we get

$$\begin{aligned} &\frac{1}{|\varepsilon\phi(w) + \varepsilon\phi(\tau) - 2d|^\alpha} \\ &= \frac{1}{(2d)^\alpha} - \alpha \int_0^1 \frac{-2d\,\varepsilon\mathrm{Re}\big[\phi(\overline{\tau}) + \phi(\overline{w})\big] + t\varepsilon^2|\phi(\tau) + \phi(w)|^2}{|t\varepsilon\phi(w) + t\varepsilon\phi(\tau) - 2d|^{2+\alpha}} dt. \end{aligned}$$

We mention that the condition $|B| < |A|$ is satisfied because

$$\begin{aligned} \big|\varepsilon\phi(w) + \varepsilon\phi(\tau)\big| &\le 2\varepsilon\|\phi\|_{L^\infty} \\ &\le 4\varepsilon \\ &< d. \end{aligned}$$

Consequently

$$I_2(\varepsilon, f(w)) = -\alpha C_\alpha \fint_{\mathbb{T}} \int_0^1 \frac{-2d\,\mathrm{Re}\big[\phi(\overline{\tau}) + \phi(\overline{w})\big] + t\varepsilon|\phi(\tau) + \phi(w)|^2}{|t\varepsilon\phi(w) + t\varepsilon\phi(\tau) - 2d|^{2+\alpha}} \phi'(\tau) d\tau dt,$$

where we have used the fact

$$\fint_{\mathbb{T}} \phi'(\tau) d\tau = 0.$$

Thus the suitable extension of this functional is

$$\begin{aligned} I_2(\varepsilon, f(w)) &= -\alpha C_\alpha \fint_{\mathbb{T}} \int_0^1 \frac{-2d\,\mathrm{Re}\big[\phi(\overline{\tau}) + \phi(\overline{w})\big] + t\varepsilon|\phi(\tau) + \phi(w)|^2}{|t\varepsilon\phi(w) + t\varepsilon\phi(\tau) - 2d|^{2+\alpha}} \phi'(\tau) d\tau dt \\ &\equiv -\alpha C_\alpha \fint_{\mathbb{T}} \int_0^1 \mathcal{K}_2(\tau, w)\phi'(\tau) d\tau, \end{aligned} \tag{29}$$

with $\phi(w) = w + \varepsilon|\varepsilon|^\alpha f(w)$. We shall now check that this extension defines a C^1 function from $(-\frac{1}{2}, \frac{1}{2}) \times B_1^\alpha$ to Y^α. First, the integral operator is well-defined since the kernel $\mathcal{K}_2$ is not singular and satisfies the hypotheses of Lemma 1 for any $f \in B_1^\alpha$,

$$|\mathcal{K}_2(\tau, w)| \le C \quad \text{and} \quad |\partial_w \mathcal{K}_2(\tau, w)| \le C$$

for some constant C and thus

$$\|I_2(\varepsilon, f)\|_{C^{1-\alpha}(\mathbb{T})} \le C\|\phi'\|_{L^\infty} \le C.$$

Taking $(\varepsilon, f) = (0,0)$ in (29) yields

$$\begin{aligned} I_2(0,0) &= \frac{\alpha C_\alpha}{(2d)^{1+\alpha}} \fint_{\mathbb{T}} \int_0^1 \overline{\tau} d\tau dt \\ &= \frac{\alpha C_\alpha}{2(2d)^{1+\alpha}}. \end{aligned} \tag{30}$$

In addition, for $f \in B_1^\alpha$ and $h \in X^\alpha$ one has

$$\begin{aligned} \partial_f I_2(0,f)h(w) &= \frac{d}{ds} I_2(0, f(w) + s\,h(w))_{|s=0} \\ &= 0. \end{aligned}$$

As $\partial_f L(0,f) = 0$ then we deduce

$$\partial_f G_3(0,f) = 0. \tag{31}$$

For a future use, we shall apply once again (28) to $I_2(\varepsilon, f)$ in order to get

$$\begin{aligned} I_2(\varepsilon, f(w)) = \frac{\alpha C_\alpha}{(2d)^{1+\alpha}} \fint_{\mathbb{T}} \mathrm{Re}\big(\phi(\overline{\tau})\big)\phi'(\tau)d\tau - \frac{\alpha C_\alpha}{(2d)^{2+\alpha}} \frac{\varepsilon}{2} \fint_{\mathbb{T}} |\phi(\tau) + \phi(w)|^2 \phi'(\tau) d\tau \\ + \alpha C_\alpha (2+\alpha)\varepsilon \fint_{\mathbb{T}} K_2(\tau, w)\phi'(\tau)d\tau, \end{aligned} \tag{32}$$

$$\begin{aligned} &K_2(\tau, w) \\ &= \iint_{[0,1]^2} \frac{\Big(-2d\,\mathrm{Re}\,\Phi(\tau,w) + t\varepsilon|\Phi(\tau,w)|^2\Big)\Big(-2d\,t\,\mathrm{Re}\,\Phi(\tau,w) + st\varepsilon|\Phi(\tau,w)|^2\Big)}{|st\varepsilon\phi(w) + st\varepsilon\phi(\tau) - 2d|^{4+\alpha}} dtds, \end{aligned}$$

with

$$\Phi(\tau, w) = \phi(\tau) + \phi(w).$$

On the other hand, since $\phi(w) = w + \varepsilon|\varepsilon|^\alpha f(w)$

$$\fint_{\mathbb{T}} \mathrm{Re}\big(\phi(\overline{\tau})\big)\phi'(\tau)d\tau = \frac{1}{2} + \varepsilon|\varepsilon|^\alpha \fint_{\mathbb{T}} \mathrm{Re}(\tau) f'(\tau)d\tau + \varepsilon|\varepsilon|^\alpha \fint_{\mathbb{T}} \mathrm{Re}\big(f(\tau)\big)\phi'(\tau)d\tau.$$

Therefore one gets

$$I_2(\varepsilon, f(w)) = \frac{\alpha C_\alpha}{2(2d)^{1+\alpha}} + \varepsilon\,\mathcal{I}_2(\varepsilon, f(w)). \tag{33}$$

Using that the kernel K_2 satisfies the conditions of Lemma 1, that is,

$$|K_2(\tau, w)| \le C, \quad |\partial_w K_2(\tau, w)| \le C,$$

one can verify that the function $I_2 : (-\frac{1}{2}, \frac{1}{2}) \times B_1^\alpha \to C^{1-\alpha}(\mathbb{T})$ is well-defined. To prove that it is indeed of class C^1 we shall look for its derivatives and study their continuity. The computations are straightforward and resemble to those done for Euler equations and thus we will skip the details. Inserting the formula (33) into the expression of G_3 allows to get the decomposition

$$G_3(\varepsilon, f) \equiv -\frac{\alpha C_\alpha}{2(2d)^{1+\alpha}} e_1(w) - \varepsilon \mathcal{R}_2(\varepsilon, f) \tag{34}$$

with $\mathcal{R}_2 : (-\frac{1}{2}, \frac{1}{2}) \times B_1^\alpha \to C^{1-\alpha}(\mathbb{T})$ being a C^1 function. Let us now move to the extension of the function G_2 defined in (23). We can split an extension of $I_1(\varepsilon, f)$ into three parts as follows,

$$\begin{aligned} I_1(\varepsilon, f(w)) &= C_\alpha \fint_{\mathbb{T}} \frac{f'(\tau)d\tau}{|\phi(w)-\phi(\tau)|^\alpha} + \frac{C_\alpha}{\varepsilon|\varepsilon|^\alpha}\fint_{\mathbb{T}} \frac{d\tau}{|\tau-w|^\alpha} \\ &+ \frac{C_\alpha}{\varepsilon|\varepsilon|^\alpha}\fint_{\mathbb{T}} \Big(\frac{1}{|\phi(w)-\phi(\tau)|^\alpha} - \frac{1}{|\tau-w|^\alpha}\Big)d\tau \\ &= C_\alpha \fint_{\mathbb{T}} \frac{f'(\tau)d\tau}{|\phi(w)-\phi(\tau)|^\alpha} + \frac{\widehat{\mu}_\alpha\, w}{\varepsilon|\varepsilon|^\alpha} + C_\alpha \fint_{\mathbb{T}} K(\varepsilon,\tau,w)d\tau \\ &= I_{11}(\varepsilon, f(w)) + \frac{\widehat{\mu}_\alpha\, w}{\varepsilon|\varepsilon^\alpha|} + I_{12}(\varepsilon, f(w)) \end{aligned} \tag{35}$$

with

$$K(\varepsilon,\tau,w) = \frac{1}{\varepsilon|\varepsilon|^\alpha}\Big(\frac{1}{|\phi(w)-\phi(\tau)|^\alpha} - \frac{1}{|\tau-w|^\alpha}\Big)$$

and

$$\widehat{\mu}_\alpha = \frac{\alpha\Gamma(1-\alpha)}{(2-\alpha)\Gamma^2(1-\frac{\alpha}{2})}C_\alpha = \frac{\Gamma(1-\alpha)\Gamma(1+\frac{\alpha}{2})}{2^{1-\alpha}\Gamma(2-\frac{\alpha}{2})\Gamma^2(1-\frac{\alpha}{2})}. \tag{36}$$

Note that we have used the identity, see [20, Lemma 2]

$$\fint_{\mathbb{T}} \frac{d\tau}{|\tau-w|^\alpha} = \frac{\alpha\Gamma(1-\alpha)}{(2-\alpha)\Gamma^2(1-\frac{\alpha}{2})}w.$$

Consequently,

$$\begin{aligned} G_2(\varepsilon, f(w)) &= \mathrm{Im}\Big\{I_{11}(\varepsilon, f(w))\overline{w}\Big(1+\varepsilon|\varepsilon|^\alpha\overline{f'(w)}\Big)\Big\} \\ &+ \mathrm{Im}\Big\{I_{12}(\varepsilon, f(w))\overline{w}\Big(1+\varepsilon|\varepsilon|^\alpha\overline{f'(w)}\Big)\Big\} - \widehat{\mu}_\alpha \mathrm{Im}(f'(w)). \end{aligned} \tag{37}$$

The last term defines a linear operator from X^α to Y^α and therefore it is smooth. It remains to study the first and second terms. This amounts to studying the terms I_{11} and I_{12}. The first part is extended as usual through the formula

$$I_{11}(\varepsilon, f(w)) = C_\alpha \fint_{\mathbb{T}} \frac{f'(\tau)\,d\tau}{\big|\phi(\tau)-\phi(w)\big|^\alpha}, \quad \text{with } \phi(w) = w + \varepsilon|\varepsilon|^\alpha f(w).$$

First to check that I_{11} is well-defined we use Corollary 1 which implies that

$$\begin{aligned} \|I_{11}(\varepsilon, f)\|_{C^{1-\alpha}(\mathbb{T})} &\le C\|f'\|_{L^\infty} \\ &\le C\|f\|_{C^{2-\alpha}(\mathbb{T})}. \end{aligned}$$

Now we shall prove that I_{11} is C^1 and for this purpose it suffices to check the existence of the partial derivatives and their continuity in strong topology. The partial derivative with respect to ε can be easily computed and we find

$$\partial_\varepsilon I_{11}(\varepsilon, f(w)) = -\alpha C_\alpha(|\varepsilon|^\alpha + \alpha\varepsilon\text{sign}(\varepsilon)|\varepsilon|^{\alpha-1}) \fint_{\mathbb{T}} \frac{\text{Re}\big[(\overline{\tau}-\overline{w})(f(\tau)-f(w))\big]}{\big|\phi(\tau)-\phi(w)\big|^{2+\alpha}} f'(\tau)\,d\tau$$
$$-\alpha C_\alpha \varepsilon|\varepsilon|^\alpha\big(|\varepsilon|^\alpha + \alpha\varepsilon\text{sign}(\varepsilon)|\varepsilon|^{\alpha-1}\big) \fint_{\mathbb{T}} \frac{|f(\tau)-f(w)|^2}{\big|\phi(\tau)-\phi(w)\big|^{2+\alpha}} f'(\tau)\,d\tau.$$

Introduce the kernels

$$K_1(\tau, w) = \frac{\text{Re}\big[(\overline{\tau}-\overline{w})(f(\tau)-f(w))\big]}{\big|\phi(\tau)-\phi(w)\big|^{2+\alpha}}, \quad K_2(\tau, w) = \frac{|f(\tau)-f(w)|^2}{\big|\phi(\tau)-\phi(w)\big|^{2+\alpha}}.$$

Then for $\tau \neq w$

$$|K_1(\tau, w)| \leq \frac{\|f\|_{Lip}}{\|\phi^{-1}\|_{Lip}^{2+\alpha}}|\tau - w|^{-\alpha}$$
$$\leq C|\tau - w|^{-\alpha}.$$

and in a similar way

$$|K_2(\tau, w)| \leq C|\tau - w|^{-\alpha}.$$

Moreover

$$|\partial_w K_1(\tau, w)| + |\partial_w K_2(\tau, w)| \leq C|\tau - w|^{-1-\alpha}.$$

Therefore using Lemma 1 we deduce that $\partial_\varepsilon I_{11}(\varepsilon, f) \in C^{1-\alpha}(\mathbb{T})$ and the dependence on $(\varepsilon, f) \in (-\frac{1}{2}, \frac{1}{2}) \times B_1^\alpha$ is continuous. More details in a similar context can be found in [20]. The partial derivative with respect to $f \in X^\alpha$ in the direction $h \in X^\alpha$ is given by

$$\partial_f I_{11}(\varepsilon, f)h = C_\alpha \fint_{\mathbb{T}} \frac{h'(\tau)d\tau}{\big|\phi(\tau)-\phi(w)\big|^\alpha}$$
$$-\alpha C_\alpha \varepsilon|\varepsilon|^\alpha \fint_{\mathbb{T}} \frac{\text{Re}\big[(\phi(\tau)-\phi(w))(h(\overline{\tau})-h(\overline{w}))\big]f'(\tau)}{\big|\phi(\tau)-\phi(w)\big|^{2+\alpha}}\,d\tau$$

which is continuous from $(-\frac{1}{2}, \frac{1}{2}) \times B_1^\alpha$ to $C^{1-\alpha}(\mathbb{T})$. In particular we get for any $h \in X^\alpha$

$$\partial_f I_{11}(0, 0)h = C_\alpha \fint_{\mathbb{T}} \frac{h'(\tau)}{\big|\tau - w\big|^\alpha}\,d\tau. \tag{38}$$

We shall now move to the extension and the regularity of I_{12} defined in (35). It can be extended through its kernel as follows,

$$K(\varepsilon, \tau, w) = \frac{1}{\varepsilon|\varepsilon|^\alpha}\Big(\frac{1}{|\phi(w)-\phi(\tau)|^\alpha} - \frac{1}{|\tau - w|^\alpha}\Big), \quad \phi(w) = w + \varepsilon|\varepsilon|^\alpha f(w).$$

Now using (28) we find that

$$K(\varepsilon,\tau,w)=-\alpha\int_0^1\frac{\operatorname{Re}\Big((\overline{\tau}-\overline{w})\big(f(\tau)-f(w)\big)\Big)}{|\tau-w+t\varepsilon|\varepsilon|^\alpha(f(\tau)-f(w))|^{2+\alpha}}dt$$
$$-\alpha\varepsilon|\varepsilon|^\alpha\int_0^1\frac{t|f(\tau)-f(w)|^2}{|\tau-w+t\varepsilon|\varepsilon|^\alpha(f(\tau)-f(w))|^{2+\alpha}}dt. \tag{39}$$

By straightforward computations we can check that there exists an absolute constant C such that for $(\varepsilon,f)\in(-\frac{1}{2},\frac{1}{2})\times\in B_1^\alpha$ and for $\tau\neq w$

$$|K(\varepsilon,\tau,w)|\leq\frac{C}{|\tau-w|^\alpha}\quad\text{and}\quad|\partial_w K(\varepsilon,\tau,w)|\leq\frac{C}{|\tau-w|^{1+\alpha}}\cdot$$

Therefore using Lemma 1 again we can conclude that $I_{12}(\varepsilon,f)$ is well-defined and belongs to $C^{1-\alpha}(\mathbb{T})$. The regularity with respect to ε is straightforward since $\varepsilon\mapsto K(\varepsilon,\tau,w)$ is C^1 and

$$|\partial_\varepsilon K(\varepsilon,\tau,w)|\leq\frac{C}{|\tau-w|^\alpha}\quad\text{and}\quad|\partial_w\partial_\varepsilon K(\varepsilon,\tau,w)|\leq\frac{C}{|\tau-w|^{1+\alpha}}$$

which implies that $\partial_\varepsilon I_{12}(\varepsilon,f)$ is well-defined for $(\varepsilon,f)\in(-\frac{1}{2},\frac{1}{2})\times\in B_1^\alpha$ and it belongs to the space $C^{1-\alpha}(\mathbb{T})$. The continuity can be done in a similar way. Moreover by (39) we have that

$$I_{12}(0,0)=0. \tag{40}$$

The existence of partial derivative with respect to f can be done without difficulty and we can check that this derivative is continuous. Thus we establish that I_{12} is C^1 and in particular we deduce that

$$\partial_f I_{12}(0,0)h(w)=-\alpha C_\alpha\fint_{\mathbb{T}}\frac{\operatorname{Re}\Big((\overline{\tau}-\overline{w})\big(h(\tau)-h(w)\big)\Big)}{|\tau-w|^{2+\alpha}}d\tau. \tag{41}$$

Putting together (37), (38) and (41) we find for $h\in X^\alpha$,

$$\partial_f G_2(0,0)h(w)=C_\alpha\operatorname{Im}\bigg\{\overline{w}\fint_{\mathbb{T}}\frac{h'(\tau)d\tau}{|\tau-w|^\alpha}\bigg\}-\widehat{\mu}_\alpha\operatorname{Im}\{h'(w)\}$$
$$-\alpha C_\alpha\operatorname{Im}\bigg\{\overline{w}\fint_{\mathbb{T}}\frac{\operatorname{Re}\Big((\overline{\tau}-\overline{w})\big(h(\tau)-h(w)\big)\Big)}{|\tau-w|^{2+\alpha}}d\tau\bigg\}. \tag{42}$$

According to (26), (31) and (42) we get for any $\Omega\in\mathbb{R}$ and $h\in X^\alpha$,

$$\begin{aligned}\partial_f G^\alpha(0,\Omega,0)h(w)&=\partial_f G_1(0,\Omega,0)h(w)-\partial_f G_2(0,0)h(w)+\partial_f G_3(0,0)h(w)\\&=-C_\alpha\operatorname{Im}\bigg\{\overline{w}\fint_{\mathbb{T}}\frac{h'(\tau)d\tau}{|\tau-w|^\alpha}\bigg\}+\widehat{\mu}_\alpha\operatorname{Im}\{h'(w)\}\\&\quad+\alpha C_\alpha\operatorname{Im}\bigg\{\overline{w}\fint_{\mathbb{T}}\frac{\operatorname{Re}\Big((\overline{\tau}-\overline{w})\big(h(\tau)-h(w)\big)\Big)}{|\tau-w|^{2+\alpha}}d\tau\bigg\}\\&\equiv\mathcal{L}_1h+\mathcal{L}_2h+\mathcal{L}_3h.\end{aligned}$$

Finally note that the extension for G is obtained by putting together (15), (25),(27), (29), (35),(37) and (39). Note that to obtain the point vortex pairs we write

$$G^\alpha(0,\Omega,0)=\Big(\Omega d-\frac{\alpha C_\alpha}{2(2d)^{1+\alpha}}\Big)e_1$$

which implies that this is a solution if and only if

$$\Omega=\Omega^\alpha_{sing}\equiv\frac{\widehat{C}_\alpha}{(2d)^{2+\alpha}}.$$

This means that two point vortices $\pi\delta_0$ and $\pi\delta_{2d}$ rotate uniformly about their center $(d,0)$ with the angular velocity Ω^α_{sing}.

It remains to compute explicitly $\partial_f G^\alpha(0,\Omega,0)$ and show that it is an isomorphism from X^α to Y^α. To this aim we will start computing $\partial_f G_2(0,0)$ whose expression is given in (42). It is easy to see that

$$\begin{aligned}\alpha C_\alpha\overline{w}\fint_{\mathbb{T}}\frac{\mathrm{Re}\Big((\overline{\tau}-\overline{w})\big(h(\tau)-h(w)\big)\Big)}{|\tau-w|^{2+\alpha}}d\tau&=\frac{1}{2}\alpha C_\alpha\overline{w}\fint_{\mathbb{T}}\frac{(\tau-w)(h(\overline{\tau})-h(\overline{w}))}{|\tau-w|^{2+\alpha}}d\tau\\&\quad+\frac{1}{2}\alpha C_\alpha\overline{w}\fint_{\mathbb{T}}\frac{(\overline{\tau}-\overline{w})(h(\tau)-h(w))}{|\tau-w|^{2+\alpha}}d\tau\\&\equiv I_4(h(w))+I_5(h(w)).\end{aligned}$$

According [20, p. 360] these terms were computed and take the form

$$\mathrm{I}_4(h(w))=\frac{\alpha(1+\frac{\alpha}{2})}{2(2-\alpha)}\frac{C_\alpha\Gamma(1-\alpha)}{\Gamma^2(1-\alpha/2)}\sum_{n\geq1}a_n\left(1-\frac{\big(2+\frac{\alpha}{2}\big)_n}{\big(2-\frac{\alpha}{2}\big)_n}\right)w^{n+1}\tag{43}$$

and

$$\mathrm{I}_5(h(w))=-\frac{\alpha C_\alpha\Gamma(1-\alpha)}{4\Gamma^2(1-\alpha/2)}\sum_{n\geq1}a_n\left(1-\frac{\big(\frac{\alpha}{2}\big)_n}{\big(-\frac{\alpha}{2}\big)_n}\right)\overline{w}^{n+1}.\tag{44}$$

It follows that

$$\mathcal{L}_3h(w)=\mathrm{Im}\{I_4(h(w))+I_5(h(w))\}=\frac{\alpha C_\alpha\Gamma(1-\alpha)}{4\Gamma^2(1-\alpha/2)}\sum_{n\geq1}a_n\beta_ne_{n+1}.\tag{45}$$

with

$$\begin{aligned}\beta_n&=\left(1-\frac{\big(\frac{\alpha}{2}\big)_n}{\big(-\frac{\alpha}{2}\big)_n}\right)+\frac{1+\frac{\alpha}{2}}{1-\frac{\alpha}{2}}\left(1-\frac{\big(2+\frac{\alpha}{2}\big)_n}{\big(2-\frac{\alpha}{2}\big)_n}\right)\\&=\left(1-\frac{\big(\frac{\alpha}{2}\big)_n}{\big(-\frac{\alpha}{2}\big)_n}\right)+\frac{1+\frac{\alpha}{2}}{1-\frac{\alpha}{2}}-\frac{\big(1+\frac{\alpha}{2}\big)_{n+1}}{\big(1-\frac{\alpha}{2}\big)_{n+1}}\\&=\frac{2}{1-\frac{\alpha}{2}}+\frac{\big(1+\frac{\alpha}{2}\big)_{n-1}}{\big(1-\frac{\alpha}{2}\big)_{n-1}}-\frac{\big(1+\frac{\alpha}{2}\big)_{n+1}}{\big(1-\frac{\alpha}{2}\big)_{n+1}}.\end{aligned}\tag{46}$$

Regarding the first term $\mathcal{L}_1(h(w))$ it may be rewritten in the form

$$\mathcal{L}_1(h(w)) = \mathrm{Im}\big\{\mathrm{I}_3(h(w))\big\}$$

with

$$\begin{aligned}\mathrm{I}_3(h(w)) &\equiv -C_\alpha \overline{w} \fint_{\mathbb{T}} \frac{h'(\tau)}{|w-\tau|^\alpha} d\tau\\ &= C_\alpha \sum_{n\geq 1} n a_n \overline{w} \fint_{\mathbb{T}} \frac{\overline{\tau}^{n+1}}{|w-\tau|^\alpha} d\tau.\end{aligned}$$

Once again we get in view of [20, p.360] that

$$\mathrm{I}_3(h(w)) = \frac{C_\alpha \Gamma(1-\alpha)}{\Gamma^2(1-\alpha/2)} \sum_{n\geq 1} n a_n \frac{\left(\frac{\alpha}{2}\right)_n}{\left(1-\frac{\alpha}{2}\right)_n} \overline{w}^{n+1}. \tag{47}$$

Therefore

$$\mathcal{L}_1(h(w)) = -\frac{C_\alpha \Gamma(1-\alpha)}{\Gamma^2(1-\alpha/2)} \sum_{n\geq 1} n a_n \frac{\left(\frac{\alpha}{2}\right)_n}{\left(1-\frac{\alpha}{2}\right)_n} e_{n+1}. \tag{48}$$

For $\mathcal{L}_2$ we readily get by (36),

$$\begin{aligned}\mathcal{L}_2 h(w) &= \widehat{\mu}_\alpha \sum_{n\geq 1} n a_n e_{n+1}\\ &= \frac{\alpha C_\alpha \Gamma(1-\alpha)}{(2-\alpha)\Gamma^2(1-\frac{\alpha}{2})} \sum_{n\geq 1} n a_n e_{n+1}.\end{aligned}$$

Putting together the preceding identities yields to

$$\begin{aligned}\partial_f G^\alpha(0,\Omega,0)h(w) &= \mathcal{L}_1 h(w) + \mathcal{L}_2 h(w) + \mathcal{L}_3 h(w)\\ &= \frac{\alpha C_\alpha \Gamma(1-\alpha)}{4\Gamma^2(1-\frac{\alpha}{2})} \sum_{n\geq 1} a_n \gamma_n e_{n+1},\end{aligned} \tag{49}$$

with

$$\begin{aligned}\gamma_n &= \beta_n - \frac{4}{\alpha} \frac{\left(\frac{\alpha}{2}\right)_n}{\left(1-\frac{\alpha}{2}\right)_n} n + \frac{4}{2-\alpha} n\\ &= \beta_n - 2n \frac{\left(1+\frac{\alpha}{2}\right)_{n-1}}{\left(1-\frac{\alpha}{2}\right)_n} + \frac{2n}{1-\frac{\alpha}{2}}\\ &= \frac{2(1+n)}{1-\frac{\alpha}{2}} + \frac{\left(1+\frac{\alpha}{2}\right)_{n-1}}{\left(1-\frac{\alpha}{2}\right)_{n-1}} - \frac{\left(1+\frac{\alpha}{2}\right)_{n+1}}{\left(1-\frac{\alpha}{2}\right)_{n+1}} - \frac{2n}{n-\frac{\alpha}{2}} \frac{\left(1+\frac{\alpha}{2}\right)_{n-1}}{\left(1-\frac{\alpha}{2}\right)_{n-1}}\\ &= \frac{2(1+n)}{1-\frac{\alpha}{2}} - \frac{\left(1+\frac{\alpha}{2}\right)_n}{\left(1-\frac{\alpha}{2}\right)_n} - \frac{\left(1+\frac{\alpha}{2}\right)_{n+1}}{\left(1-\frac{\alpha}{2}\right)_{n+1}}.\end{aligned}$$

Now we shall prove that $\partial_f G^\alpha(0,\Omega,0): X^\alpha \to \widehat{Y}^\alpha$ is an isomorphism. The case $\alpha=0$ is elementary since

$$\partial_f G^0(0,\Omega,0)h(w)=\frac{1}{2}\mathrm{Im}(h'(w))$$

and one can easily check that this operator is an isomorphism from X^0 to $\widehat{Y}^0$. So it remains to check the case $\alpha\in(0,1)$. Thus to verify that $\partial_f G^\alpha(0,\Omega,0)$ is one-to-one it is enough to prove the following: There exist two constants $C_1>0$ and $C_2>0$ such that for any $n\geq 1$

$$C_1 n\leq \gamma_n\leq C_2 n. \tag{50}$$

It easy to check that

$$\frac{(1+\frac{\alpha}{2})_n}{(1-\frac{\alpha}{2})_n}<\frac{n+\frac{\alpha}{2}}{1-\frac{\alpha}{2}}\beta,$$

where $\beta=\frac{1+\frac{\alpha}{2}}{2-\frac{\alpha}{2}}<1$. Therefore we deduce by simple computations that for $\alpha\in[0,1]$

$$\gamma_n>\frac{2(1+n)}{1-\frac{\alpha}{2}}-\beta\frac{n+\frac{\alpha}{2}}{1-\frac{\alpha}{2}}-\beta\frac{n+1+\frac{\alpha}{2}}{1-\frac{\alpha}{2}}\geq C_1(\alpha)n.$$

On the other hand, we readily get

$$\gamma_n\leq C_2(\alpha)n,$$

and hence the proof of (50) is achieved. It remains to prove that $\partial_f G^\alpha(0,\Omega,0)$ is onto. Let $g\in\widehat{Y}^\alpha$, we shall prove that the equation $\partial_f G^\alpha(0,\Omega,0)h=g$ admits a solution $h\in X^\alpha$. The functions g and h take the form

$$g(w)=\frac{\widehat{C}_\alpha\Gamma(1-\alpha)}{4\Gamma^2(1-\frac{\alpha}{2})}\sum_{n\geq 1}b_n e_{n+1}(w)\quad\text{and}\quad h(w)=\sum_{n\geq 1}a_n\overline{w}^n.$$

Therefore using (49) and (50) the equation $\partial_f G^\alpha(0,\Omega,0)h=g$ is equivalent to

$$a_n=\frac{b_n}{\gamma_n},\quad n\geq 1.$$

The only point to check is $h\in C^{2-\alpha}(\mathbb{T})$, that is

$$w\in\mathbb{T}\mapsto\sum_{n\geq 1}\frac{b_n}{\gamma_n}\overline{w}^n\in C^{2-\alpha}(\mathbb{T}).$$

From [20, p. 358], there exists a constant $C>0$ such that

$$\frac{\big(1+\frac{\alpha}{2}\big)_n}{\big(1-\frac{\alpha}{2}\big)_n}=Cn^\alpha+O\Big(\frac{1}{n^{1-\alpha}}\Big)$$

which implies in turn that

$$\gamma_n=n\Big(\frac{2}{1-\frac{\alpha}{2}}-2C\frac{1}{n^{1-\alpha}}+O\Big(\frac{1}{n^{2-\alpha}}\Big)\Big).$$

It is not difficult to show that $h \in L^\infty(\mathbb{T})$ and thus it remains to check that $h' \in C^{1-\alpha}(\mathbb{T})$. Note that

$$-wh'(w) = \sum_{n\geq 1} \frac{nb_n}{\gamma_n}\overline{w}^n$$

and

$$\begin{aligned}\frac{nb_n}{\gamma_n} &= \frac{b_n}{\frac{2}{1-\frac{\alpha}{2}} - 2C\frac{1}{n^{1-\alpha}} + O\big(\frac{1}{n^{2-\alpha}}\big)}\\ &= \frac{(1-\frac{\alpha}{2})b_n}{2} + \frac{C(1-\frac{\alpha}{2})}{n^{1-\alpha}\big(\frac{2}{1-\frac{\alpha}{2}} - 2C\frac{1}{n^{1-\alpha}}\big)}b_n + O\big(\frac{1}{n^{2-\alpha}}\big)b_n\\ &\equiv \frac{(1-\frac{\alpha}{2})b_n}{2} + \alpha_n b_n + O\big(\frac{1}{n^{2-\alpha}}\big)b_n.\end{aligned}$$

Using the continuity of Szegö projector Π in $C^{1-\alpha}(\mathbb{T})$ we obtain easily that

$$\tilde{h} : w \mapsto \sum_{n\geq 1} b_n\overline{w}^n \in C^{1-\alpha}(\mathbb{T}).$$

Define the kernels

$$K_1(w) = \sum_{n\geq 1}\alpha_n\overline{w}^n \quad \text{and} \quad K_2(w) = \sum_{n\geq 1} O\big(\frac{1}{n^{2-\alpha}}\big)\overline{w}^n.$$

The remainder term of $-wh'$ is given by

$$K_1 \star \tilde{h} + K_2 \star \tilde{h}.$$

As the kernel $K_2 \in L^\infty(\mathbb{T}) \subset L^1(\mathbb{T})$ then $K_2 \star \tilde{h} \in C^{1-\alpha}(\mathbb{T})$. In [20, p.363-366] we established that $K_1 \in L^1(\mathbb{T})$ and therefore we obtain $K_1 \star \tilde{h} \in C^{1-\alpha}(\mathbb{T})$ and this gives finally $h' \in C^{1-\alpha}(\mathbb{T})$ which concludes the proof. □

4.2. Relationship between the angular velocity and the boundary shape. As we have seen in Proposition 1 the linear operator $\partial_f G^\alpha(0, \Omega, 0)$ is an isomorphism from X^α to $\widehat{Y}^\alpha$ and not to the space Y^α. However the functional G^α has its range in Y^α which contains strictly $\widehat{Y}^\alpha$. The strategy will be to choose carefully Ω in such way that the range of G^α is contained in $\widehat{Y}^\alpha$. This condition is strong enough to uniquely determine Ω and by this way Ω plays the role of a Lagrangian multiplier with respect to the range constraint. The main result reads as follows.

Proposition 2. *Let* $\alpha \in [0, 1)$. *There exists a* C^1 *function* $\mathcal{R}^\alpha : (-\frac{1}{2}, \frac{1}{2}) \times B_1^\alpha \to \mathbb{R}$ *such that, for*

$$\Omega = \Omega^\alpha(\varepsilon, f) = \Omega^\alpha_{sing} + \mathcal{R}^\alpha(\varepsilon, f),$$

the modified function $\widehat{G}^\alpha : (-\frac{1}{2}, \frac{1}{2}) \times B_1^\alpha \to \widehat{Y}^\alpha$ *given by*

$$\widehat{G}^\alpha(\varepsilon, f) = G^\alpha\big(\varepsilon, \Omega^\alpha(\varepsilon, f), f\big)$$

is well-defined and is of class C^1. Moreover,

$$\forall\, f \in B_1^\alpha,\ \mathcal{R}^\alpha(0, f) = 0 \quad and \quad \widehat{G}^\alpha(0,0) = 0.$$

Notice that Ω^α_{sing} was defined in (24).

Proof. The proof will be separated in different parts: the case $\alpha = 0$ and the case $\alpha \in (0,1)$.

Part I: case $\alpha = 0$ A sufficient and necessary condition to guarantee that G^α admits a range contained in the space $\widehat{Y}^\alpha$ is that its first Fourier coefficients vanishes. This condition amounts to

$$\fint_{\mathbb{T}} G^\alpha(\varepsilon, \Omega, f(w))dw = 0$$

or equivalently

$$\fint_{\mathbb{T}} F^\alpha\big(\Omega, \varepsilon, f(w)\big)\big(\overline{w}^2 - 1\big)dw = 0. \tag{51}$$

We recall that F^α was defined in (9) and (13). For $\alpha = 0$ one may use the residue theorem to get

$$\fint_{\mathbb{T}} F_1\big(\Omega, \varepsilon, f(w)\big)\overline{w}^2 dw = 2\Omega\bigg(-d + \varepsilon^3 \fint_{\mathbb{T}} f(\overline{w})\overline{w} f'(w)dw\bigg)$$

and

$$\fint_{\mathbb{T}} F_1\big(\Omega, \varepsilon, f(w)\big)dw = 2\Omega\bigg(-d\varepsilon \fint_{\mathbb{T}} wf'(w)dw + \varepsilon^3 \fint_{\mathbb{T}} f(\overline{w})wf'(w)dw\bigg).$$

This last identity can be written in the form

$$\fint_{\mathbb{T}} F_1\big(\Omega, \varepsilon, f(w)\big)dw = 2\Omega\bigg(d\varepsilon \fint_{\mathbb{T}} f(w)dw + \varepsilon^3 \fint_{\mathbb{T}} f(\overline{w})wf'(w)dw\bigg).$$

Consequently

$$\fint_{\mathbb{T}} F_1\big(\Omega, \varepsilon, f(w)\big)\big(\overline{w}^2 - 1\big)dw = 2\Omega\bigg(-d\Big(1 + \varepsilon \fint_{\mathbb{T}} f(w)dw\Big) + \varepsilon^3 \fint_{\mathbb{T}} f(\overline{w}) f'(w)\big(\overline{w} - w\big)dw\bigg).$$

Now we shall evaluate the contribution of F_3. First, observe that

$$F_3(\varepsilon, f(w)) = -\widehat{F}_3(w)w\big(1 + \varepsilon f'(w)\big),$$

with the notation

$$\widehat{F}_3(\varepsilon, f(w)) \equiv \fint_{\mathbb{T}} \frac{\overline{\tau} + \varepsilon f(\overline{\tau})}{\varepsilon(\tau + w) + \varepsilon^2\big(f(\tau) + f(w)\big) - 2d}\big(1 + \varepsilon f'(\tau)\big)d\tau.$$

We write

$$
\begin{aligned}
\frac{\overline{\tau}+\varepsilon f(\overline{\tau})}{\varepsilon(\tau+w)+\varepsilon^2\big(f(\tau)+f(w)\big)-2d} &= -\frac{\overline{\tau}}{2d}+\varepsilon\frac{f(\overline{\tau})}{\varepsilon(\tau+w)+\varepsilon^2\big(f(\tau)+f(w)\big)-2d}\\
&\quad+\frac{\varepsilon}{2d}\frac{\tau+w+\varepsilon(f(\tau)+f(w))}{\varepsilon(\tau+w)+\varepsilon^2\big(f(\tau)+f(w)\big)-2d}\overline{\tau}\\
&\equiv -\frac{\overline{\tau}}{2d}+\varepsilon g_3(\varepsilon,\tau,w).
\end{aligned}
$$

Thus

$$
\widehat{F_3}(\varepsilon, f(w)) = -\frac{1}{2d}+\varepsilon\fint_{\mathbb{T}} g_3(\varepsilon,\tau,w)\big(1+\varepsilon f'(\tau)\big)d\tau.
$$

Hence

$$
\begin{aligned}
\fint_{\mathbb{T}} F_3\big(\Omega,\varepsilon,f(w)\big)\overline{w}^2dw &= \frac{1}{2d}-\varepsilon\fint_{\mathbb{T}}\fint_{\mathbb{T}} g_3(\varepsilon,\tau,w)\big(1+\varepsilon f'(\tau)\big)\overline{w}\big(1+\varepsilon f'(w)\big)d\tau dw\\
\fint_{\mathbb{T}} F_3\big(\Omega,\varepsilon,f(w)\big)dw &= -\frac{\varepsilon}{2d}\fint_{\mathbb{T}} f(\tau)d\tau-\varepsilon\fint_{\mathbb{T}}\fint_{\mathbb{T}} g_3(\varepsilon,\tau,w)\big(1+\varepsilon f'(\tau)\big)w\big(1+\varepsilon f'(w)\big)d\tau dw.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
\fint_{\mathbb{T}} F_3\big(\Omega,\varepsilon,f(w)\big)\big(\overline{w}^2-1\big)dw &= \frac{1}{2d}+\frac{\varepsilon}{2d}\fint_{\mathbb{T}} f(\tau)d\tau\\
&\quad-\varepsilon\fint_{\mathbb{T}}\fint_{\mathbb{T}} g_3(\varepsilon,\tau,w)\big(1+\varepsilon f'(\tau)\big)(\overline{w}-w)\big(1+\varepsilon f'(w)\big)d\tau dw.
\end{aligned}
$$

On the other hand using residue theorem we get

$$
\begin{aligned}
F_2(\varepsilon, f(w)) &= \varepsilon\fint_{\mathbb{T}}\frac{A\overline{B}-\overline{A}B}{A(A+\varepsilon B)}f'(\tau)d\tau\, w\big(1+\varepsilon f'(w)\big)\\
&\quad+\varepsilon\fint_{\mathbb{T}}\frac{\big(\overline{A}B-A\overline{B}\big)B}{A^2(A+\varepsilon B)}d\tau\, w\big(1+\varepsilon f'(w)\big)\\
&\equiv \varepsilon g_2(\varepsilon,w)w\big(1+\varepsilon f'(w)\big).
\end{aligned}
$$

Therefore

$$
\fint_{\mathbb{T}} F_2\big(\Omega,\varepsilon,f(w)\big)\big(\overline{w}^2-1\big)dw = \varepsilon\fint_{\mathbb{T}} g_2(\varepsilon,w)(\overline{w}-w)\big(1+\varepsilon f'(w)\big)dw.
$$

The Eq. (51) becomes

$$
\begin{aligned}
2\Omega\bigg(d\Big[1+\varepsilon\fint_{\mathbb{T}} f(w)dw\Big]-\varepsilon^3\fint_{\mathbb{T}} f(\overline{w})f'(w)\big(\overline{w}-w\big)dw\bigg) &= \frac{1}{2d}+\frac{\varepsilon}{2d}\fint_{\mathbb{T}} f(\tau)d\tau\\
&\quad+\varepsilon\fint_{\mathbb{T}} g_2(\varepsilon,w)(\overline{w}-w)\big(1+\varepsilon f'(w)\big)dw\\
&\quad+\varepsilon\fint_{\mathbb{T}}\fint_{\mathbb{T}} g_3(\varepsilon,\tau,w)\big(1+\varepsilon f'(\tau)\big)(w-\overline{w})\\
&\quad\times\big(1+\varepsilon f'(w)\big)d\tau dw\\
&\equiv \frac{1}{2d}+\frac{\varepsilon}{2d}T_1(\varepsilon,f)
\end{aligned}
$$

which can be written in the form

$$\begin{aligned}\Omega &= \Omega^0(\varepsilon, f)\\ &= \frac{1}{4d^2}\frac{1+\varepsilon T_1(\varepsilon, f)}{1-\varepsilon T_2(\varepsilon, f)}\\ &= \Omega^0_{sing} + \frac{\varepsilon}{4d^2}\frac{T_1(\varepsilon, f)+T_2(\varepsilon, f)}{1-\varepsilon T_2(\varepsilon, f)}\\ &\equiv \Omega^0_{sing} + \mathcal{R}^0(\varepsilon, f), \end{aligned}\tag{52}$$

where

$$T_2(\varepsilon, f) = -\fint_{\mathbb{T}} f(w)dw + \frac{\varepsilon^2}{d}\fint_{\mathbb{T}} f(\overline{w})f'(w)\big(\overline{w}-w\big)dw.$$

Now we intend to prove that $(\varepsilon, f)\mapsto \Omega^0(\varepsilon, f)$ is C^1. For this aim it is enough to check that the functions $(\varepsilon, f)\mapsto T_1(\varepsilon, f)$ and $(\varepsilon, f)\mapsto T_2(\varepsilon, f)$ are C^1 functions and that $|T_2(\varepsilon, f)|<2$. Since f has real coefficients it is clear that $T_2(\varepsilon, f)\in\mathbb{R}$ and

$$|T_2(\varepsilon, f)| \le \|f\|_{C^{1+\beta}(\mathbb{T})} + \frac{\varepsilon^2}{d}2\|f\|^2_{C^{1+\beta}(\mathbb{T})} < 2.$$

On the other hand, T_2 is polynomial in the variables ε, f and f' and so it should be a C^1 function from $(\frac12, \frac12)\times B_1^0$ to $\mathbb{R}$. Let's take now the functional

$$\begin{aligned}T_1(\varepsilon, f) = &\fint_{\mathbb{T}} f(\tau)d\tau + 2d\fint_{\mathbb{T}} g_2(\varepsilon, w)(\overline{w}-w)(1+\varepsilon f'(w))dw\\ &+ 2d\fint_{\mathbb{T}}\fint_{\mathbb{T}} g_3(\varepsilon, \tau, w)(1+\varepsilon f'(\tau))(w-\overline{w})(1+\varepsilon f'(\tau))d\tau dw,\end{aligned}$$

where

$$g_2(\varepsilon, f) = \fint_{\mathbb{T}}\frac{A\overline{B}-\overline{A}B}{A(A+\varepsilon B)}f'(\tau)d\tau + \fint_{\mathbb{T}}\frac{(\overline{A}B-A\overline{B})B}{A^2(A+\varepsilon B)}d\tau,$$

with $A=\tau-w$, $B=f(\tau)-f(w)$ and

$$g_3(\varepsilon, f) = \frac{f(\overline{\tau})}{\varepsilon(\tau+w)+\varepsilon^2(f(\tau)+f(w))-2d} + 2d\frac{\tau+w+\varepsilon(f(\tau)+f(w))}{\varepsilon(\tau+w)+\varepsilon^2(f(\tau)+f(w))-2d}\overline{\tau}.$$

Since $|\varepsilon|<\frac12$ and $\|f\|_{C^{1+\beta}}<1$ we get that g_3 is a bounded function. Moreover

$$\begin{aligned}|g_2(\varepsilon, f)(w)| \le\ & 2\int_{\mathbb{T}}\Big|\frac{\operatorname{Im}\{(\tau-w)(f(\overline{\tau})-f(\overline{w}))\}}{(\tau-w)(\tau-w+\varepsilon(f(\tau)-f(w)))}f'(\tau)\Big||d\tau|\\ &+2\int_{\mathbb{T}}\Big|\frac{\operatorname{Im}\{(\overline{\tau}-\overline{w})(f(\tau)-f(w))\}(f(\tau)-f(w))}{(\tau-w)^2(\tau-w+\varepsilon(f(\tau)-f(w)))}\Big||d\tau| \le C,\end{aligned}$$

where in the last inequality we use again that $|\varepsilon|<\frac12$ and $\|f\|_{C^{1+\beta}}<1$. To prove that T_1 is a C^1 function it suffices to check that the partial derivatives of $g_2(\varepsilon, f)$ and $g_3(\varepsilon, f)$ are continuous functions on $(-\frac12, \frac12)\times B_1^0$. Observe that,

$$\begin{aligned}\partial_\varepsilon g_2(\varepsilon, f) = &-2i\fint_{\mathbb{T}}\frac{\operatorname{Im}\{(\tau-w)(f(\overline{\tau})-f(\overline{w}))\}}{(\tau-w)(\tau-w+\varepsilon(f(\tau)-f(w)))^2}(f(\tau)-f(w))f'(\tau)d\tau\\ &-2i\fint_{\mathbb{T}}\frac{\operatorname{Im}\{(\overline{\tau}-\overline{w})(f(\tau)-f(w))\}}{(\tau-w)^2(\tau-w+\varepsilon(f(\tau)-f(w)))^2}(f(\tau)-f(w))^2d\tau.\end{aligned}$$

It is easy to verify that the kernels involved in the above integral operators satisfy the conditions of Lemma 1 and so we can conclude that $\partial_\varepsilon g_2(\varepsilon, f)$ is a continuous function from $(-\frac{1}{2}, \frac{1}{2}) \times B_1^0$ to $\mathbb{R}$. For any direction $h \in X^0$ straightforward computations yield

$$\begin{aligned}
\partial_f g_2(\varepsilon, f)(h) &= 2i \fint_{\mathbb{T}} \frac{\operatorname{Im}\{(\tau-w)(h(\overline{\tau})-h(\overline{w}))\}}{(\tau-w)(\tau-w+\varepsilon(f(\tau)-f(w)))} f'(\tau)d\tau \\
&+ 2i \fint_{\mathbb{T}} \frac{\operatorname{Im}\{(\tau-w)(f(\overline{\tau})-f(\overline{w}))\}}{(\tau-w)(\tau-w+\varepsilon(f(\tau)-f(w)))} h'(\tau)d\tau \\
&- 2i\varepsilon \fint_{\mathbb{T}} \frac{\operatorname{Im}\{(\tau-w)(f(\overline{\tau})-f(\overline{w}))\}}{(\tau-w)(\tau-w+\varepsilon(f(\tau)-f(w)))^2} (h(\tau)-h(w))f'(\tau)d\tau \\
&+ 2i \fint_{\mathbb{T}} \frac{\operatorname{Im}\{(\overline{\tau}-\overline{w})(h(\tau)-h(w))\}}{(\tau-w)^2(\tau-w+\varepsilon(f(\tau)-f(w)))} (f(\tau)-f(w))d\tau \\
&+ 2i \fint_{\mathbb{T}} \frac{\operatorname{Im}\{(\overline{\tau}-\overline{w})(f(\tau)-f(w))\}}{(\tau-w)^2(\tau-w+\varepsilon(f(\tau)-f(w)))} (h(\tau)-h(w))d\tau. \\
&- 2i\varepsilon \fint_{\mathbb{T}} \frac{\operatorname{Im}\{(\overline{\tau}-\overline{w})(f(\tau)-f(w))\}}{(\tau-w)^2(\tau-w+\varepsilon(f(\tau)-f(w)))^2} \\
&\times (f(\tau)-f(w))(h(\tau)-h(w))d\tau.
\end{aligned}$$

Again the kernels involved in the integral operators satisfy the conditions in Lemma 1 and so $\partial_f g_2(\varepsilon, f)(h)$ defines a continuous function from $(-\frac{1}{2}, \frac{1}{2}) \times B_1^0$ to $\mathbb{R}$. Reproducing similar computations one can prove that $g_3(\varepsilon, f)$ is a C^1 function from $(-\frac{1}{2}, \frac{1}{2}) \times B_1^0$ to $\mathbb{R}$, and so we get that the function $\Omega^0(\varepsilon, f)$ is C^1.

Part II: case $\alpha \in (0, 1)$ As in the first part of the proof, the condition imposed to Ω to guarantee that the component of e_1 in the Fourier expansion G^α vanishes is

$$\fint_{\mathbb{T}} G^\alpha\big(\varepsilon, \Omega, f(w)\big)dw = 0.$$

According to the decomposition (15) this assumption is equivalent to

$$A_1 = A_2 - A_3, \tag{53}$$

with

$$A_j = -2i \fint_{\mathbb{T}} G_j\big(\varepsilon, \Omega, f(w)\big)dw.$$

Note that A_j is the Fourier coefficient of $e_1 = \operatorname{Im}(w)$ in G_j and when $G_j = \operatorname{Im}(F_j)$ then

$$A_j = \fint_{\mathbb{T}} F_j\big(\varepsilon, \Omega, f(w)\big)\big(\overline{w}^2 - 1\big)dw.$$

Looking for the Fourier expansion of $\operatorname{Im}\left\{\Omega\Big(\varepsilon w - d\Big)\,\overline{w}\,\Big(1 + \varepsilon|\varepsilon|^\alpha \overline{f'(w)}\Big)\right\}$ we deduce that the coefficient of e_1 is

$$\Omega d\Big(1 + \varepsilon|\varepsilon|^\alpha \fint_{\mathbb{T}} f(\tau)d\tau\Big).$$

In a similar way the Fourier coefficient of e_1 in $\Omega\varepsilon^2|\varepsilon|^\alpha \mathrm{Im}\Big(f(w)\overline{w}\big(1+\varepsilon|\varepsilon|^\alpha\overline{f'(w)}\big)\Big)$ is

$$\Omega\varepsilon^3|\varepsilon|^{2\alpha}\fint_{\mathbb{T}} f(w)\overline{f'(w)}\big(\overline{w}^3-\overline{w}\big)dw.$$

Consequently,

$$\begin{aligned} A_1 &= \Omega d\Big(1+\varepsilon|\varepsilon|^\alpha\fint_{\mathbb{T}} f(\tau)d\tau\Big)\\ &\quad+\Omega\varepsilon^3|\varepsilon|^{2\alpha}\fint_{\mathbb{T}} f(w)\overline{f'(w)}\big(\overline{w}^3-\overline{w}\big)dw\\ &\equiv \Omega d+\Omega\varepsilon|\varepsilon|^\alpha\mathcal{R}_0(\varepsilon,f) \end{aligned}\tag{54}$$

with $\mathcal{R}_0:(-\frac12,\frac12)\times B_1^0\to\mathbb{R}$. We point out that for any $(\varepsilon,f)\in(-\frac12,\frac12)\times B_1^0$

$$\begin{aligned} |\mathcal{R}_0(\varepsilon,f)| &\le d\|f\|_{L^\infty}+2|\varepsilon|^{2+\alpha}\|f\|_{L^\infty}\|f'\|_{L^\infty}\\ &\le d+2|\varepsilon|^{2+\alpha} \end{aligned}\tag{55}$$

which means that the function $\mathcal{R}_0$ is well-defined. It is clear that it is differentiable with continuity in the ε-variable and moreover the function is polynomial in f and f' and so its derivative satisfies the required assumption. Therefore we conclude that $\mathcal{R}_0$ is a C^1 function from $(-\frac12,\frac12)\times B_1^\alpha$ to $\mathbb{R}$. Now we shall compute the quantity A_1 associated to G_1 which is described by (37) and (35). Thus

$$\begin{aligned} -A_2 &= \fint_{\mathbb{T}} I_{11}(\varepsilon,f(w))\big(\overline{w}-\overline{w}^3\big)dw+\fint_{\mathbb{T}} I_{12}(\varepsilon,f(w))\big(\overline{w}-\overline{w}^3\big)dw\\ &\quad+\varepsilon|\varepsilon|^\alpha\fint_{\mathbb{T}}\Big[I_{11}(\varepsilon,f(w))+I_{12}(\varepsilon,f(w))\Big]\overline{f'(w)}\big(\overline{w}-\overline{w}^3\big)dw\\ &\equiv A_{11}+A_{12}+\varepsilon|\varepsilon|^\alpha A_{13}. \end{aligned}$$

To calculate A_{11} we use (35) which yields

$$A_{11}=C_\alpha\fint_{\mathbb{T}}\fint_{\mathbb{T}}\frac{f'(\tau)\big(\overline{w}-\overline{w}^3\big)}{|\tau-w|^\alpha}dwd\tau+C_\alpha\fint_{\mathbb{T}}\fint_{\mathbb{T}} f'(\tau)\big(\overline{w}-\overline{w}^3\big)K(\tau,w)dwd\tau$$

with

$$K(\tau,w)=\frac{1}{|\phi(\tau)-\phi(w)|^\alpha}-\frac{1}{|\tau-w|^\alpha},\quad \phi(w)=w+\varepsilon|\varepsilon|^\alpha f(w).$$

From the Fourier expansion (47) we conclude that

$$\fint_{\mathbb{T}}\fint_{\mathbb{T}}\frac{f'(\tau)\big(\overline{w}-\overline{w}^3\big)}{|\tau-w|^\alpha}dwd\tau=0$$

and therefore

$$A_{11}=C_\alpha\fint_{\mathbb{T}}\fint_{\mathbb{T}} f'(\tau)\big(\overline{w}-\overline{w}^3\big)K(\tau,w)dwd\tau.$$

According to (39) we get

$$A_{11} = C_\alpha \varepsilon|\varepsilon|^\alpha \fint_{\mathbb{T}} \fint_{\mathbb{T}} f'(\tau)\big(\overline{w} - \overline{w}^3\big) K(\varepsilon, \tau, w) dw d\tau. \tag{56}$$

For the term A_{12} recall from (35) that

$$I_{12}(\varepsilon, f(w)) = C_\alpha \fint_{\mathbb{T}} K(\varepsilon, \tau, w) d\tau.$$

Combining (39) with (28) we get

$$\begin{aligned} K(\varepsilon, \tau, w) = & -\alpha \frac{\text{Re}\Big((\overline{\tau} - \overline{w})\big(f(\tau) - f(w)\big)\Big)}{|\tau - w|^{2+\alpha}} \\ & + \alpha(2+\alpha)\varepsilon|\varepsilon|^\alpha \int_0^1 \int_0^1 \frac{\widehat{K}(t, \varepsilon, \tau, w)}{|\tau - w + st\varepsilon|\varepsilon|^\alpha (f(\tau) - f(w))|^{4+\alpha}} dt ds \\ & - \alpha\varepsilon|\varepsilon|^\alpha \int_0^1 \frac{t|f(\tau) - f(w)|^2}{|\tau - w + t\varepsilon|\varepsilon|^\alpha (f(\tau) - f(w))|^{2+\alpha}} dt, \end{aligned}$$

with

$$\begin{aligned} \widehat{K}(t, \varepsilon, \tau, w) \equiv & \, \text{Re}\Big((\overline{\tau} - \overline{w})\big(f(\tau) - f(w)\big)\Big) \\ & \times \Big[t\text{Re}\Big((\overline{\tau} - \overline{w})\big(f(\tau) - f(w)\big)\Big) + st^2\varepsilon|\varepsilon|^\alpha |f(\tau) - f(w)|^2\Big]. \end{aligned}$$

Thus

$$I_{12}(\varepsilon, f(w)) = -\alpha C_\alpha \fint_{\mathbb{T}} \frac{\text{Re}\Big((\overline{\tau} - \overline{w})\big(f(\tau) - f(w)\big)\Big)}{|\tau - w|^{2+\alpha}} d\tau + \varepsilon|\varepsilon|^\alpha \mathcal{I}_{12}(\varepsilon, f(w)).$$

Using that the function $\mathcal{I}_{12}$ is a sum of terms defined by integral operators and those operators have kernels satisfying the conditions of Lemma 1, we can conclude that $I_{12} : (-\frac{1}{2}, \frac{1}{2}) \times B_1^\alpha \to Y^\alpha$ is a C^1 function. Now using (45) we deduce that

$$\fint_{\mathbb{T}} \fint_{\mathbb{T}} \frac{\text{Re}\Big((\overline{\tau} - \overline{w})\big(f(\tau) - f(w)\big)\Big)}{|\tau - w|^{2+\alpha}} (\overline{w} - \overline{w}^3) d\tau dw = 0$$

which implies that

$$\begin{aligned} A_{12} &= \fint_{\mathbb{T}} I_{12}(\varepsilon, f(w))\big(\overline{w} - \overline{w}^3\big) dw \\ &= \varepsilon|\varepsilon|^\alpha \fint_{\mathbb{T}} \mathcal{I}_{12}(\varepsilon, f(w))\big(\overline{w} - \overline{w}^3\big) dw. \end{aligned}$$

Finally we get

$$\begin{aligned} -A_2 &= A_{11} + A_{12} + \varepsilon|\varepsilon|^\alpha A_{13} \\ &= C_\alpha \varepsilon|\varepsilon|^\alpha \fint_{\mathbb{T}} \fint_{\mathbb{T}} f'(\tau)\big(\overline{w} - \overline{w}^3\big) K(\varepsilon, \tau, w) dw d\tau \\ &\quad + \varepsilon|\varepsilon|^\alpha \fint_{\mathbb{T}} \mathcal{I}_{12}(\varepsilon, f(w))\big(\overline{w} - \overline{w}^3\big) dw + \varepsilon|\varepsilon|^\alpha A_{13} \\ &\equiv -\varepsilon|\varepsilon|^\alpha \mathcal{R}_1(\varepsilon, f). \end{aligned} \tag{57}$$

Analyzing carefully all the terms in A_2, as in the foregoing cases, one may conclude that $\mathcal{R}_1 : (-\frac{1}{2}, \frac{1}{2}) \times B_1^\alpha \to \mathbb{R}$ is C^1. So, it remains to compute A_3 which is described by (53) and (34),

$$\begin{aligned} A_3 &= \fint_{\mathbb{T}} F_3(\varepsilon, f(w))(\overline{w}^2 - 1)dw \\ &= \fint_{\mathbb{T}} \frac{\alpha C_\alpha}{2(2d)^{1+\alpha}}(\overline{w}^3 - \overline{w})dw - \varepsilon\mathcal{R}_2(\varepsilon, f) \\ &= -\frac{\alpha C_\alpha}{2(2d)^{1+\alpha}} - \varepsilon\mathcal{R}_2(\varepsilon, f) \end{aligned} \tag{58}$$

and we can check that $\mathcal{R}_2 : (-\frac{1}{2}, \frac{1}{2}) \times B_1^\alpha \to \mathbb{R}$ is well-defined and C^1. Combining (53) with (54), (57) and (58) we deduce that,

$$\Omega\Big(d + \varepsilon|\varepsilon|^\alpha \mathcal{R}_0(\varepsilon, f)\Big) = \frac{\alpha C_\alpha}{2(2d)^{1+\alpha}} + \varepsilon|\varepsilon|^\alpha \mathcal{R}_1(\varepsilon, f) + \varepsilon\mathcal{R}_2(\varepsilon, f).$$

According to (55), since $d > 2$ then for any $(\varepsilon, f) \in (-\frac{1}{2}, \frac{1}{2}) \times B_1^\alpha$ we obtain

$$\begin{aligned} d + \varepsilon|\varepsilon|^\alpha \mathcal{R}_0(\varepsilon, f) &\geq d - d|\varepsilon|^{1+\alpha} - 2|\varepsilon|^{3+2\alpha} \\ &\geq \frac{d}{4}. \end{aligned}$$

Therefore

$$\begin{aligned} \Omega &= \Omega(\varepsilon, f) \\ &= \frac{\frac{\alpha C_\alpha}{2(2d)^{1+\alpha}} + \varepsilon|\varepsilon|^\alpha \mathcal{R}_1(\varepsilon, f) + \varepsilon\mathcal{R}_2(\varepsilon, f)}{d + \varepsilon|\varepsilon|^\alpha \mathcal{R}_0(\varepsilon, f)} \\ &\equiv \Omega_{sing} + \mathcal{R}^\alpha(\varepsilon, f), \qquad \Omega_{sing} = \frac{\widehat{C}_\alpha}{(2d)^{2+\alpha}}, \end{aligned}$$

where $\mathcal{R}^\alpha : (-\frac{1}{2}, \frac{1}{2}) \times B_1^\alpha \to \mathbb{R}$ is C^1 because it is obtained as an algebraic combination of C^1 functions without zeros in the denominator. Obviously $\mathcal{R}^\alpha(0, f) = 0$ for any $f \in B_1^\alpha$ and this achieves the proof of the proposition. □

4.3. Proof of the main Theorem-(i). In this section we will give a precise statement of the first part of the main theorem which describes the structure of the solution in a neighborhood of the point vortex pairs. Recall from Proposition 2 that the existence of solutions to the V-states equation can be transformed into the resolution of

$$\widehat{G}^\alpha(\varepsilon, f) = 0, \quad (\varepsilon, f) \in (-\frac{1}{2}, \frac{1}{2}) \times B_1^\alpha$$

with $\widehat{G}^\alpha$ being the functional defined by

$$\widehat{G}^\alpha\big(\varepsilon, f(w)\big) = G^\alpha\big(\varepsilon, \Omega^\alpha(\varepsilon, f), f\big)$$

and $\Omega(\varepsilon, f)$ has been introduced in Proposition 2. The main result is the following.

Proposition 3. *Let* $\alpha \in [0, 1)$*, then the following holds true.*

1. *The linear operator* $\partial_f \widehat{G}^\alpha(0,0) : X^\alpha \to \widehat{Y}^\alpha$ *is an isomorphism and*

$$\partial_f \widehat{G}^\alpha(0,0)h(w) = \sum_{n\geq 1} a_n \widehat{\gamma}_n e_{n+1}$$

with

$$\widehat{\gamma}_n = \frac{\widehat{C}_\alpha \Gamma(1-\alpha)}{4\Gamma^2(1-\frac{\alpha}{2})}\left(\frac{2(1+n)}{1-\frac{\alpha}{2}} - \frac{\big(1+\frac{\alpha}{2}\big)_n}{\big(1-\frac{\alpha}{2}\big)_n} - \frac{\big(1+\frac{\alpha}{2}\big)_{n+1}}{\big(1-\frac{\alpha}{2}\big)_{n+1}}\right).$$

2. *There exists* $\varepsilon_0 > 0$ *such that the set*

$$\Big\{(\varepsilon, f) \in [-\varepsilon_0, \varepsilon_0] \times B_1^\alpha,\ s.t.\ \ \widehat{G}^\alpha(\varepsilon, f) = 0\Big\}$$

is parametrized by one-dimensional curve $\varepsilon \in [-\varepsilon_0, \varepsilon_0] \mapsto (\varepsilon, f_\varepsilon)$ *and*

$$\forall\ \varepsilon \in [-\varepsilon_0, \varepsilon_0]\backslash\{0\},\ f_\varepsilon \neq 0.$$

3. *If* (ε, f) *is a solution then* $(-\varepsilon, \tilde{f})$ *is also a solution, where*

$$\tilde{f}(w) = f(-w), \quad \forall\ \ w \in \mathbb{T}.$$

4. *For all* $\varepsilon \in [-\varepsilon_0, \varepsilon_0]\backslash\{0\}$, *the domain* D_1^ε *is strictly convex.*

Proof. **(i)** From the composition rule

$$\partial_f \widehat{G}^\alpha(0,0)h(w) = \partial_\Omega^\alpha G^\alpha(0, \Omega_{sing}^\alpha, 0)\, \partial_f \Omega^\alpha(0,0)h(w) + \partial_f G^\alpha(0, \Omega_{sing}^\alpha, 0)h(w).$$

By virtue of the expansion in ε of $\Omega^\alpha(\varepsilon, f)$ given in Proposition 2 we deduce that

$$\begin{aligned}\partial_f \Omega^\alpha(0,0) &= \frac{d}{dt}\Omega^\alpha(0, th(w))_{|t=0}\\ &= 0\end{aligned}$$

and therefore

$$\partial_f \widehat{G}^\alpha(0,0)h(w) = \partial_f G^\alpha(0, \Omega_{sing}^\alpha, 0)h(w).$$

Combining this identity once again with Proposition 1 we deduce the desired result.

(ii) As we have seen before in Proposition 2, $\widehat{G}^\alpha : (-\frac{1}{2}, \frac{1}{2}) \times B_1^\alpha \to \widehat{Y}^\alpha$ is C^1 and the linear operator $\partial_f \widehat{G}^\alpha(0,0) : X^\alpha \to \widehat{Y}^\alpha$ is invertible. Therefore we can conclude using the implicit function theorem. It remains to check that $f_\varepsilon \neq 0$ for $\varepsilon \neq 0$. For this purpose, we will prove that for any ε small enough and for any Ω we can not get a solution with $f = 0$. So, it means that for $\varepsilon \neq 0$ we should get

$$G^\alpha(\varepsilon, \Omega, 0) \neq 0.$$

We shall start with the case $\alpha = 0$ which is much more simpler. It is easy to check from (9) that

$$F_1(\varepsilon, \Omega, 0) = 2\Omega\big(\varepsilon - d\, w\big) \quad \text{and} \quad F_2(\varepsilon, 0) = 0.$$

However to compute F_3 we proceed by Taylor expansion as follows,

$$\begin{aligned}F_3(\varepsilon,0)&=-w\fint_{\mathbb{T}}\frac{\overline{\tau}}{\varepsilon(\tau+w)-2d}d\tau\\&=w\sum_{n\in\mathbb{N}}\frac{\varepsilon^n}{(2d)^{n+1}}\fint_{\mathbb{T}}\overline{\tau}(\tau+w)^n\,d\tau\\&=\sum_{n\in\mathbb{N}}\frac{\varepsilon^n}{(2d)^{n+1}}w^{n+1},\end{aligned}$$

which gives in turn

$$F_3(\varepsilon,0)=\frac{w}{2d-\varepsilon w}\cdot \tag{59}$$

Consequently

$$G^0(\varepsilon,\Omega,0)=\mathrm{Im}\Big\{-2d\Omega w+\frac{w}{2d-\varepsilon w}\Big\}.$$

and this quantity is not zero if $\varepsilon\neq 0$ is small enough.

Let us now move to the case $\alpha\in(0,1)$. One finds using (15), (25), (37) and (40), that

$$G_1(\varepsilon,\Omega,0)=-\Omega d\,\mathrm{Im}(\overline{w})\quad\text{and}\quad G_2(\varepsilon,\Omega,0)=0.$$

To compute $G_3(\varepsilon,0)$ it is enough to calculate $I_2(\varepsilon,0)$ because $I_1(\varepsilon,0)=0$. The exact computations turn out to be much more involved. Thus we shall give the expansion of $I_2(\varepsilon,0)$ at the order one in ε. Applying (32) one gets

$$\begin{aligned}I_2(\varepsilon,0)&=\frac{\alpha C_\alpha}{2(2d)^{1+\alpha}}-\frac{\alpha C_\alpha\varepsilon}{2(2d)^{2+\alpha}}\fint_{\mathbb{T}}|\tau+w|^2d\tau\\&\quad+\frac{\alpha C_\alpha(2+\alpha)\varepsilon}{2(2d)^{2+\alpha}}\fint_{\mathbb{T}}\Big(\mathrm{Re}\big[\tau+w\big]\Big)^2d\tau+\varepsilon^2O(1)\\&=\Omega_{sing}^\alpha d-\frac{\widehat{C}_\alpha\varepsilon}{2(2d)^{2+\alpha}}w+\frac{\widehat{C}_\alpha(2+\alpha)\varepsilon}{4(2d)^{2+\alpha}}\big(w+\overline{w}\big)+\varepsilon^2O(1),\end{aligned}$$

and so

$$G_3(\varepsilon,0)=\mathrm{Im}\{I_2(\varepsilon,0)\overline{w}\}.$$

Therefore the V-states equations becomes

$$\begin{aligned}\mathrm{Im}\Big\{\big(I_2(\varepsilon,0)-\Omega d\big)\overline{w}\Big\}&=d(\Omega_{sing}^\alpha-\Omega)\mathrm{Im}(\overline{w})+\varepsilon\frac{\widehat{C}_\alpha(2+\alpha)}{4(2d)^{2+\alpha}}\mathrm{Im}(\overline{w}^2)+\varepsilon^2O(1)\\&=0\end{aligned}$$

and this equation is impossible for $0<\varepsilon\leq\varepsilon_0$ with ε_0 small enough depending on d and α.

(iii) We shall only present the proof for the case $\alpha=0$ and for the same proof works as well for $\alpha\in(0,1)$. Using the definition of $\tilde{f}$ one can check that $T_i(-\varepsilon,\tilde{f})=-T_i(\varepsilon,f)$, for $i=1,2$ and so by (52) we obtain that

$$\Omega(\varepsilon,f)=\Omega(-\varepsilon,\tilde{f}).$$

Taking the decomposition of $F^0 = F_1 + F_2 + F_3$ given in (9) we only need to check that $F_i(\varepsilon, \Omega, f)(-w) = -F_i(-\varepsilon, \Omega, \tilde{f})(w)$, for $i = 1, 2, 3$. Since $\tilde{f}'(w) = -f'(-w)$ we have

$$\begin{aligned}F_1(-\varepsilon, \Omega, \tilde{f})(w) &= 2\Omega\big(-\varepsilon\overline{w} + \varepsilon^2\tilde{f}(\overline{w}) - d\big)w\big(1 - \varepsilon\tilde{f}'(w)\big) - \tilde{f}'(w)\\ &= -\big[2\Omega\big(\varepsilon(-\overline{w}) + \varepsilon^2 f(-\overline{w}) - d\big)(-w)\big(1 + \varepsilon f'(-w)\big) - f'(-w)\big]\\ &= -F_1(\varepsilon, \Omega, f)(-w).\end{aligned}$$

Straightforward computations will lead to the same properties for the functions F_2 and F_3. It follows that,

$$F^0(\varepsilon, \Omega, f)(w) = -F^0(-\varepsilon, \Omega, \tilde{f})(-w)$$

and therefore $(-\varepsilon, \tilde{f})$ defines a curve of solutions for $0 < \varepsilon < \varepsilon_0$.

(iv) As before we shall restrict the discussion to the case $\alpha = 0$ because the proof in the case $\alpha \in (0, 1)$ follows exactly the same lines. First we shall make the following comment about the regularity of the conformal mapping. As it was mentioned in Remark 6 one can reproduce the preceding proofs and replace β by $n + \beta$ with $n \in \mathbb{N}$. Therefore the implicit function theorem ensures that the function f_ε belongs to $C^{n+1+\beta}$ for any fixed n. Of course, the size of ε_0 is not uniform with respect to n and it shrinks to zero as n grows to infinity. Now to prove the convexity of the domain D_1^ε we shall implement the same arguments of [22]. Recall that the exterior conformal mapping associated to this domain is given by

$$\phi(w) = \varepsilon w + \varepsilon^2 f_\varepsilon(w)$$

and the curvature can be expressed by the formula

$$\kappa(\theta) = \frac{1}{|\phi'(w)|}\,\mathrm{Re}\left(1 + w\frac{\phi''(w)}{\phi'(w)}\right).$$

It is plain that

$$1 + w\frac{\phi''(w)}{\phi'(w)} = 1 + \varepsilon w\frac{f''(w)}{1 + \varepsilon f'(w)}$$

and so

$$\mathrm{Re}\left(1 + w\frac{\phi''(w)}{\phi'(w)}\right) \geq 1 - |\varepsilon|\frac{|f''(w)|}{1 - |\varepsilon||f'(w)|} \geq 1 - \frac{|\varepsilon|}{1 - |\varepsilon|},$$

which is non-negative if $|\varepsilon| < 1/2$. Thus the curvature is strictly positive and therefore the domain is strictly convex. □

5. Existence of Counter-Rotating Vortex Pairs

In this section we will prove the existence of planar translating pairs of vortex patches with velocity U in the direction (OY) for the $(\mathrm{SQG})_\alpha$ with $\alpha \in [0, 1)$. The proof is similar to that of the corotating pairs and therefore we shall skip many details and focus on the significant variations.

5.1. Extension and regularity of G^α. This section is devoted to the study of the regularity of the functions G^α appearing in (18) and (23).

Proposition 4. *Let $\alpha \in [0,1)$, then the following holds true.*

(i) *The function G^α can be extended from $(-\frac12,\frac12)\times\mathbb{R}\times B_1^\alpha \to Y^\alpha$ as a C^1 function. Moreover, for any $U\in\mathbb{R}$, the operator $\partial_f G^\alpha(0,U,0): X^\alpha \to \widehat{Y}^\alpha$ is an isomorphism. More precisely, for $h=\sum_{n\geq 1} a_n w^{-n}\in X^\alpha$, we get*

$$\partial_f G^\alpha(0,U,0)h(w) = -\sum_{n\geq 1} a_n\widehat{\gamma}_n e_{n+1},$$

with

$$\widehat{\gamma}_n = \frac{\widehat{C}_\alpha\Gamma(1-\alpha)}{4\Gamma^2(1-\frac{\alpha}{2})}\left(\frac{2(1+n)}{1-\frac{\alpha}{2}} - \frac{\left(1+\frac{\alpha}{2}\right)_n}{\left(1-\frac{\alpha}{2}\right)_n} - \frac{\left(1+\frac{\alpha}{2}\right)_{n+1}}{\left(1-\frac{\alpha}{2}\right)_{n+1}}\right).$$

(ii) *Two initial point vortex $\pi\delta_0$ and $-\pi\delta_{(2d,0)}$ move uniformly in the direction (OY) with the speed*

$$U_{sing}^\alpha \equiv \frac{\widehat{C}_\alpha}{2(2d)^{1+\alpha}}.$$

Proof. (**i**) The proof is quite similar to (i) of Proposition 1. The only slight difference is in the treatment of G_1 which is clearly C^1 in the variable ε and moreover it has a polynomial dependence in Ω, f, f', and so its derivatives in these variables are also continuous. Note that G_2 and G_3 are exactly the same functions of the rotating case, see (15). Now we shall compute the linearized operator and we restrict ourselves only to $\alpha\in(0,1)$. The case $\alpha=0$ can be done separately or by taking the limit when $\alpha\to 0^+$. It is easy to see that

$$\partial_f G_1(0,U,0) = 0,$$

and so

$$\begin{aligned}\partial_f G^\alpha(0,U,0) &= \partial_f G_1(0,U,0) + \partial_f G_2(0,0) + \partial_f G_3(0,0)\\ &= \partial_f G_2(0,0) + \partial_f G_3(0,0).\end{aligned}$$

On the other hand by (31) $\partial_f G_3(0,0)=0$, and so this operator coincides, after a change of sign, with the linearized operator in the rotating case and whose formula was stated in Proposition 1.

(ii) Obvious computations yield

$$G_1(0,U,0) = Ue_1.$$

According to (34) one finds

$$G_3(0,0) = -\frac{\alpha C_\alpha}{2(2d)^{1+\alpha}}e_1.$$

Using (35), (37) and (40) we obtain

$$G_2(0,0) = 0.$$

Therefore we get

$$G^\alpha(0, U, 0) = \Big(U - \frac{\alpha C_\alpha}{2(2d)^{1+\alpha}}\Big)e_1$$

and consequently $G^\alpha(0, U, 0) = 0$ if and only if

$$U = U^\alpha_{sing} \equiv \frac{\widehat{C}_\alpha}{2(2d)^{1+\alpha}}\cdot$$

□

5.2. Relationship between the speed and the boundary shape. As for the rotating case the image of the space X^α by $G^\alpha(\varepsilon, U, \cdot)$ is contained in Y^α and not necessary in $\widehat{Y}^\alpha$. Therefore to apply the implicit function theorem we should impose a constraint between U, ε and f which guarantees a range contained in $\widehat{Y}^\alpha$. The main result reads as follows.

Proposition 5. *Let* $\alpha \in [0, 1)$. *There exists a* C^1 *function* $\mathcal{R}^\alpha : (-\frac{1}{2}, \frac{1}{2}) \times B^\alpha_1 \to \mathbb{R}$ *such that with the choice*

$$U = U^\alpha(\varepsilon, f) = U^\alpha_{sing} + \mathcal{R}^\alpha(\varepsilon, f),$$

the function $\widehat{G}^\alpha : (-\frac{1}{2}, \frac{1}{2}) \times B^\alpha_1 \to \widehat{Y}^\alpha$ *given by*

$$\widehat{G}^\alpha(\varepsilon, f) \equiv G^\alpha\big(\varepsilon, U(\varepsilon, f), f\big)$$

is well-defined and is C^1. *Moreover,*

$$\forall\ f \in B^\alpha_1,\ \mathcal{R}^\alpha(0, f) = 0 \quad \text{and} \quad \widehat{G}^\alpha(0, 0) = 0.$$

Proof. We shall follow the same strategy of the Sect. 4.2. According to Proposition 4 the linear operator $\partial_f G^\alpha(0, U, 0)$ is an isomorphism from X^α to $\widehat{Y}^\alpha$, and the latter space is strictly contained in Y^α. Note also that the image of the functional G^α lies in Y^α and therefore we shall choose U in such a way that the range of G^α is contained in $\widehat{Y}^\alpha$. To this end we should impose a nonlinear constraint on U such that the coefficient of e_1 vanishes in the Fourier expansion of $G^\alpha(U, \varepsilon, f)$. This constraint reads

$$\fint_{\mathbb{T}} G^\alpha\big(\varepsilon, U, f(w)\big)dw = 0. \tag{60}$$

Case $\alpha = 0$**.** According to (18), the above condition is equivalent to

$$\fint_{\mathbb{T}} F^0\big(U, \varepsilon, f(w)\big)\big(\overline{w}^2 - 1\big)dw = 0. \tag{61}$$

Note that by residue theorem

$$\fint_{\mathbb{T}} F_1\big(U, \varepsilon, f(w)\big)\overline{w}^2 dw = 2U$$

and

$$\fint_{\mathbb{T}} F_1\big(U,\varepsilon,f(w)\big)dw = 2U\varepsilon \fint_{\mathbb{T}} w\, f'(w)dw$$
$$= -2U\varepsilon \fint_{\mathbb{T}} f(w)dw.$$

Consequently

$$\fint_{\mathbb{T}} F_1\big(\Omega,\varepsilon,f(w)\big)\big(\overline{w}^2-1\big)dw = 2U\Big(1+\varepsilon\fint_{\mathbb{T}} f(w)dw\Big).$$

Now we shall calculate the contribution of F_3. First we make the decomposition

$$F_3(\varepsilon,f(w)) = \widehat{F_3}(w)w\big(1+\varepsilon f'(w)\big),$$

with

$$\widehat{F_3}(\varepsilon,f(w)) \equiv \fint_{\mathbb{T}} \frac{\overline{\tau}+\varepsilon f(\overline{\tau})}{\varepsilon(\tau+w)+\varepsilon^2\big(f(\tau)+f(w)\big)-2d}\big(1+\varepsilon f'(\tau)\big)d\tau.$$

Now one may write

$$\frac{\overline{\tau}+\varepsilon f(\overline{\tau})}{\varepsilon(\tau+w)+\varepsilon^2\big(f(\tau)+f(w)\big)-2d} = -\frac{\overline{\tau}}{2d}+\varepsilon\frac{f(\overline{\tau})}{\varepsilon(\tau+w)+\varepsilon^2\big(f(\tau)+f(w)\big)-2d}$$
$$+\frac{\varepsilon}{2d}\frac{\tau+w+\varepsilon(f(\tau)+f(w))}{\varepsilon(\tau+w)+\varepsilon^2\big(f(\tau)+f(w)\big)-2d}\overline{\tau}$$
$$\equiv -\frac{\overline{\tau}}{2d}+\varepsilon g_3(\varepsilon,\tau,w).$$

Thus we find

$$\widehat{F_3}(\varepsilon,f(w)) = -\frac{1}{2d}+\varepsilon\int_{\mathbb{T}} g_3(\varepsilon,\tau,w)\big(1+\varepsilon f'(\tau)\big)d\tau.$$

Hence

$$\fint_{\mathbb{T}} F_3\big(\Omega,\varepsilon,f(w)\big)\overline{w}^2dw = -\frac{1}{2d}+\varepsilon\fint_{\mathbb{T}}\fint_{\mathbb{T}} g_3(\varepsilon,\tau,w)\big(1+\varepsilon f'(\tau)\big)\overline{w}\big(1+\varepsilon f'(w)\big)d\tau dw$$
$$\fint_{\mathbb{T}} F_3\big(\Omega,\varepsilon,f(w)\big)dw = \frac{\varepsilon}{2d}\fint_{\mathbb{T}} f(\tau)d\tau+\varepsilon\fint_{\mathbb{T}}\fint_{\mathbb{T}} g_3(\varepsilon,\tau,w)\big(1+\varepsilon f'(\tau)\big)w\big(1+\varepsilon f'(w)\big)d\tau dw.$$

It follows that

$$\fint_{\mathbb{T}} F_3\big(\Omega,\varepsilon,f(w)\big)\big(\overline{w}^2-1\big)dw = -\frac{1}{2d}-\frac{\varepsilon}{2d}\fint_{\mathbb{T}} f(\tau)d\tau$$
$$+\varepsilon\fint_{\mathbb{T}}\fint_{\mathbb{T}} g_3(\varepsilon,\tau,w)\big(1+\varepsilon f'(\tau)\big)$$
$$\times\ (\overline{w}-w)\big(1+\varepsilon f'(w)\big)d\tau dw.$$

On the other hand using residue theorem we get

$$\begin{aligned}F_2(\varepsilon, f(w)) &= \varepsilon \fint_{\mathbb{T}} \frac{A\overline{B} - \overline{A}B}{A(A+\varepsilon B)} f'(\tau)d\tau\, w\big(1+\varepsilon f'(w)\big)\\ &\quad + \varepsilon \fint_{\mathbb{T}} \frac{\big(\overline{A}B - A\overline{B}\big)B}{A^2(A+\varepsilon B)} d\tau\, w\big(1+\varepsilon f'(w)\big)\\ &\equiv \varepsilon g_2(\varepsilon, w) w\big(1+\varepsilon f'(w)\big).\end{aligned}$$

Thus

$$\begin{aligned}&\fint_{\mathbb{T}} F_2\big(\Omega, \varepsilon, f(w)\big)\overline{w}^2 dw - \fint_{\mathbb{T}} F_2\big(\Omega, \varepsilon, f(w)\big)dw\\ &= \varepsilon \fint_{\mathbb{T}} g_2(\varepsilon, w)(\overline{w} - w)\big(1+\varepsilon f'(w)\big)dw.\end{aligned}$$

The Eq. (61) becomes

$$\begin{aligned}2U\Big(1+\varepsilon \fint_{\mathbb{T}} f(w)dw\Big) &= \frac{1}{2d} + \frac{\varepsilon}{2d}\fint_{\mathbb{T}} f(\tau)d\tau\\ &\quad + \varepsilon \fint_{\mathbb{T}} g_2(\varepsilon, w)(w-\overline{w})\big(1+\varepsilon f'(w)\big)dw\\ &\quad + \varepsilon \fint_{\mathbb{T}}\fint_{\mathbb{T}} g_3(\varepsilon, \tau, w)\big(1+\varepsilon f'(\tau)\big)(w-\overline{w})\big(1+\varepsilon f'(w)\big)d\tau dw\\ &\equiv \frac{1}{2d} + \frac{\varepsilon}{2d} T_1(\varepsilon, f)\end{aligned}$$

which may be written in the form

$$\begin{aligned}U &= U^0(\varepsilon, f)\\ &= \frac{1}{4d}\frac{1+\varepsilon T_1(\varepsilon, f)}{1+\varepsilon T_2(f)}\\ &= U^0_{sing} + \frac{\varepsilon}{4d}\frac{T_1(\varepsilon, f) - T_2(f)}{1+\varepsilon T_2(f)}\\ &\equiv U^0_{sing} + \mathcal{R}^0(\varepsilon, f),\end{aligned} \tag{62}$$

with

$$T_2(f) = \fint_{\mathbb{T}} f(w)dw.$$

Case $\alpha \in (0, 1)$. From the splitting (23) the assumption (60) is equivalent to

$$A_0 = -A_1 - A_2, \tag{63}$$

with

$$A_j = -2i \fint_{\mathbb{T}} G_j\big(\varepsilon, \Omega, f(w)\big))dw.$$

Note that A_j is the Fourier coefficient of $e_1 = \mathrm{Im}(w)$ in $G_j \equiv \mathrm{Im}(F_j)$, then

$$A_j = \fint_{\mathbb{T}} F_j\big(\varepsilon, \Omega, f(w)\big)\big(\overline{w}^2 - 1\big)dw.$$

The computation of A_1 is easy,

$$A_1 = -U \fint_{\mathbb{T}} \big(1 + \varepsilon|\varepsilon|^\alpha f'(\overline{w})\big)\big(\overline{w}^3 - \overline{w}\big)dw.$$

Since

$$f(w) = \sum_{n\geq 1} a_n \overline{w}^n \quad \text{and} \quad f'(w) = -\sum_{n\geq 1} n a_n \overline{w}^{n+1},$$

then

$$\begin{aligned}\fint_{\mathbb{T}} f'(\overline{w})\big(\overline{w}^3 - \overline{w}\big)dw &= -a_1\\ &= -\fint_{\mathbb{T}} f(\tau)d\tau.\end{aligned}$$

Consequently

$$A_1 = U\Big(1 + \varepsilon|\varepsilon|^\alpha \fint_{\mathbb{T}} f(\tau)d\tau\Big). \tag{64}$$

Combining (63) with (64), (57) and (58) one gets

$$U\Big(1 + \varepsilon|\varepsilon|^\alpha \fint_{\mathbb{T}} f(\tau)d\tau\Big) = \frac{\alpha C_\alpha}{2(2d)^{1+\alpha}} + \varepsilon\mathcal{R}_2(\varepsilon, f) - \varepsilon|\varepsilon|^\alpha \mathcal{R}_1(\varepsilon, f).$$

Thus

$$\begin{aligned} U &= U^\alpha(\varepsilon, f)\\ &\equiv \frac{\frac{\alpha C_\alpha}{2(2d)^{1+\alpha}} + \varepsilon\mathcal{R}_2(\varepsilon, f) - \varepsilon|\varepsilon|^\alpha\mathcal{R}_1(\varepsilon, f)}{1 + \varepsilon|\varepsilon|^\alpha \fint_{\mathbb{T}} f(\tau)d\tau}\\ &\equiv \frac{\alpha C_\alpha}{2(2d)^{1+\alpha}} + \mathcal{R}^\alpha(\varepsilon, f).\end{aligned}$$

Similarly to the rotating case, $\mathcal{R}^\alpha : (-\frac{1}{2}, \frac{1}{2}) \times B_1^\alpha \to \mathbb{R}$ is well-defined and C^1. Moreover, by construction one can see that $\mathcal{R}^\alpha(0, f) = 0$. This will imply that $\widehat{G}^\alpha(0, 0) = 0$. □

5.3. Proof of the main Theorem-(ii). Recall that the V-states equation can be written in the form

$$\widehat{G}^\alpha(\varepsilon, f) = 0, \quad (\varepsilon, f) \in (-1/2, 1/2) \times B_1^\alpha,$$

with $\widehat{G}^\alpha$ being the functional defined by

$$\widehat{G}^\alpha\big(\varepsilon, f(w)\big) = G^\alpha\big(\varepsilon, U^\alpha(\varepsilon, f), f\big).$$

The proof of the existence of translating pairs stated in the Main Theorem follows from the next proposition whose proof is quite similar to that of Proposition 3 and so is left to the reader.

Proposition 6. *Let* $\alpha \in [0, 1)$. *The following holds true.*

(i) *The linear operator* $\partial_f \widehat{G}^\alpha(0, 0) : X^\alpha \to \widehat{Y}^\alpha$ *is an isomorphism and*

$$\partial_f \widehat{G}^\alpha(0, 0)h(w) = -\sum_{n\geq 1} a_n \widehat{\gamma}_n e_{n+1}$$

with

$$\widehat{\gamma}_n = \frac{\widehat{C}_\alpha \Gamma(1-\alpha)}{4\Gamma^2(1-\frac{\alpha}{2})}\left(\frac{2(1+n)}{1-\frac{\alpha}{2}} - \frac{\big(1+\frac{\alpha}{2}\big)_n}{\big(1-\frac{\alpha}{2}\big)_n} - \frac{\big(1+\frac{\alpha}{2}\big)_{n+1}}{\big(1-\frac{\alpha}{2}\big)_{n+1}}\right).$$

(ii) *There exists* $\varepsilon_0 > 0$ *such that the set*

$$\Big\{(\varepsilon, f) \in [-\varepsilon_0, \varepsilon_0] \times B_1^\alpha,\ s.t.\ \ \widehat{G}^\alpha(\varepsilon, f) = 0\Big\}$$

is parametrized by one-dimensional curve $\varepsilon \in [-\varepsilon_0, \varepsilon_0] \mapsto (\varepsilon, f_\varepsilon)$ *and*

$$\forall\ \ \varepsilon \in [-\varepsilon_0, \varepsilon_0]\backslash\{0\}, \quad f_\varepsilon \neq 0.$$

(iii) *If* (ε, f) *is a solution then* $(-\varepsilon, \tilde{f})$ *is also a solution, where*

$$\tilde{f}(w) = f(-w), \quad \forall\ \ w \in \mathbb{T}.$$

(iv) *For all* $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, *the domain* D_1^ε *is strictly convex.*

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References

1. Aref, H.: Integrable, chaotic, and turbulent vortex motion in two-dimensional flows. Ann. Rev. Fluid Mech. **15**, 345–389 (1983)
2. Bertozzi, A.L., Constantin, P.: Global regularity for vortex patches. Commun. Math. Phys. **152**(1), 19–28 (1993)
3. Burbea, J.: Motions of vortex patches. Lett. Math. Phys. **6**, 1–16 (1982)
4. Burton, G.R.: Steady symmetric vortex pairs and rearrangements. Proc. R. Soc. Edinb. Sect. A. **108**, 269–290 (1988)
5. Burton, G.R., Nussenzveig Lopes, H.J., Lopes Filho, M.C.: Nonlinear Stability for Steady Vortex Pairs. Commun. Math. Phys. **324**, 445–463 (2013)
6. A. Castro, D. Córdoba, J. Gómez-Serrano, Uniformly rotating analytic global patch solutions for active scalars. Ann. PDE 2(1) (2016) (Art. 1)
7. Castro, A., Córdoba, D., Gómez-Serrano, J.: Existence and regularity of rotating global solutions for the generalized surface quasi-geostrophic equations. Duke Math. J. **165**(5), 935–984 (2016)

8. Chae, D., Constantin, P., Córdoba, D., Gancedo, F., Wu, J.: Generalized surface quasi-geostrophic equations with singular velocities. Commun. Pure Appl. Math. **65**(8), 1037–1066 (2012)
9. Chemin, J.-Y.: Fluides Parfaits Incompressibles, Astérisque 230 (1995) (Perfect Incompressible Fluids translated by I. Gallagher and D. Iftimie, Oxford Lecture Series in Mathematics and Its Applications, Vol. 14, Clarendon Press-Oxford University Press, New York (1998))
10. Constantin, P., Majda, A.J., Tabak, E.: Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar. Nonlinearity **7**(6), 1495–1533 (1994)
11. Córdoba, D., Fontelos, M.A., Mancho, A.M., Rodrigo, J.L.: Evidence of singularities for a family of contour dynamics equations. Proc. Natl. Acad. Sci. USA **102**(17), 5949–5952 (2005)
12. Deem, G.S., Zabusky, N.J.: Vortex waves: stationary "V-states", interactions, recurrence, and breaking. Phys. Rev. Lett. **40**(13), 859–862 (1978)
13. de la Hoz, F., Hassainia, Z., Hmidi, T.: Doubly connected V-states for the generalized surface quasi-geostrophic equations. Arch. Ration. Mech. Anal. **220**(3), 1209–1281 (2016)
14. de la Hoz, F., Hmidi, T., Mateu, J., Verdera, J.: Doubly connected V-states for the planar Euler equations. SIAM J. Math. Anal. **48**(3), 1892–1928 (2016)
15. Denisov, S.A.: The centrally symmetric V-states for active scalar equations. Two-dimensional Euler with cut-off. Commun. Math. Phys. **337**, 955–1009 (2015)
16. Dritschel, D.G.: A general theory for two-dimensional vortex interactions. J. Fluid Mech. **293**, 269–303 (1995)
17. Gallay, Th.: Interaction of vortices in weakly viscous planar flows. Arch. Ration. Mech. Anal. **200**, 445–490 (2011)
18. Gallay, Th.: Interacting vortex pairs in inviscid and viscous planar flows. In: J. Robinson, J. Rodrigo, et W. Sadowski (eds.) Mathematical Aspects of Fluid Mechanics. London Math. Soc. Lecture Notes Series, vol. 402. Cambridge University Press, Cambridge (2012)
19. Gancedo, F.: Existence for the α-patch model and the QG sharp front in Sobolev spaces. Adv. Math. **217**(6), 2569–2598 (2008)
20. Hassainia, Z., Hmidi, T.: On the V-States for the generalized quasi-geostrophic equations. Commun. Math. Phys. **337**(1), 321–377 (2015)
21. Held, I., Pierrehumbert, R., Garner, S., Swanson, K.: Surface quasi-geostrophic dynamics. J. Fluid Mech. **282**, 1–20 (1995)
22. Hmidi, T., Mateu, J., Verdera, J.: Boundary regularity of rotating vortex patches. Arch. Ration. Mech. Anal. **209**(1), 171–208 (2013)
23. Hmidi, T., Mateu, J.: Bifurcation of rotating patches from Kirchhoff vortices. Discret. Contin. Dyn. Syst. **36**(10), 5401–5422 (2016)
24. Hmidi, T., Mateu, J.: Degenerate bifurcation of the rotating patches. Adv. Math. **302**, 799–850 (2016)
25. Juckes, M.: Quasigeostrophic dynamics of the tropopause. J. Armos. Sci. (1994) 2756–2768
26. Kamm, J.R.: Shape and stability of two-dimensional uniform vorticity regions. PhD thesis, California Institute of Technology (1987)
27. Keady, G.: Asymptotic estimates for symmetric vortex streets. J. Aust. Math. Soc. Ser. B **26**, 487–502 (1985)
28. Kirchhoff, G.: Vorlesungen uber mathematische Physik. Teubner, Leipzig (1874)
29. Lamb, H.: Hydrodynamics. Dover Publications, New York (1945)
30. Lapeyre, G., Klein, P.: Dynamics of the upper oceanic layers in terms of surface quasigeostrophic theory. J. Phys. Oceanogr. **36**, 165–176 (2006)
31. Luzzatto-Fegiz, P., Williamson, C.H.K.: Stability of elliptical vortices from "Imperfect-Velocity-Impulse" diagrams. Theor. Comput. Fluid Dyn. **24**(1–4), 181–188 (2010)
32. Marchioro, C., Pulvirenti, M.: Euler evolution for singular initial data and vortex theory. Commun. Math. Phys. **91**, 563–572 (1983)
33. Mateu, J., Orobitg, J., Verdera, J.: Extra cancellation of even Calderón-Zygmund operators and quasiconformal mappings. J. Math. Pures Appl. **9**(1), 402–431 (2009)
34. Newton, P.K.: The N-Vortex Problem, Analytical Techniques. Springer, New York (2001)
35. Norbury, J.: Steady planar vortex pairs in an ideal fluid. Commun. Pure Appl. Math. **28**, 679–700 (1975)
36. Pierrehumbert, R.T.: A family of steady, translating vortex pairs with distributed vorticity. J. Fluid Mech. **99**, 129–144 (1980)
37. Rodrigo, J.L.: On the evolution of sharp fronts for the quasi-geostrophic equation. Commun. Pure Appl. Math. **58**(6), 821–866 (2005)
38. Saffman, P.G.: Vortex Dynamics. Cambridge Monographs on Mechanics and Applied Mathematics. Cambridge University Press, New York (1992)
39. Saffman, P.G., Szeto, R.: Equilibrium shapes of a pair of equal uniform vortices. Phys. Fluids **23**(12), 2339–2342 (1980)
40. Smets, D., Schaftingen, J.V.: Desingularization of Vortices for the Euler Equation. Arch. Ration. Mech. Anal. **198**, 869–925 (2010)

41. Turkington, B.: Corotating steady vortex flows with n-fold symmety. Nonlinear Anal. Theory Methods Appl. **9**(4), 351–369 (1985)
42. Wu, H.M., Overman, E.A. II., Zabusky, N.J.: Steady-state solutions of the Euler equations in two dimensions: rotating and translating V-states with limiting cases I. Algorithms ans results. J. Comput. Phys. **53**, 42–71 (1984)
43. Yudovich, V.I.: Non-stationnary flows of an ideal incompressible fluid. Zhurnal Vych Matematika **3**, 1032–1106 (1963)

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