A handy Jacobian criterion for uniqueness of solution to systems of equations

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Abstract

One of the most famous sufficiency theorems for the existence and uniqueness of a solution to an equation of the form $f(x) = 0$, when f is a real-valued function defined on a real interval I, is the following:

Theorem 1. *Consider a differentiable function* $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$*, such that*

(I) $f(a) \cdot f(b) < 0$*, i.e.,* $f(a)$ *and* $f(b)$ *have opposite signs,*

(II) $f'(x) \neq 0$ *for any* $x \in (a, b)$.

Then, the equation $f(x) = 0$ *has exactly one solution* $x \in [a, b]$.

A natural question is how this result can be generalized to a wider context, specifically to functions in several variables. In other words, we aim to find a similar result that ensures global injectivity for a function of several variables. This generalization process is not straightforward since, for example, the notion of an "interval" could be extended to either a "(hyper)rectangle" or a "convex region." Moreover, there are multiple ways to generalize the condition "$f' \neq 0$." While some of these possible statements are true, others are not: in $\mathbb{R}^n$ with $n > 1$, the classical inverse function theorem only guarantees local injectivity.

In our work, after providing suitable motivation, we prove the following criterion for continuously differentiable multivariate functions: if the symmetric part of the Jacobian matrix is definite at a point and its determinant is non-zero in a convex region, then the original function is injective in that region. This supposes a slight improvement of a result mentioned in [1]. Besides, from this result, we derive a practical criterion that ensures the asymptotic global stability of certain continuous dynamical systems. This result is similar to one by L. Markus and H. Yamabe [2], but it is neither weaker nor stronger. Finally, we discuss possible relaxations of the hypotheses, such as replacing definiteness with semi-definiteness.

Keywords: Poincaré-Miranda Theorem; Jacobian matrix; definite matrices; symmetric and skew-symmetric matrices.

MSC 2020: 26B12; 37C75.

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