Convex Integration, Staircase Laminates and Applications 2025

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Preface 1

The method of convex integration starts with Nash construction of a local C^1 isometric inmersion of the standard round sphere in $\mathbb{R}^4$. That is solving a partial differential relation, $u = \mathbb{R}^2 \to \mathbb{R}^4$ such that

$$Du^t Du = g$$

The solution u is built on a short map, u_0

$$Du^t Du < g$$

and $u = \lim u^N$ where
$u_0 + \sum_{q=1}^{N} (1 - \delta_q)^{\frac{1}{2}} \frac{a_q(x)}{\lambda_q} \left(\sin(\lambda_q x \cdot \xi^q)\zeta_q(x) + \cos(\lambda_q x \cdot \xi^q)\eta_q(x)\right),$
where the parameters λ_q, δ_q, the function a_q and the vectors ξ^q, ζ_q, η_q are carefully chosen in an inductive way so that at each step u^N is still short but

$$\|g - (Du^N)^T Du^N\|_{C^0(\Omega)} \leq \varepsilon \tag{1}$$

$$\|Du^{N+1} - Du^N\|_{C^0(\Omega)} \leq C\|g - (Du^N)^T Du\|_{C^0(\Omega)}^{1/2} \tag{2}$$

By now we call convex integration scheme or a solution obtained by convex integration to a PDE where one starts with a subsolution (a coarse grained solution) u_0 and one obtains the limit map
$u^\infty = \lim u^N$
where $u^{N+1} - u^N = \omega_q$ and ω_q has some oscillation and concentration parameters λ_q, τ_q, some especial direction $\eta_k^1 \eta_k^2$ and some simple oscillating functions ϕ which replaced *cos*.

- Nash idea is very robust and was developed massively by Gromov and coauthors in the framework of differential geometry.
- Müller and Sverák realized that it could be combined with Tartar compensated compactness theory to provided unexpected solutions to variational problems where the direct method fails. It also provided nowhere C^1 solutions to non linear smooth elliptic systems.
- De Lellis and Székelyhidi realize that the method could be applied to the Euler equation and thereby to many other equations in fluid dynamics. This culminated with the celebrated solution of Onsager conjecture by Iset on $C^{\frac{1}{3}}$ weak solutions of Euler dissipating kinetic energy.
- In this course we will consider a version of convex integration adapted to obtain critical integrability properties as oppose to differentiability.

Preface II: Staircase Laminates

Staircase laminates were first introduced by Faraco in his 2002 thesis, [F03, F04] within the context of isotropic elliptic equations. His work was based on a microstructure proposed by Graeme Milton in the field of homogenization. Milton aimed to construct an isotropic material in which the corresponding electric field was either extremely strong or very weak. Since then, and particularly when combined with the convex integration method adapted to partial differential equations by Müller and vSverák, this approach has proven to be highly versatile and have been rediscovered in the last 5 years. Loosely speaking, the method consists of three main steps.

- First, the problem is reformulated as a differential inclusion, $Du \in K$, where K is a euclidean closed set represents the data of the problem.
- Secondly, the interaction of K with the rank-one geometry (or the relevant wave cone geometry) determines the presence of a corresponding staircase laminate. The integrability of this laminate depends on the geometry of K. Laminates provide approximate solutions

$$\int dist_K(Du)^p \leq \epsilon$$

- Thirdly we need to develop a convex integration scheme a la Nash to combine the approximate solutions to obtain an exact solutions.

Preface III. Applications

We will present various applications of the theory.

- Non L^1 inequalities in C-Z theory. The first one was developed together with Conti-Maggi and Müller, ([CFM05, CFMM05] see also the work of Kirchheim and Kristensen and a recent simplification with Guerra.
- Critical solutions to elliptic PDES.
  The second one corresponds to the existence of critical solutions to various elliptic equations as in the original application of Faraco. The method was developed together with Astala and Szekelyhidi, [AFSz08], and it has been much more recently adapted to non linear autonomous equations in [ACFJKM] A recent smart variant was found by Colombo and Tione, [CT24] to solve an old conjecture of Iwaniec and Sbordone concerning the existence of very weak p-harmonic functions.
- Patological homemorphisms and solutions to Ohm-Faraday dissipating helicity.
  The last application emerges in electromagnetism and plasma relaxation has appeared in the work with Lindberg and Szekelyhidi, [FLSz24]. This is actually related to the construction of Sobolev maps with zero determinant, see [F05, FM018]

Menu of the course

Scenario 1: Calderón Zygmund theory

Recall that:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\, dx$$

$$\widehat{\partial_{x_j} f} = i\xi_j \hat{f}(\xi)$$

$$\widehat{\partial_{x_i x_j} f} = -\xi_i \xi_j \hat{f}(\xi)$$

where $|\xi|^2 = \xi_1^2 + \xi_2^2$. Therefore,

$$|\xi|^2 \hat{f} = \widehat{\partial^2_{x_1 x_1} f} + \widehat{\partial_{x_2 x_2} f} = \Delta f$$

By Plancherel:

$$\begin{aligned}
\int |\partial_{x_1x_2} f|^2 \, dx &= \int |\xi_1|^2 |\xi_2|^2 |\hat{f}|^2 \, d\xi \\
&= \int \frac{|\xi_1|}{|\xi|} \frac{|\xi_2|}{|\xi|} |\xi|^2 |\hat{f}|^2 \, d\xi \\
&= \int |\xi|^2 |\hat{f}|^2 \, d\xi \\
&= \int |\Delta f|^2 \, dx
\end{aligned}$$

i.e C-Z Theory $\forall 1 < p < \infty \; \exists \; c(p)$ such that :

$$\int |\partial_{x_1x_2} f|^p \leq c_p \int |\Delta f|^p dx$$

Ornstein Non - Inequalities

Our first application shows that the estimates is not true when $p = 1$

Theorem

$\forall N, \Omega$ *regular* $\exists$ f_N *with* $\int |\partial_{x_1 x_2} f| \geq N, sup\{|\partial x_1 x_1 f|, |\partial x_2, x_2 f|\} \leq 1$

Onstein, Intrincated ad hoc example
F-Conti-Maggi, Staircase Laminates
Kirchheim-Kristensen. Corollary of the following geometric fact: Positive Homegenous rank-one convex functions are convex.
F-Guerra. Simple proof based on a second order laminate.

Elliptic Equations and Quasiconformal Mappings

Electrostatics We consider the basic conductivity equation. Here u is the electric potential

- $\operatorname{div}(\rho \nabla u) = 0$ where ρ is the conductivity.
- Boundary condition: $u\big|_{\partial\Omega} = g$
- When $\rho = 1$: $\operatorname{div}(\rho \nabla u) = \operatorname{div}(\nabla u) = \Delta u$

Quantitative Ellipticity

$$\frac{1}{K} I \leq \rho(x) \leq K I$$

Weak solution interpret the equation in distributional form

$$\int_\Omega \rho \nabla u \nabla \phi \, dx = 0, \forall \phi \in C_0^\infty(\Omega)$$

Indeed,we can arrive to the the above equation as the Euler Lagrange equation (The first Variation) of the following energy functional.

$$I[u] = \int_\Omega \rho |\nabla u|^2 dx$$

Thus, the natural domain of definition of the equation is $W^{1,2}(\Omega)$. However, the distributional solution makes sense for mappings just in $W^{1,1}$.....

The question

Are all these distributional solutions honest weak solutions?
T.Iwaniec call the distributional solutions " very weak solutions".

Weyl Lemma

Classical Weyl for harmonic functions If $u \in W^{1,1}$ is weakly harmonic,i.e $\forall \phi \in C_0^\infty(\Omega)$

$$\int_\Omega \langle \nabla u, \nabla \phi \rangle \, dx = 0$$

Then automatically $u \in C^\infty$ and real analytic.

Solving an elliptic equation yields an gain of regularity.(Paradigm of the theory of elliptic p.d.e)

Sharp Weyl lemma for elliptic equations in the plane

Theorem (K.Astala Acta 94,Leonetti-Nessi,Volberg-Petermichl)

Let $\frac{2K}{K+1} \leq q < 2 < p < \frac{2K}{K-1}$. Then if $u \in W_{loc}^{1,\frac{2K}{K+1}} \Rightarrow u \in W^{1,p}$ and indeed $\nabla u \in L^{p,\infty}$

comments

-Recall Sobolev Embedding $W^{1,p} \to C^{1-\frac{2}{p}}$ if $p > 2$.Thus, our Weyl Lemma implies than apriori unbounded function becomes continuous.
-Recall that a function f $\in L^{p,\infty}$,the Markiencewitz space,weak L^p space if:

$$|x : |f(x)| \geq t| \leq t^{-p}$$

Exercise

Show that $L^p \subset L^{p,\infty} \subset L^q$ *for all* $q < p$

Sharpness of exponents in Weyl Lemma

In the following B is an arbitrary ball, contained in an arbitrary domain Ω, and the equations are solved weakly, that is in the distributional sense.

Theorem (Astala-F-Széleyhidi 08)

1. $\exists\ u \in W^{1,2}(\Omega)$, $\rho(x) \in \{\frac{1}{K}, K\} : \mathrm{div}(\rho \nabla u) = 0$, *but*

$$\int_B |\nabla u|^{\frac{2K}{K-1}} = \infty$$

2. $\exists u \ \in C^{\alpha}(\Omega)$, $0 < \alpha < 1$, $\rho(x) \in \{\frac{1}{K}, K\} : \mathrm{div}(\rho \nabla u) = 0$,

$$\nabla u \in L^{\frac{2K}{K+1},\infty}, \ \ but, \ \int_B |\nabla u|^{\frac{2K}{K+1}} dx = \infty \ \forall B \in \Omega$$

Beltrami Equations - ρ Harmonic Conjugates

Harmonic conjugates Recall $\Delta u = 0 \Rightarrow \exists v : f = u + iv$ holomorphic.
That is, it solves the Cauchy -Riemann equations:
$\partial_{\bar{z}} f = 0$ where, , $\partial_{\bar{z}} = \partial_x + i\partial_y$

Lemma

If $\operatorname{div}(\rho\nabla \mathrm{u}) = 0 \ \exists \ v : \Omega \to \mathbb{R}$ *such that* $f : \Omega \to \mathbb{C}$ *with*

$$f = u + iv, \partial_{\bar{z}} f = \underbrace{\frac{1+\rho}{1-\rho}}_{v} \overline{\partial_z f} \quad \textit{(Beltrami Equation)}$$

Moreover $|\frac{1+\rho}{1-\rho}| \leq \kappa < 1, \|Df\|^2 \leq KJ_f$ *where*
$1 \leq K < \infty, J_f = det(D_f), \kappa = \frac{K-1}{K+1}$

The Hodge star operator

How to find the ρ harmonic conjugate v? Set,

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \star.$$

Then (Exercise!)

$$\star \operatorname{div}^{-1}(0) = \operatorname{curl}^{-1}(0)$$

and therefore

$$\operatorname{div}(\rho \nabla u) = 0 \iff \nabla \times \star(\rho \nabla u) = 0$$

(Recall that the Curl vanishes is equivalent to the fact that mixed derivatives commute. By approximation the argument works in simply connected domains and maps with weak derivatives $W^{1,1}$)

$$\iff \exists v \text{ such that } \nabla v = J\rho\nabla u,$$

Complex Notation for a Matrix

Complex notation for a matrix : We can describe the action of 2×2 by using complex notation, in the following way

$$Az = a_+ z + a_- \bar{z}$$

$A = (a_+, a_-)$
Composition is a bit tricky but we do the following identification

$$A = (a_+, a_-) = \begin{bmatrix} a_+ & a_- \\ \bar{a}_+ & \bar{a}_- \end{bmatrix}$$

$$A \circ B = \begin{bmatrix} a_+ & a_- \\ \bar{a}_+ & \bar{a}_- \end{bmatrix} \begin{bmatrix} b_+ & b_- \\ \bar{b}_+ & \bar{b}_- \end{bmatrix} = \begin{bmatrix} a_+ b_+ + a_- \bar{b}_- & a_+ b_- + a_- \bar{b}_+ \\ \bar{a}_- b_+ + \bar{a}_+ \bar{b}_- & \bar{a}_- b_- + \bar{a}_- \bar{b}_+ \end{bmatrix}$$

$det A = det \begin{bmatrix} a_+ & a_- \\ \bar{a}_- & \bar{a}_+ \end{bmatrix} = |a_+|^2 - |a_-|^2$

Exercise

$A^{-1} =?$

Wirtinger derivatives

Then if we take $A = Df(z)$, and compute its conformal and anticonformal coordinates we arrive to the celebrated Wirtinguer derivatives.

$$a_+ = \partial_z f = (\partial_x - i\partial_y)f$$

$$a_- = \partial_{\overline{z}} f = (\partial_x + i\partial_y)f$$

Lemma

Let $f = u + iv \in W^{1,1}_{loc}$ then the weak solutions satisfy

$$\partial_{\overline{z}} f = \nabla u + \star\nabla v$$

$$\overline{\partial_z} f = \nabla u - \star\nabla v$$

Since $\nabla v = \star\rho\nabla u$, the lemma is an exercise.

Critical solutions to Beltrami equations

Theorem

For every μ_1, μ_2 with $dist_h(\mu_1, \mu_2) = logK, \partial_{\bar{z}} f = \mu \partial_z f$ or $\partial_{\bar{z}} f = v \overline{\partial_z} f$

1. $\exists\, \mu \in \{\mu_1, \mu_2\}$ *a.e and* $f \in W^{1,q}(\Omega)\ \forall\ q < \frac{2K}{K+1}$. *Similarly for the upper exponent but* $f \notin W^{1,\frac{2K}{K+1}}$

Open question : Consider now a linear elliptic system, which can be written as
$\partial_{\bar{z}} f = \mu \partial_z f + v \overline{\partial_z f}$ what is the right condition in terms of μ and ν to obtain the same critical integrability values $\frac{2K}{K+1}, \frac{2K}{K-1}$
(divergence setting solved by Nesi and coauthors)

Non linear Beltrami Equations

It turns out that any nonlinear planar elliptic system can be brought into the shape : $\partial_{\bar{z}} f = H(z, \partial_{\bar{z}} f)$ where the structure function $H : \Omega \times \mathbb{C} \to \mathbb{C}$ satisfies :

1. $\forall \zeta \ H(z, \zeta)$ is measurable.
2. $\forall z : |\mathcal{H}(z, \zeta_1) - \mathcal{H}(z, \zeta_2)| \leq \kappa |\zeta_1 - \zeta_2|$ with $\kappa < 1$

Autonomous non Linear Beltrami

$$\partial_{\bar{z}} f = \mathcal{H}(\partial_z f)$$

1. Sverak : all $W^{1,2}$ solutions are $C^{1,\frac{1}{K}}$
2. Astala,Clop,F,Jääskeläinen,Koski : $\exists\ \frac{1}{K} < \psi(K) < \beta(K) < 1$ such that all solutions are $C^{\psi(K)}$ but some are not $C^{\beta(K)}$.
3. $D^2 f \in L^p\ \forall p < \frac{2K}{K-1}$

Open Problem: Is the integrability of second derivative optimal?

Weyl Exponent for Autonomous Equations

1. G.Martin : If $\mathcal{H}(\zeta) = h_+\zeta + h_-\bar{\zeta} + O(|h|^\alpha)$ Then every $W^{1,1}_{\mathrm{loc}}$ solution becomes $W^{1,p}_{\mathrm{loc}}$ solution for $1 < p < \infty$ In other words linear at infinity $\Rightarrow W^{1,1} \to W^{1,\infty}$
2. ACFJK + G.Martin 2025:If $\mathcal{H}(\zeta) = \kappa|\zeta| + O(|\zeta|^\alpha)$, then there exists infinitely many solutiosn such that $\nabla f \in L^{\frac{2K}{K+1},\infty}$ but $\int_B |\nabla f|^{\frac{2K}{K+1}} dx = \infty$

Remark An example is $\mathcal{H}_\varepsilon(\zeta) = \kappa(\sqrt{|\zeta|^2+1} - 1)$ which is real analytic and still not improvement of the integrability!

Summarizing for the question of Weyl exponent for autonomous Beltrami equations:
linear at infinity leads do a major self improvement of regularity.
$\Rightarrow W^{1,1} \to W^{1,\infty}$

However even the simplest κ-Lipschitz map does not suffice even if being Linear analytic !
This is an interesting question on its own, but it arises in L^p Teichmuller theory.

P-Laplacian

Q:Iwaniec-Sbordone are distributional solutions to the p-laplace equation, p-harmonic functions for some exponent?

Theorem (Colombo-Tione JEMS 22)

Let $p \in (1, \infty) \setminus \{2\}$. *Then, there exists* $q \in \max\{(1, p-1), p\}$ *and a continuous solution* $v \in W^{1,q}(B) \cap C(\tilde{B})$ *of the p-Laplace equation* $\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$ *with affine boundary such that:*

$$\int_{B'} |\nabla u|^p \, dx = \infty$$

Exercise

Relate the p Laplace equations with the differential inclussion given by the set $K_p = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{p-1} \end{pmatrix} R : \lambda \geq 0, R \in SO(2) \right\}$.

Faraday System and Ohm Law

In the following, $E(x,t) \equiv$Electric Field, $B(x,t) \equiv$Magnetic Field

Faraday Law of Induction :

$$\partial_t B = \nabla \times E$$
$$\operatorname{div} B = 0$$

$(E, B) : \Omega \times [0, T] \to \mathbb{R}^3 \to \mathbb{R}^3$

Faraday 2 Form

$$\omega = \varepsilon^{i,j,k} B^i dx^j \wedge dx^k + E^i dx^i \wedge dt$$

Exercise

$$d\omega = 0 \iff \begin{cases} \partial_t B = \nabla \times E \\ \text{div } B = 0 \end{cases}$$

Potentials

$$\nabla \times \psi = B$$

where ψ the Magnetic Potential.

$$\nabla \times (\partial_t \psi - E) = 0 \Rightarrow E = \nabla g + \partial_t \psi$$

where g the electric potential.

$$d\omega = 0 \iff \omega = \alpha$$

Ideal Ohm's Law

$J \equiv$ electric current.
If the frame is moving, Ohm´ law reads as,

$$\eta J = E + v \times B$$

where $\eta = \frac{1}{\sigma}$ the resistivity.

Plasmas are very good conductors and those $\sigma \approx \infty$, $\eta \approx 0$ Thus, Ideal Ohm Law is accepted to hold. $E = -v \times B$ which implies that $E \cdot B = 0$

$$\begin{cases} \partial_t B = \nabla \times E \\ \operatorname{div} B = 0 \\ E \cdot B = 0 \end{cases}$$

(Ideal Ohm Faraday)

MHD

If (E,B) are carried by a fluid, interfere with the fluid through the Lorentz force.
For non relativistic fluids(away from black holes),Lorentz force is :

$$(\nabla \times B) \times B$$

.
When we put it in Navier Stokes we get full MHD equations:

$$\begin{cases} & \partial_t B = \nabla \times (v \times B) + (\eta \Delta B) \\ & \operatorname{div} B = 0 \\ \partial_t v + v \cdot \nabla v + \nabla p & = B \times (\nabla \times B) + (\nu \Delta v) \\ \operatorname{div} v & = 0 \end{cases}$$

η is the resistivity which arises if σ is not zero, and ν the kinematical viscosity which can be assume to be zero at high Reynolds number.

Magnetic Helicity and Plasma Faraday equation

Big Issue: In plasma physics understanding the possible final states.In reality typically force free.

Magnetic Helicity

$$\mathcal{H}(B) = \mathcal{H}(t) = \int \psi \cdot B$$

(In some examples measures the topology of Magnetic fields)

$$\partial_t \mathcal{H} = \int \partial_t \psi \cdot B + \psi \cdot \partial_t B$$

$$= \int (E - \nabla g) \cdot B + \int \psi \cdot \nabla \times E$$

$$= \int E \times B + \int \underbrace{\nabla \times \psi}_{B} \cdot E = 2 \int E \cdot B$$

Thus if we are solving Ideal Ohm-Faraday $E \cdot B = 0$ and $\mathcal{H}$ is constant!

Let $1 < p < \infty, p' = \frac{p}{p-1}$ Obs If $E \in L^p, B \in L^{p'} \Rightarrow E \cdot B \in L^1$

Theorem

If $E \in L^p, B \in L^p$ and E,B solve Ideal Ohm-Faraday, MH is constant

Corollary If $B, v \in L^3$ solve M.H.D thy preserve Magnetic Helicity.

Plasma Relaxation

Q: What happens with solutions to MHD when t goes to ∞?
In the 1950's, it was observed suggested that various cosmic magnetic fields are approximately force-free, that is, the Lorentz force $(\nabla \times \mathcal{B}) \times \mathcal{B} \approx 0$ (e.g Crab Nebula ,about to celebrate its millennial birthday; the supernova was observed by Chinese astronomers in 1054).
Woltjer Plasma relaxation.
P: Minimise magnetic energy $2^{-1} \int_\Omega |\mathcal{B}|^2 \underset{\cdot}{x}$ under the constraint that the magnetic helicity $\int_\Omega \mathcal{A} \cdot \mathcal{B} \underset{\cdot}{x}$ is given.
S:The minimiser is a linear force-free (Beltrami) field: $\nabla \times \mathcal{B} = \alpha \mathcal{B}$ for some $\alpha \in \mathbb{R}$.
Various other models, Arnold, Moffat, Taylor Nuñez, Komedanzick and counterexamples based on topology (Boorromean Rings, Hopf Links) and many questions. For example in which sense we take limit as $t \to \infty$. Strong, weak? In which sense we velocity field is as rest?

Evolution of Magnetic Helicity in Turbulent regimes

- Theorem 1 [CPAM 2024] There exists solutions $v, B \in L^\infty[(0,T), L^{3,\infty} \times L^{3,\infty})]$ which do not preserve magnetic helicity, nor energy nor cross helicity.
- Theorem 2 [ARMA 2021] There exists bounded solutions which do not preserve energy, nor cross helicity but whose helicity (constant a forteriori) is an arbitrary constant h.
- Rigorous result on conserved and dissipated quantities in ideal MHD Turbulence . Geophysical and Astrophysical fluid dynamics.

For this course, Theorem 1 is proved by an staircase laminate!
Beckie, Bukmaster and Vicol were the first to show that MH is not preserved in L^2 in the ideal case, which is an stark contrast with the vanishing resistivity limit (F-Lindberg proved of Taylor conjecture).

Ideal Ohm-Faraday in the two forms formalism

A computation shows that,

$$\omega \wedge \omega = E \cdot B dV$$

$\omega \wedge \omega = d(\alpha d\alpha)$
This has two important implications.

1. $E \cdot B$ is a compensated compactness (weakly continuous) Quantity!
2. In Ideal Ohm-Faraday $\omega \wedge \omega = 0$, which implies that

   $$\omega = \alpha_1 \wedge \alpha_2$$

   for suitable 1 forms

In any case, we write shortly Ohm-Faraday saying that there exists and space-time two form ω such that

$$d\omega = 0, \omega \wedge \omega = 0$$

Gradient Distribution & Differential Inclusions

The idea to solve such problems comes from the vectorial calculus of variations.

Let $W : M^{2\times 2} \to \mathbb{R}$ with $W \geq 0, W^{-1}(0) = \{A, B\}$

Ω regular bounded domains open and connected on $\partial\Omega$ has zero measure.

We will rephrase all our problems as a differential inclusion.

That is,given a set $K \subset M^{m\times n}$

$$\nabla u(x) \in K \ a.e \ x \in \Omega$$

Thus the relevant information is the range of the gradient.

Definition (Gradient Distribution (g.d))

$u : \Omega \to \mathbb{R}^m$ (Lipschitz on Sobolev map)

$v_u =$ Pushforward of the Lebesgue measure by the gradient $\equiv$ Gradient Distribution of u

$v_u \in \mathcal{M}(M^{m\times n})$

$$\int F(\lambda)\, dv_u(\lambda) = \frac{1}{|\Omega|}\int_\Omega F(\nabla u)\, dx$$

$$v_u(E) = \frac{|x \in \Omega : \nabla u(x) \in E|}{|\Omega|}$$

e.g.

$$\nabla u = A, v_u = \delta_A$$

Definition (Affine maps)

Given $A \in M^{m\times n}, b \in \mathbb{R}^m, l_{A,b}(x) = Ax + b$.

f Lipschitz if $\exists$ $M : |f(x) - f(y)| \leq M|x - y|$

f Holder α if $\exists$ M :$|f(x) - f(y)| \leq |x - y|^\alpha$

$W^{k,p}_{loc}(\Omega) = \cap W^{k,p}(\bar{\Omega})$ with $\tilde{\Omega} \subset \Omega$ where k = Numero of derivatives ,p integrability.
Given a matrix A, we will denote by Ω^A an open set where $Du\chi_{\Omega^A} = A$

Definition (Piecewise affine Sobolev maps)

$u \in W^{1,1}(\Omega)$ is piecewise affine if there exist pointwise disjoint sets $\{\Omega_i\}_{i=1}^{\infty}$ and N with $|N| = 0$ such that:

$$\Omega = \bigcup_i \Omega_i \cup N,$$
$$u = \sum l_{A_i,b_i} \chi_{\Omega_i},$$
$$Du = \sum A_i \chi_{\Omega_i}.$$

In particular,

$$\nu_u = \sum_{i=1}^{\infty} \frac{|\Omega_i|}{|\Omega|} \delta_{A_i}$$

$$\nu_u = \sum \lambda_i \delta_{A_i}$$

is purely atomic. $\lambda_i = \frac{|\Omega_i|}{|\Omega|}$

Gluing Argument

Let $u \in W^{1,1}(\Omega) \cap C^{\alpha}(\overline{\Omega})$ p.a.

For each v_i let $v_i \in W^{1,1}(\Omega_i) \cap C^{\alpha}(\overline{\Omega_i})$ such that $v_i - u \in W_0^{1,1}(\Omega_i)$

Then, $\tilde{u} = \begin{cases} v^i(x), x \in \Omega_i \\ u, otherwise \end{cases}$

$\tilde{u} \in W^{1,1}(\Omega)$ and $\|\tilde{u} - u\|_{C^{\alpha}(\overline{\Omega})} \leq 2sup\,\|v_i - u\|_{C^{\alpha}(\overline{\Omega_i})}$

Proof: Both of the claims are Useful Exercises.

Rescaling and covering argument

Assume $v_{\Omega^1} \in W^{1,1}(\Omega^1) \cap C^\alpha(\overline{\Omega^1})$ with $|\partial\Omega^1| = 0$, $v_{\Omega^1}\big|_{\partial\Omega^1} = l_{A,b}$

A standard covering argument yields that given Ω^2, there exists $\{r_i\}_{i=1}^\infty$ with $r_i^{1-\alpha} \leq \varepsilon$ such that if we declare $\Omega_i = r_i\Omega^1 + x_i$ it holds that, $|\Omega^2 \setminus \cup\Omega_i| = 0$
Then we set,

$$v_{\Omega_2}(x) = v_i(x) = r_i v_{\Omega_1}\left(\frac{x - x_i}{r_i}\right) + Ax + (1 - r_i)b, x \in \Omega_i,\ l_{A,b} \text{ otherwise.}$$

Lemma

Then,

1. $v_i = l_{A,b} \in \partial\Omega_i$
2. $\|v_i - l_{A,b}\|_{C^\alpha(\overline{\Omega_i})} \leq r_i^{1-\alpha}\|u - l_{A,b}\|_{C^\alpha(\overline{\Omega^2})}$

Corollary: The gluing lemma applies yielding $v_{\Omega^2} \in W^{1,1}(\Omega^2) \cap C^\alpha(\Omega^2)$ with:

1. $\nu_{v_{\Omega^1}} = \nu_{v_{\Omega^2}}$
2. $\|v_{\Omega^2} - l_{A,b}\|_{C^\alpha(\Omega^1)} \leq \varepsilon$

Proof of Lemma.

1.

$$x = x_i + yr_i, \quad y \in \Omega_i \Rightarrow \frac{x - x_i}{r_i} = y$$

$$\begin{aligned}
r_i v_{\Omega_1}\left(\frac{x - x_i}{r_i}\right)\Big|_{\partial\Omega_i} &= r_i(Ay + b) = A(x - x_i) + r_i b \\
&= Ax - Ax_i + r_i b + Ax_i + (1 - r_i)b \\
&= Ax + b \\
\Rightarrow v_i\Big|_{\partial\Omega_i} &= Ax + b
\end{aligned}$$

2. Hölder Improvement

$$\begin{aligned}
&\text{for } x = x_i + r_i y \\
[v_i - l_{A,b}](x) &= r_i v\left(\frac{x - x_i}{r_i}\right) + Ax_i + (1 - r_i)b_i - l_{A,b}(x) \\
&= r_i v_{\Omega_1}(y) - A(x - x_i) - br_i \\
&= r_i[v_{\Omega_1}(y) - l_{A,b}(y)]
\end{aligned}$$

$\square$

Proof of Lemma.

Thus, let $x_1 = x_i + r_i y_1, x_2 = x_i + r_i y_2$

$$|x_1 - x_2| = r_i |y_1 - y_2|$$
$$|y_1 - y_2| \leq \frac{1}{r_i} |x_1 - x_2|$$

Then,

$$\begin{aligned}
|[v^i - l_{A,b}](x_1) - [v^i - l_{A,b}](x_2)| &= |r_i(v^i - l_{A,b}[y_1]) - r_i(v^i - l_{A,b}[y_2]))| \\
&\leq r_i |y_1 - y_2|^\alpha \\
&= r_i \cdot r_i^{-\alpha} |r_i(y_1 - y_2)|^\alpha
\end{aligned}$$

$$= r_i^{1-\alpha} |x_1 - x_2|^\alpha$$

□

Proof of the corollary.

The first item is straigtforward For the second Observe that

$$\nabla v_{\Omega^2} = \nabla v_{\Omega^1}(\frac{x - x_i}{r_i})\chi_{x_i + r_i\Omega_1}$$

If we declare $S_i : \Omega_i \to \Omega^1; S_i(x) = \frac{x - x_i}{r}$, a similarity such that $S(\Omega_i) = \Omega$ Notice that [1]
$\{x \in S^{-1}(\Omega) = \Omega_i : \nabla v_{\Omega_1} \circ S \in E\} = S^{-1}\{y \in \Omega : \nabla v_{\Omega_1}(y) \in E\}$
Then for $E \subset \mathbb{M}^{m\times n}$
$\nu_{v_{\Omega_2}}(E) = \sum_i \frac{|S^{-1}\{y \in \Omega : \nabla v_{\Omega_1}(y) \in E\}|}{|\Omega_2|} = \sum_i r_i^n \nu_{v_{\Omega_1}}(E)\frac{|\Omega_1|}{|\Omega_2|} = \nu_{v_{\Omega_1}}(E)\sum \frac{|\Omega_i|}{|\Omega_2|}$
and we conclude because since $|\Omega_2 \setminus \cup_i \Omega_i| = 0$, $\sum_i \Omega_i| = |\Omega_2|$ $\square$

[1] In terms of push forwards measures, $(\nabla v_{\Omega_1} \circ S)_\#(\mathcal{L}) = (\nabla u_\#(S_\#(\mathcal{L}))$

Gradients supported on rank-one lines

In order to solve differential inclusions our approach, we need to construct functions such that the range of the gradients (in other words the support of the gradient distribution is prescribed).
Recall that a matrix is rank-one if and only if it can be expressed as a tensor product

$$A = a \otimes n$$

Exercise:Show that $Av = (v \cdot n)a$
Remark: In dimension 2, A is of rank-one iff $\det(A) = 0$.
Rank-one connections
Two matrices A, B are rank-one connected if and only if $A - B = a \otimes n$
What means this for diagonal matrices?

Scalar functions as vector value functions.

Let $s : \mathbb{R} \to \mathbb{R}$ and $\xi \in \mathbb{R}^n$. Then if we set $f(x) = s(\langle x, \xi \rangle)$, $\partial_i f = s'(\langle x, n \rangle)\xi_i$. Moreover if $a \in \mathbb{R}^n$ and we declare $f_a : \mathbb{R}^n \to \mathbb{R}, f_a(x) = f(x)a$

$$\partial_j (f_a)^i = s'(\langle x, \xi \rangle) a_i \otimes \xi_j, i.e\, Df_a = s'(\langle x, \xi \rangle) a \otimes \xi$$

The range lies in the line spanned by the rank-one matrix $a \otimes \xi$
Prescribed gradient: Thus if for example $s' = \{\lambda, (\lambda - 1)\}$, $D(f_a)(x) \in \{\lambda a \otimes \xi, (\lambda - 1) a \otimes \xi\}$ a.e
Prescribed gradient but small function We notice that if s[2] periodic we can rescale the construction declaring

$$(f_a)_j(x) = \frac{1}{j} s(j \langle x, \xi \rangle) a$$

This oscillating wave has the same gradient, but it is very small in L^∞.
Problem No control on boundary data, then we can not iterate

[2] $s \equiv$ saw-tooth function

The Roof construction: With control on the boundary

We start by prescribing gradients of functions with zero boundary values. It is not possible to have an exact value for two gradients.

Lemma (Roof-Lemma)

Let $\xi \in S^{n-1}$ a direction and let $\lambda, \alpha \in (0,1)$. Let $\xi^{(1)}, \ldots, \xi^{(J)} \in \mathbb{R}^n$ be such that $0 \in int\ conv\{\xi, -\xi, \xi^{(1)}, \ldots, \xi^{(J)}\}$. Denote,

$$\Omega^\lambda = \{x \in \Omega : \nabla f(x) = -\lambda\xi\}, \Omega^{(1-\lambda)} = \{x \in \Omega : \nabla f(x) = (1-\lambda)\xi\}$$

Then, for any open bounded set $\Omega \subset \mathbb{R}^n$ with$|\partial\Omega| = 0$ and any $\delta > 0$ there exists a piecewise affine Lipschitz function $f \in Lip_0(\Omega)$ $[f]_\alpha \leq \delta$ with

$$\nabla f(x) \in \left\{-\lambda\xi, (1-\lambda)\xi, \xi^{(1)}, \ldots, \xi^{(J)}\right\} \quad a.e.\ x \in \Omega, \tag{3}$$

and

$$|\Omega^\lambda| \geq (1-\lambda)(1-\epsilon)|\Omega|, |\Omega^{(1-\lambda)}| \geq \lambda(1-\epsilon)|\Omega| \tag{4}$$

Notice that we could choose the auxiliar vectors $\xi^{(1)}, \ldots, \xi^{(J)} \in \mathbb{R}^n$ as close to zero as we like. On the other hand $\pm\xi^\perp$ is a trivial choice.

Proof of the Roof-Lemma

Let s be the saw tooth function as before and set

$$P = \left\{ x \in \mathbb{R}^n : \, x \cdot \xi^{(j)} > -1 \text{ for all } j = 1 \ldots J \text{ and } |x \cdot \xi| < 1 \right\}.$$

Then P is a convex open set containing 0. Moreover, for any $N \in \mathbb{N}$ the function

$$f_N(x) = \min \left\{ \min_j (1 + x \cdot \xi^{(j)}), \, \frac{1}{N} s(Nx \cdot \xi) \right\}$$

is Lipschitz, satisfies (3), and $f_N = 0$ on ∂P.
Moreover, by choosing N, sufficiently large in terms of ϵ

$$\frac{1}{N} h(Nx \cdot \xi) < \min_j (1 + x \cdot \xi^{(j)}) \quad \text{ on } (1 - \epsilon)P,$$

and thus, $f_N(x) =, \frac{1}{N} s(Nx \cdot \xi)$ from which (4) follows.

- Exercise 1. For a general Ω we apply a standard rescaling and covering argument
- Exercise 2. For $a \in \mathbb{R}^m$, if we declare $f_{N a}(x) = f_N a$ $D(f_N)_a(x) \in \{-\lambda a \otimes \xi, (1 - \lambda) a \otimes \xi, a \otimes \xi^j\}$ a.e

Simple Laminates

Def:Let A, B rank-one connected and $C = \lambda A + (1-\lambda)B$. Then $\lambda\delta_A + (1-\lambda)\delta_B$ is called a *s*imple laminate.
The roof construction allows to prove the following lemma. For every δ, ϵ and every domain Ω, There exists $u : \Omega \to \mathbb{R}^m$ piecewise affine such that

1. $u\big|_{\partial\Omega} = Cx$
2. $|x : Du(x) = A| \geq \lambda(1-\epsilon)$
3. $|x : Du(x) = B| \geq (1-\lambda)(1-\epsilon)$
4. $Du \in \{A, B, \mathtt{B}(C, \delta)\}$ a.e.

In terms of the gradient distribution

$$\|\nu_u - (\lambda\delta_A - (1-\lambda)\delta_B)\| \leq \epsilon$$

Indeed we can construct a sequence u^j such that

- $\nu_{u_j} \to \lambda\delta_A + (1-\lambda)\delta_B$
- $u_j \to l_{C,b}$ uniformly and weakly in $W^{1,p}$

Since $|C| \leq \max |A|, |B|$, the mapping is Lipschitz.

Constraints

In order to iterate the construction a very important technical detail is being able to handle constraints. That is to say if A, B are rank-one connected and $A, B \in \mathcal{M} \subset \mathbb{M}^{m\times n}$ can be find u such that ν_u approximate $\lambda\delta_A + (1-\lambda)\delta_B$ and $\nu_u \subset \mathcal{M}$ and still u is piecewise affine.

- $\mathcal{M} = \mathbb{M}^{m\times n}_{sym}$. OK. Kirhcheim. Use Hessians D^2 and a ingenuous approximation of smooth Hessians by piecewise quadratic ones
- $\mathcal{M} = \{A : \det A = a > 0\}$ Müller vSverák.
- $\mathcal{M}$ diagonal matrices. nop.
- $\mathcal{M} = \overline{\mathbb{M}^{2\times 2}_{sym}}$. Use $D(\partial_z u)$ and take conjugates to reduce the situation to the case of symmetric matrices.

PreLaminates(Splitting)

The previous game, gets much more interesting when we iterate. In terms of measures code the above process by the notation,

$$\underbrace{\delta_C}_{\mu_1} \rightsquigarrow \underbrace{\lambda\delta_A + (1-\lambda)\delta_B}_{\mu_2}$$

and we say that μ_2 is obtained by splitting μ_1.
Given a discrete probability measure with finite support i.e

$$\mu = \sum_{i=1}^{n} \lambda_i \delta_{A_i}$$

if $A_1 = \lambda B + (1-\lambda)C, A - C \in \Lambda$ such that $A - C \in \Lambda$, we say that

$$\mu_1 \rightsquigarrow \mu_2 = \lambda_1 \lambda \delta_B + (1-\lambda)\lambda \delta_C + \sum_{i=2} \lambda_i \delta_{A_i}$$

We say that μ_1 splits in μ_2,

Laminates:Definition

Definition

The class of Prelaminates $\mathcal{PL}$ is the smallest class of probability measures such that

- It contains all Dirac masses δ_A
- It is closed after splitting.

Given set $K^{lc} = \{A \in \mathbb{M}^{m\times n} : \text{ exists } \nu \in PL(K), \int \lambda d\nu = A\}$

Def: More general Splitting Clearly the above process is closed by taking convex combinations. If $A_1 = \overline{\omega}$ and $\omega \in \mathcal{PL}$, then $\nu_2 = \lambda_1\omega + \sum_{j=2}^{n} \lambda_j\delta_{A_j} \in \mathcal{PL}$ as well and we denote it by

$$\nu_1 \overbrace{\rightsquigarrow}^{\omega} \nu_2$$

Aproximation of Prelaminates

If we combine the approximation of simple laminate, with the gluing lemma, we obtain the following approximation of a finite order laminate by the gradient distribution of a Lipschitz map.

Lemma

Let $\nu = \sum \lambda_i \delta_{A_i} \in \mathcal{PL}$*, and* $\epsilon > 0, 0 < \alpha < 1$*. Then there exists* $u \in Lip_{p.a,C}(\Omega)$ *such that*

- $\nu(A_i) - \nu_u(A_i) \leq \epsilon$
- $\|Du\| \leq (1 + \epsilon) \max |A_i|$

In particular $\nu_u(\mathbb{M}^{m\times n} \setminus suppt(\nu)) \leq \epsilon$

Laminates

Definition

A measure is called a laminate if It is the weak star limit of a laminate. Laminates are a prime example of Homogeneous Gradient Young measures.
$\mathcal{L}(K)$ is the set of laminates supported in K

Tartar and Morrey conjecture

Suppose that for every $A, B \in K$, $A - B$ is not rank-one are laminates trivial, are gradient distributions trivial?

- Ok if K is compact connected. (vSverák, F-Kristensen proof based on regularity).
- False for four matrices. T_4 configurations.
- No T_4 and no rank-one connections.
  - Laminates are trivial (Székelyhidi),
  - Gradient distribution with affine boundary values are trivial (F-Székelyhidi Acta. 08),

  1. But are all gradient distributions, laminates. This is equivalent to Morrey conjecture, arguably the most important open problem in the vectorial Calculus of Variations.

Wave Cones

A corner stone of Murat-Tartar compensated compactness theory is the concept of Wave cone. This generalizes the role of rank-one matrices in the case of gradients, when we are dealing with a general linear system of p.d.e with constant coefficients

$$\mathcal{L}(z) = A_{ijk}\partial_j z^k$$

Example

$$\mathcal{L}(z) = \nabla \times z = \varepsilon_{ijk}\partial_j z^k$$

where ε_{ijk} are the Levi-Civita symbols.

Definition

Let $\mathcal{L}$ be a first order differential operator.Then $l \in \Lambda_L \subset \mathbb{R}^N$ if there exists a direction $\xi \in \mathbb{R}^d$ such that for every $h : \mathbb{R} \to \mathbb{R} : \mathcal{L}(h(x)l) = 0$

Explicit computation

Lemma

Let $l = (l^1, l^2, \ldots, l^k)$. Then $l \in \Lambda$ iff $\exists \xi = (\xi_1, \xi_2, \ldots, \xi_k)$ such that

$$\forall i \; A_{i,j,k} \xi_j l^k = 0$$

Proof.

$A_{i,j,k} \partial_j (h l^k) = A_{i,j,k} l^k \xi_j h'(x \cdot \xi) = 0$ for an arbitrary function h iff

$$A_{i,j,k} l^k \xi_j = 0$$

for every i $\square$

The rank-one cone as a wave cone

Exercise

Show that if for a matrix field $Z = Z_{ij}$ we define its curl as the curl of all its rows, that is $(\mathcal{L}Z)_{il} = \nabla \times Z_{ij} = \varepsilon_{ljk}\partial_j Z_{ik}$, Then, the corresponding wave cone is the rank-one cone. That is, for such $\mathcal{L}$

$$\Lambda^{\mathcal{L}} = RC$$

Staircase Laminates $\mathcal{SL}$

Informally, they are purely atomic Laminates supported in unbounded sets and with critical integrability properties. As in the case of Laminates we will define first simple staircase Laminates and then we will close them under splitting.

Definition (K step laminates)

$K \subset M^{d\times m}$ and $A \notin K$. Suppose $\exists$ increasing sequence $A_n \subset M^{d\times m} \setminus K, A_0 = A, \mu_n \in \mathcal{P}(K)$ and γ_n such that:

1. K Step Laminates

   $\omega_n = (1-\gamma_n)\mu_n + \gamma_n \delta_{A_n}$ are laminates of finite order with $\bar{\omega}_n = A_{n-1}$
2. $\beta_n = \prod_{k=1}^{n} \gamma_k, \lim_{n\to\infty} \prod_{k=1}^{n} \gamma_k = 0$

We will create an iterative sequence ν^N, where $\nu^N \overbrace{\rightsquigarrow}^{\omega_n} \nu^{N+1}$

Set $\nu^N = \underbrace{\sum_{k=1}^{N} \beta_{k-1}(1-\gamma_k)\mu_k}_{\tilde{v}_n} + \beta_N \delta_{A_N}$ If we keep track of the iteration we arrive to the explicit expression,

$\nu^\infty(E) = \lim_{N\to\infty} \nu^N(E)$

Remark $\nu^\infty \in \mathcal{M}(K)$ satisfies that $supp\nu^\infty \subset K, \bar{\nu} = A$.

Definition (Staircase Laminates)

- We say that $\nu \in \mathcal{SL}(K)$ if there exists K steps laminates ω_n such that

$$\nu^\infty = \lim \nu^N$$

and $\nu^N \overbrace{\rightsquigarrow}^{\omega_n} \nu^{N+1}$

- For $1 \leq p < \infty$ We say that $\nu \in \mathcal{SL}^p(K)$ if there exists a constant M such that

$$\frac{1}{M}(1+|\bar{\nu}|)^p \leq t^p|\nu\{X \in K : |X| \geq t\}| \leq M(1+|\bar{\nu}|)^p$$

Arguably , the above are called simple staircase laminates, and staircase laminates is the smallest class of probability measure close under splitting along simple laminates or simple staircase laminates. For some applications we need that the sequence $A_n \in \mathcal{U}$ and we will speak of $\mathcal{SL}(K; U)$

Invariance

Lemma (Invariance under Rank-one cone preserving Maps)

Let $T : \mathbb{R}^{d\times m} \to \mathbb{R}^{d\times m}$ be a linear map preserving rank-one matrices. If

$$v^\infty = \sum \lambda_i \delta_{A_i} \in \mathcal{SL}(K), \overline{v^\infty} = A$$

Then,

$$T_\# v^\infty = \sum \lambda_i \delta_{T(A_i)} \in \mathcal{SL}(T(K))$$

$$\overline{T_\# v^\infty} = T(\overline{\nu^\infty})$$

Integrability of Laminates

Lemma (Upper bound)

Assume that for $1 \leq p < \infty$ *:*

1. $\exists\ c_0, \mu_0 \geq 1$ *such that supp* $\mu_n \subset \{X \in \mathbb{R}^{d \times m} : |X| \leq c_0 |A_n|\}$
2. $\beta_n |A_n|^p \leq M_0$
3. $\exists\ c : |A_n| \leq |A_{n+1}| \leq c|A_n|$

Then ,

$$v^{\infty}(X : |x| > t) \leq M_0 c^p c_0^p t^{-p}$$

Let

$$t_n = c_0|A_n|$$
$$t_{n+1} = c_0|A_{n+1}| \leq ct_n$$

By assumption for $k < n$

$$\mu_k(\{X : X \geq t_n\}) = 0$$

Thus, for $N > n$,as μ_k are subprobability measures

$$v^N(\{X : |X| \geq t_n\}) \leq \sum_{k=n+1}^{N} \beta_{k-1}(1-\gamma_k) + \beta_N$$

But recall that $\beta_{k-1}\gamma_k = \beta_k$

$$\beta_{k-1}(1-\gamma_k)+\beta_k(1-\gamma_{k+1}) = \beta_{k-1}-\underbrace{\beta_{k+1}\gamma_k + \beta_k}_{0}-\beta_k\gamma_{k+1} = \beta_{k-1}-\beta_{k+1}$$

Thus telescoping we get that for every $N > n$

$$\sum_{k=n+1}^{N-1}(\beta_{k-1} - \beta_{k+1}) = \beta_n - \beta_N$$

Thus

$$v^N(\{X : X \geq t_n\}) = \beta_n \leq M_0|A_n|^{-p} = M_0 c_0^p t_n^{-p}$$

Where the first inequality is by assumption and the second by definition of t_n

Now if $t_n \leq t \leq t_{n+1}$:

$$\begin{aligned} v^N\{X : |X| \geq t\}) &\leq v^N\{X : |X| \geq t_n\}) \\ &\leq M_0 c_0^p t_n^{-p} \\ &= M_0 c_0^p c^p (ct_n)^{-p} \\ &\leq M_0 c_0^p c^p t^{-p} \end{aligned}$$

Lower bound

Lemma

$$\beta_n|A_n|^p \geq M_1$$
$$\mu_n(\{X : |X| \geq c_0|A_n|\}) \geq c$$

For constants $0 < c_0 < 1, M_1 > 0, c_1 \geq 0$
Then,

$$v^\infty(\{X : |X| \geq t\}) \leq c_1 c_0^p M_1 t^{-p}$$

Set $t_n = c_0|A_n|$.Observe that for every $k \geq n$:

$$\mu_k(\{A : |A| \geq t_n\}) \geq \mu_k(\{A : \|A| \geq t_k\}) \geq c_1$$

$$
\begin{aligned}
v^N(\{A : |A| \geq t_n\}) &\geq \sum \beta_{k-1}(1-\gamma_k)\mu_k + \beta_N \\
&\geq c_1 \sum_{k=n} \beta_{k-1}(1-j_k) + \beta_N \\
&\geq c_1\beta_{n-1} + (1-c_1)\beta_N \\
&\geq c_1 M_1 |A_{n-1}|^p \\
&\geq M_1 c_1^p c_0^p t_{n-1}^p \\
&\geq M_1 c_1 c_0^p t_n^{-p}
\end{aligned}
$$

Set $t_0 = c_0|A|$.Choose $t_n \leq t \leq t_{n+1}$.Then:

$$
\begin{aligned}
v(\{X : |X| \geq t\}) &\geq v(\{X : |X| \geq t\}) \\
&\geq M_1 c_1 c_0^p t_n^{-p} \\
&\geq M_1 c_1 c_0^p t^{-p}
\end{aligned}
$$

Examples of Laminates

In practise, most of the examples in the literature sit on 2×2 diagonal laminates and the basic step is a second order laminate. In this situation the rank-one geometry is that of separately convexity and thus rank-one direction are horizontal and vertical lines. We will write

$$(x, y) \equiv diag(x, y)$$

and use $(x, 0), (0, y)$ as rank-one matrices. Moroever we will use the norm $|(x, y)| = \max x, y$. Often, the construction will leave in a different two dimensional subspace of matrices also spanned by two rank-one directions.

An L^1 step

Consider $A_n = \mathrm{diag}(n, n)$ $K = \{(x, -x), x \in \mathbb{R}\}$. We write

$$(k, k) = \frac{1}{3}(k, -k) + \frac{2}{3}(k, 2k)$$

$$(k, 2k) = \frac{1}{4}(-2k, 2k) + \frac{3}{4}(2k, 2k)$$

Thus

$$\omega_k = \frac{1}{3}\delta_{k,-k} + \frac{1}{4}\delta_{-2k,2k} + \frac{1}{2}\delta_{2k,2k}$$

is a second order laminate and the step of our staircase.

The staircase

Therefore with the notation we are using.

$$A_n = 2^n(1,1), \gamma_n = \frac{1}{2}$$

$$\mu_n = \frac{1}{3}\delta_{(2^{n-1},-2^{n-1})} + \frac{1}{4}\delta_{(-2^n,2^n)}$$

which is supported in $K \cap \{|(x,y)| \leq |A_n|\}$ Now we check the conditions $A_n \to \infty$

$$|A_{n+1}| = 2|A_n|$$

and

$$\beta_n = 2^{-n}$$

tends to zero. Moreover

$$\beta_n|A_n| = 1$$

Thus $\omega_n = \omega_n(A_n, \mu_n, \gamma_n)$ defines a weak L^1 laminate.

Since in the application to Calderón Zygmund theory, we are looking for a sequence of mappings violating the $L^1 - L^1$ estimate it will suffices to consider the sequence v^N.

Notice that laminate is in weak L^1. However if $diag = (CO_+(2) \oplus CO_(2)$ with $CO_+(2) = \{(x,x)\}, CO_-(2) = \{x, -x\}$.

$$\int_{CO_+(2)} |\lambda| d\nu^N = 1$$

Exercise

Relate the construction with second order derivatives by inserting the above construction in symmetric matrices.

Now the sets are

$$E = \{(a, Ka), (Ka, a) : a \in \mathbb{R}$$

There exists a $\frac{2K}{K-1}$ laminate ν^{upper} supported on E and a $\frac{2K}{K+1}$ laminate supported on E, ν^{lower}. However they are very different as the $\frac{2K}{K-1}$ laminate the corresponding sequence A_n are matrices such that $l_{A_n,b}$ is an honest quasiconformal map. In the case $\frac{2K}{K+1}$ A_n must be matrices with negative determinant.
This is an important observation when we construct the corresponding solution to the laminate as in the case $\frac{2K}{K-1}$ is a limit of homeomorphism and in the case $\frac{2K}{K+1}$ is not and it can not be since weak $W^{1,1}$ quasiconformal maps are quasiconformal
Indeed

$$\nu^{upper} \in \mathcal{SL}^{\frac{2K}{K-1}}(E, \mathcal{U})$$

for $\mathcal{U} = E^{pc}$, the polyconvex hull of E. However,

$$\nu^{lower} \in \mathcal{SL}^{\frac{2K}{K-1}}(E, \mathbb{R}^{2\times 2})$$

Weak quasiregular steps

We just provide the weak quasiregular steps for illustration, but also because we will need them also for the nonlinear Beltrami. The steps are a second order laminate as before $A_n = (-n, n)$ this time.

$$((-n, n) = \frac{K}{n(K+1)+K}\delta_{(\frac{n}{K}, n)} + (1 - \frac{K}{n(K+1)+K})\delta_{(-(n+1), n)}$$

$$(-(n+1), n) = \frac{K}{n+1(K)}\delta_{(n+1, \frac{-(n+1)}{K})} + (1 - \frac{K}{n+1(K)})\delta_{(-(n+1), n+1)}$$

Thus $\omega_n = \mu_n + \gamma_n \delta_{A_n}$ with

$$\gamma_n = (1 - \frac{K}{n(K+1)+K})(1 - \frac{K}{n+1(K)})$$

Precise integrals

It is easy to see that all properties are satisfied except for the integrability. In the exercises (assisted by the appendix about infinite products) it is proven that,

$$(\Pi_{n=1}^{N}(1-\frac{z}{n+z})(1-\frac{z}{n}))N^{2z} \to \frac{\Gamma(z)}{-\Gamma(-z)}$$

Thus,

$$(\Pi_{n=1}^{N}(1-\frac{z}{n+z})(1-\frac{z}{n+1}))N^{z}(N+1)^{z} \to (1-z)\frac{\Gamma(z)}{-\Gamma(-z)}$$

Writing $z=\frac{K}{K+1}, 1-z=\frac{1}{K+1}$ Thus

$$\beta_N|A_n|^{\frac{2K}{K+1}} \to (K+1)\frac{\Gamma(\frac{K}{K+1})}{-\Gamma(\frac{-K}{K+1})} \tag{5}$$

If we know let K to infinity, (or repeat the process) in the limit, we get a weak L^2 $\nu^\infty \in \mathcal{SL}(E)$ where E are two by two matrices with determinant equal to 0

The same process works in all dimensions, and also with matrices of given rank.

Q: It is clear that such mappings will not satisfy that the distributional Jacobian coincides with the pointwise Jacobian. Therefore they will not satisfy condtion N or $N-1$. Can we use laminates to construct such pathological maps where K is in weak L^1?

p-Laplacian Laminate

In this situation the data will be
1-$K_p = \{(a, a^{p-1}), (-a, -a^{p-1}) : a \geq 0\}$ For a parameter b
2-$A_n^b = (bn, -n^{p-1})$
Then

$$\gamma_n = (1 - \frac{b}{(b+1)(n+1)})(1 - \frac{(1+\frac{1}{n})^{p-1} - 1}{b^{p-1} + (1+\frac{1}{n})^{p-1}})$$

Exercise

Prove that

$$\lim_{n\to\infty} \Pi_{n=1}^{N}(1 - \frac{(1+\frac{1}{n})^{q} - 1}{a + (1+\frac{1}{n})^{q}})N^{\frac{q}{a+1}} = C$$

So we obtain integrability for

$$q = \frac{b}{b+1} + \frac{p-1}{b^{p-1}+1}$$

for each p we obtain an optimal b and a corresponding $p - 1 < q < p$

Our aim is to support the measure in the Faraday cone $\Lambda^F = \{(E, B) : E \cdot B = 0$. However, as often, a much smaller set suffices, Namely we will consider $K = \{(E, B) : E \cdot B = 0$ and moreover since we will want the laminate to have an arbitrary center of mass (E_0, B_0) we wil use a two dimesional construction and embedded through the isomorphism $(x, y) \to (xE_0, yB_0)$

Thus if $p = 2$, our two dimensional situation would do the job.

Anistropic norms and spaces

The easiest way to explain the construction is by introducing the anisotropic norm

$$\|(x,y)\|_p = |x|^p + |y|^{p'}$$

Then if the step laminates satisfy that

$$\beta_N \|A_n\|_p \leq C$$

we would get that

$$\nu^\infty(\{A : |A|_p \geq t) \leq \frac{C}{t}$$

The construction will be based on two steps laminates as in the previous cases. The sequence $A_n = (2^n, (2^n)^{(p-1)})$. Then it is easy to see that

$$\gamma_n = \frac{1}{2^p}, \beta_N = \frac{1}{(2^N)^p}$$

and thus

$$\beta_N \|A_n\|_p \leq 2$$

Differential Inclusions

Laminates provide examples of approximate solutions to differential inclusions.Starting from the work of Nash in the setting of isometric inmersions of Riemmanian Manifolds,Convex integration denotes a general process to combine suitably localized approximate solutions to find exact solutions.

In the setting of Sobolev or Lipschitz maps there are three main techniques available :

1. In approximation
2. Piecewise affine approximation
3. Baire category arguments

In this course, we will mainly follow 2 and in particular an adaptation of the recent preprint [KMSzX24]

Lemma

Let A, B be rank-one connected.

$$\mathcal{C} = \lambda A + (1-\lambda)B.$$

Then $\forall \delta, \varepsilon > 0,\ \exists\ u \in Lip_{p.a,C}(\Omega)$

$$Du \in B(A,\varepsilon) \cup B(B,\varepsilon),$$

$$|\{x : Du \in B(A,\varepsilon)\}| = \lambda,$$

$$\|u - Cx + b\|_{C^\alpha} \leq \delta,$$

$$u\big|_{\partial\Omega} = Cx + b$$

Baby Convex Integration

Proof For the case of gradients, the statement follows indeed by the so-called tent construction of Müller and vSverák. However the baby convex integration gives us an idea of how convex integration works in general and at the same time is adaptable to suitable linear and nonlinear constraints. Firstly the approximation of a laminate by a gradient distribution through tthe roof construction provides us with u^1 with the following properties :

1. $u^1\big|_{\partial\Omega} = Cx$
2. $|x : Du^1(x) = A| \geq \lambda(1 - 2^{-i})$
3. $|x : Du^1(x) = B| \geq (1 - \lambda)(1 - 2^{-i})$
4. $Du^1 \in \{A, B, \mathtt{B}(C, \delta)\}$ a.e.

We claim that there exists a sequence u^k with the following properties:

$$U^k = \{x \in \Omega : dist(Du_k, \{A, B\}\} \leq (1 - 2^{-k})\delta$$

$$u_1 = \frac{1}{2}\langle Cx\ ,\ x\rangle \tag{2.1}$$

$$|\{x \in \Omega : Du_i = A\}| \geq (1-\varepsilon)\lambda|\Omega| \tag{2.2}$$

$$|\{x \in \Omega : Du_i = B\}| \geq (1-\varepsilon)\lambda|\Omega| \tag{2.3}$$

$$u_{k+n} = u_k, \quad x \in U_i \tag{2.4}$$

$$|\Omega \setminus U_{i+1}| \leq |\Omega \setminus U_i| \tag{2.5}$$

$$\mathrm{dist}(Du_i, [A, B]) \leq (1 - 2^{-i})\delta \tag{2.6}$$

Set $\Omega^{error} = \Omega \setminus U^k$ Given u^k notice that $\Omega^{error} = \bigcup \Omega_i \cup N$ where $|N| = 0$ and Ω_i are open sets such that :

$$Du_k \chi_{\Omega_i} = C_i$$

with

$$dist[C_i, [A - B]] \leq (1 - 2^{-k})\delta$$

Then the roof construction applied to (C_i, Ω_i) yields v_i such that :

$$\tilde{C} = \lambda A + (1 - \lambda)B + D = \lambda(A + D) + (1 - \lambda)B$$

1. $v_i\big|_{\partial\Omega_i} = \langle C_i x, x\rangle +$ lower order (l.o)
2. $$\underbrace{\Big|x : Dv_i \notin \mathtt{B}(\{A + D, B + D\}, 2^{-k}\delta)\Big|}_{\Omega_{k,i}} \leq 2^{-k}|\Omega_i|$$
3. $\|v_i - (C + l.o)\|_{C^\alpha} \leq 2^{-k}|\Omega_i|$

Set

$$u^{k+1} = \begin{cases} u^k & x \in \Omega_k \\ v_i & x \in \Omega_i \end{cases}$$

$$\Omega_{k+1} = \bigcup_i \Omega_{k,i}$$

$$|\Omega_{k+1}| = \sum |\Omega_{k,i}| \le 2^{-k} \sum |\Omega_i| \le 2^{-k} |\Omega_k|$$

$$\left\| u^{k+1} - u^k \right\|_{C^\alpha} = \max \|v_i - C_i + \text{l.o.}\| \le 2^{-k}$$

Then

$$u^k \text{ is a Cauchy sequence in } C^\alpha$$
$$u^k \to u^0 \text{ weakly in } W^{1,p}$$

$$U^{k+1} \subset U^k$$

$$\left| \bigcap_{k=1}^{\infty} U^k \right| = 0$$

$\int(dist(Du_k, \{A, B\}) - \delta)^+ = \lim \int(dist(Du^k, \{A, B\}) - \delta)^+ \leq \lim |\Omega_k| = 0$

- Fixing the volume property :

$$\mu_{u_\lambda} = \frac{|x \in \Omega : |D^2 u - A| \leq \delta|}{\Omega}$$

$$\frac{\lambda}{2} \leq \mu_{u_\lambda} < \lambda$$

$$C = \hat{\lambda} A + (1 - \hat{\lambda}) B$$

$$\mu_{\hat{\lambda}} > \lambda$$

$$\lambda = t\mu_{\hat{\lambda}} + (1 - t)\mu_{\hat{\lambda}}$$

Then if we declare,

$$u = \begin{cases} u_{\lambda} & \text{on } \Omega_{\lambda} \\ u_{\hat{\lambda}} & \text{on } \Omega_{\bar{\lambda}} \end{cases}$$

$$|x : D^u \in \mathtt{B}(A, \varepsilon)| = t\mu_{\hat{\lambda}} + (1-t)\mu_{\hat{\lambda}} = \lambda$$

as required.

A similar strategy works in the bounded case where we replace simple laminates with laminates of finite order.However,one needs to be careful with the splitting sequence of the laminate.

Theorem

Let $K \subset M^{mxn}$.Suppose that $\forall A \subset M^{mxn}$ $\exists$ a staircase laminate v_A such that :

- $\overline{\nu_A} = A$
- v_A *is supported in* K
- $c_2|A|^p t^{-p} \leq v_A(B : |B| \geq t) \leq c_1|A|^p t^{-p}$

Then, there exists $u \in W^{1,1}(\Omega)$ such that:

$$Du(x) \in K$$

$$\|u - l_{A,b}\|_{C^\alpha} \leq \delta$$

$$ct^{-p} \leq |x : Du(x) \geq t| \leq ct^{-p}$$

Staircase Laminates and gradient distributions

The proof is similar to the case of finding a gradient distribution which approximates a simple laminate conceptually. However,though the resulting mapping is highly more intricate intricate.
It consists of three steps analogous to the case of a simple Laminate.

1. Replace Laminate by a gradient distribution of a piecewise affine solution u^1_{Id} almost supported in K.
2. At each subdomain , Ω_i where $Du^1 = A_i \notin K$ replace by $v_{A_i} \to u_{A_i}$
3. Prove convergence of this iterative process.

Gradients distribution and finite order laminates

The following is an exercise, combining the approximation of simple laminates and the gluing lemma. Approximation of finite order laminates Let $\nu \in L(\mathbb{M}^{d\times m})$ be a laminate of finite order with center of mass A such that $\nu = \sum_{j=1}^{J} \lambda_j \delta_{A_j}$ where $\lambda_j > 0$ and $A_j \neq A_k$ for $j \neq k$. Then for any vector $b \in \mathbb{R}^d$, any $\epsilon > 0$, and any regular domain $\Omega \subset \mathbb{R}^m$, we can construct $u \in Lip_{p.a}$ such that

- $u = l_{A,b}$ on $\partial\Omega$.
- The map u satisfies $\|\nabla u\|_{L^\infty(\Omega)} \leq \max_i |A_i|$.
- Moreover, for each $j = 1, \ldots, J$, the following holds:

$$(1-\epsilon)\lambda_j|\Omega| \leq |\{x \in \Omega : \nabla u(x) = A_j\}| \leq (1+\epsilon)\lambda_j|\Omega| \qquad (7)$$

- Since $\sum_{j=1}^{J} \lambda_j = 1$, this estimate also implies:

$$|\Omega^{error}| = |\{x \in \Omega : \nabla u(x) \notin \operatorname{supp}\nu\}| \leq \epsilon|\Omega| \qquad (8)$$

Staircase laminates approximated by gradient distributions

Proposition ($\mathcal{SL}(K)$ approximation by g.d)

Suppose that for $A \in \mathbb{M}^{m\times n}$ there exists $v_A \in \mathcal{SL}(K)$ with $\bar{v}_A = A$. Then for each $b \in R^d, \varepsilon \in (0,1), \alpha \in (0,1), s \in (1,\infty)$ and each regular domain $\Omega \subseteq \mathbb{R}^m$ $\exists$ p.a $u_A \in W^{1,p}(\Omega) \cap C^\alpha(\bar{\Omega})$ with $u = l_{A,b}$ on $\partial\Omega$ and

$$\Omega_{error} = \{x \in \Omega : \nabla u(x) \in K\}$$

we have :

1. $\int_{\Omega_{error}} 1 + |\nabla u|^s dx \leq \varepsilon|\Omega|$
2. $(1-\varepsilon)v_A(E) \leq \frac{|x\in\Omega:\nabla u(x)\in E|}{|\Omega|} \leq (1+\epsilon)\nu_A(E)$

Proof

Recall that ν^∞ is the limit of measures $\nu^N = \tilde{\nu}^N + \beta_N\delta_{A_N}$ with *supp* $\tilde{\nu}^N \subset K, A_N \notin K$ and

$$\nu^{N+1} = \tilde{\nu}^N + \beta_N\omega^{N+1}$$

We will approximate each ν^N by the g.d ν_{u^N} of a p.a Lipchitz map u^N with the following properties. Set,

$$\Omega_{\text{ind}}^{(N)} = \{x \in \Omega : \nabla u^N = A_N\}$$
$$\Omega_{\text{error}}^{(N)} = \{x \in \Omega : \nabla u^N(x) \notin \operatorname{supp} \nu^N\} \cup N$$

The sequence will satisfy,

1. $u^N = u^k$ in $\Omega \setminus \Omega_{\text{ind}}^{(k)}$ for $1 \leq k \leq (N-1)$
2. $$c_N^{-1} \nu^N(E) \leq \frac{|\{x \in \Omega : \nabla u^N(x) \in E\}|}{|\Omega|} \leq c_N \nu^N(E)$$

   where $c_N = \prod(1 + 2^{-j}\eta)$ and η is such : $e^\eta \leq 1 + \varepsilon$ and thus $e^{-\eta} \geq 1 - \varepsilon$

Growth of the Laminate

In particular, take $E = A_N$ since $\Omega_{ind}^{(N)} = \{x : \nabla u^N = A_N\}$ and $\nu^N(\{A_N\}) = \beta_N$

$$\frac{\beta_N}{c_N} \leq \frac{|\Omega_{ind}^N|}{|\Omega|} \leq c_N \beta_N$$

Let us elaborate on the construction of this sequence.
u^1 is obtained from applying the roof Lemma to ν^1 and the basic approximation Lemma. Similarly, u_N will emerge from u_{N-1} by applying the roof construction to the step laminate ω_n on the region Ω^{A_n}

Suppose now that u_N is constructed and write $\Omega_{ind}^{(N)} = \cup \Omega_i$ such that $\nabla u_N = A_N$ on Ω_i

Recall now, that the $\mathcal{SL}$ laminate is built by adding steps ω_n and let ω_n be such that:

$$\omega_{N+1} = (1 - \gamma_{N+1})\mu_{N+1} + \gamma_{N+1}\delta_{A_{N+1}}$$

Then, in each Ω_i, we replace u^N by the corresponding v^i approximating ω_{N+1} with $v^i\big|_{\partial\Omega} = l_{A_N}$ Then,

where

$$\begin{aligned}
&|x \in \Omega : \nabla u_{N+1} \in E| = \\
&\left|x \in \Omega \setminus \Omega_{ind}^N : \nabla u_N \in E \setminus \{A_N\}\right| + \\
&\left|x \in \Omega_{ind}^N : \nabla u_{N+1} \in E\right|
\end{aligned}$$

The first term is taken care of by the induction assumption:

$$c_N^{-1} v^N(E \setminus \{A_N\}) \leq \frac{|\{x \in \Omega : \nabla u_N(x) \in E \setminus \{A_N\}\}|}{|\Omega|} \leq c_N v^N(E \setminus \{A_N\})$$

$$\frac{\omega_{N+1}}{c} \leq \frac{|x \in \Omega_i : \nabla v_i(x) \in E|}{|\Omega|} \leq c\omega_{N+1}(E)$$

with $c = (1 + 2^{-(N+1)})\eta$

Let us put the estimates together:

$$\begin{aligned}
|\{x \in \Omega : \nabla u^{N+1} \in E\}| &\leq c_N \nu^N(E \setminus \{A_N\})|\Omega| + \sum_i (1 + 2^{-(N+1)}\eta)|\Omega_i|\omega_{N+1}(E) \\
&= c_N \tilde{\nu}^N(E)|\Omega| + 1 + 2^{-(N+1)}\omega_{N+1}(E)|\Omega^N_{ind}| \\
&\leq c_{N+1}[\tilde{\nu}^N(E) + \beta_N \omega_{N+1}(E)] \\
&\leq c_{N+1}[\nu^{N+1}(E)]
\end{aligned}$$

Since $\nu^{N+1} = \tilde{\nu}^N + \beta_N \omega_{N+1}$

The estimate of error

Notice that

$$\Omega^N_{error} \subset \Omega^{N+1}_{error} \subset \Omega^N_{error} \cup \Omega^N_{ind}$$

$$\int_{\Omega^{N+1}_{error}} (1+|\nabla u_{N+1}|^s) dx \leq \int_{\Omega^N_{error}} 1+|\nabla u_{N+1}|^s dx + \int_{\Omega_{error}{}^{N+1} \cap \ \Omega^{N+1}_{ind}} 1+|\nabla u_{N+1}|^s) dx$$

By induction of the first term and piecewise affinity on the second :

$$\leq \eta(1-2^{-N})|\Omega| + \sum_i \int_{\Omega_i \cap \Omega^{N+1}_{error}} 1+|\nabla u|^s dx$$

Since at this point $|\nabla u|\big|_{\Omega_{ind}} \leq C$ and we can make $|\Omega_i \cap \Omega^{N+1}_{error}|$ arbitrarily small ,we can make the last term smaller than 2^{-N+1} and since $-2^{-N} + 2^{-(N+1)} = 2^{-N}$ the result follows.

Notice that $e^{-N} \leq c_N^{-1} \quad (c_N \leq e^N)$

Then ,

$$e^{-N}\nu^N(E) \leq \nu_{u_N}(E) \leq e^{\eta}\nu^N(E)$$

Thus for example ,

- $|\Omega_{ind}^N| \to 0$
- $u_N = u$ a.e on $\Omega \setminus \Omega_{ind}^N$
- $\nabla u_N \to \nabla u$ in measure.
- $E = \{X : |X| \geq t\} \Rightarrow |x \in \Omega : |\nabla u| \geq t| \leq \underbrace{|\Omega||A|^p t^{-p}}_{v^\infty(E)}$

$$\nabla u_N \to \nabla u$$

Since $\Omega^N_{error} \subset \Omega^{N+1}_{error}, \quad \Omega_{error} = \cup\, \Omega^N_{error}$

$$\int_{\Omega_{error}} 1+|\nabla u|^s dx = \lim_{N\to\infty} \int_{\Omega^N_{error}} 1+|\nabla u|^s = \lim_{N\to\infty} \int_{\Omega^N_{error}} 1+|\nabla u^N|^s dx \leq \eta|\Omega|$$

and the proposition is proven.

Reducibility of unbounded sets

The next proposition in fact has nothing to do with Laminates but is about gluing piecewise affine approximate solutions to obtain an exact solutions.

Definition

We say that $M^{d\times m}$ can be reduced to K if $\forall A \in M^{d\times m}, b \in \mathbb{R}^m, \delta \in (1,\infty), \varepsilon, \psi \in (0,1) \exists p.a\ u \in W^{1,1}(\Omega) \cap C^\alpha(\bar{\Omega})$ with:

- $u = l_{A,b}$ on $\partial\Omega$
- $\int_{\Omega^{\text{error}}} (1 + |\nabla u|^s) dx < \varepsilon|\Omega|$
- $|x \in \Omega : \nabla u(x) \geq t| \leq M^p(1 + |A|^p)|\Omega| t^{-p}$

From reducibility to Exact Solutions

Theorem

Suppose that $\mathbb{R}^{d\times m}$ can be reduced to K.Let Ω be a domain.For any $A \in \mathbb{R}^{d\times m}, b \in \mathbb{R}^d, \delta > 0, \alpha \in (0,1)\ \exists\ u \in W^{1,1}(\Omega) \cap C^\alpha(\bar{\Omega})$ such that:

- $\nabla u(x) \in K$
- $\|u - l_{A,b}\|_{C^\alpha} \leq \delta$,
- $t^{-p} \leq |x : \nabla u(x) \geq t| \leq Ct^{-p}$

An interesting twist, is that it would suffice to have a different power r for each matrix. Then the integrability obtained at the end would be that of the

The proof

The idea is, once more, an iterative construction based on the idea that if $Du\chi_{\Omega_A} = A \notin K$, we create a new map such that if: $\tilde{u} = u_A$ on Ω_A.

With this idea, we create a sequence of piecewise affine maps such that:

$$\int_{\Omega^k_{\text{error}}} (1 + |\nabla u_k|^s)\ dx \leq 2^{-k}|\Omega|$$

$$\|u_k - l_{A,B}\|_{C^\alpha} < \delta\left(1 - 2^{-k}\right)$$

$$|\{x \in \Omega : |\nabla u_k > t|\}| \leq M^p(1 + |A|^R)|\Omega|t^{-p}\sum_{i=0}^{k-1} 2^{-i}$$

It is important to deal with the error first:

$\int_{\Omega^{k+1}_{error}} 1 + |\nabla u_i^{k+1}|^R dx = \sum \int_{\Omega_i} 1 + |\nabla v_i|^R dx < 2^{-(k+1)}|\Omega|$

u^1 is an exercise.Given u^k write:

$$\Omega^k_{error} = \cup_i \Omega_i \cup N$$

where $\nabla u_k \chi_{\Omega_i} = A_i$
Declare:

$$u^{k+1} = \begin{cases} u_{A_i} \text{ on } \Omega_i \\ u^k \text{ otherwise} \end{cases}$$

C^α easy

$$|\{x : \nabla u^{k+1} > t\}| = |\{x : \nabla u_k > t\}| + \sum |\{x \in \Omega_i : \nabla u_{A_i} > t\}|$$
$$\leq M^p t^{-p}[(|\Omega||A|^p + 1) \sum_{j=1}^{k-1} 2^{-j} + \sum |\Omega_i|(|A_i|^p + 1)]$$

To deal with the last term, notice that $\nabla u^k X_{\chi_{\Omega_i}} = |A_i|$
Therefore , $\sum |\Omega_i|(|A_i|^p + 1) \leq \int_{\Omega_k^{error}} (|\nabla u^k|^p + 1) \leq 2^{-k}|\Omega|$

and then passing to the limit is straightforard.
The lower bound is easier.By construction:

$$\frac{|x \in \Omega : |\nabla u_1| > t|}{|\Omega|} \geq \frac{2}{3} M^p (1 + |A|^p)$$

Moreover, $|x \in \Omega^1_{error} : |\nabla u| > t| t^{-p} \leq \int_{\Omega^1_{error}} |\nabla u|^p \leq \int 1 + |\nabla u|^p \leq \varepsilon$

$$\Rightarrow \frac{|x \in \Omega \setminus \Omega^1_{error} |\nabla u_1| > t|}{|\Omega|} \geq \frac{2}{3} M^p (1 + |A|^p)$$

Abundance of Staircase Laminates yields exact solutions

Theorem

Let $K \subset \mathbb{M}^{m\times n}$ be such for every $A \in \mathbb{M}^{m\times n}$, there exists $\nu \in \mathcal{SL}(K)$ in weak L^p. Then there exists $u \in W^{1,q}(\Omega)$ such that

- $\nabla u(x) \in K$
- $\|u - l_{A,b}\|_{C^\alpha} \leq \delta$,
- $t^{-p} \leq |x : \nabla u(x) \geq t| \leq Ct^{-p}$

The theorem follows from the above.

Linear Beltrami Equations

It turns out that the quasiregular staircases laminates are very versatil but the process of solving the differential inclussion is very different as explained when we introduce them.

- For the linear very weak staircase, it can be shown that there exists a very weak staircase laminate for every A
- For the upper weak staircase, the laminates are supported in the quasiconvex hull of K, which corresponds to the G-closure of (μ_1, μ_2). The set can be shown to be reduced to K. We will explain the situation in the setting of the non linear Beltrami equation where it is easier to see the geometry.

For the case of NonLinear Beltrami equations the proof is a bit more cumbersome. The set $K = E_{\mathcal{H}} = \{A : a_- = \mathcal{H}(a_+)$

- If $\mathcal{H}$ is real, then for every $A \in E_{\mathcal{H}}^c$, there is $\nu_A \in \mathcal{SL}(E_{\mathcal{H}})$
- If $\mathcal{H}$ is complex, a different argument using non linear Beltrami Operatos allows to reduce the situation to the case of real $\mathcal{H}$

Important Difference: For real $\mathcal{H}$, $E_{\mathcal{H}}^c \subset \overline{\mathbb{M}^{2\times 2}_{\mathrm{sym}}}$, this is a three dimensional subspace. It is more convenient to deal with a version of the inn-approximation of Müller-Sverák. Alternatively we can try push our strategy based on piecewise affine maps. There is a difficulty however as the barycenters of the correct laminates must lie in $E_{\mathcal{H}}^c$. An aesthetic compromise between both strategies is to use the inn-approximation to replace the step laminates by gradient distribution, and then apply the convex integration procedure explained above which is what we do next. This is as far as we know, new material.

Subdomains

Let

- $\Omega = \Omega^K \cup \Omega^{\mathcal{U}}$
- $\Omega^{\mathcal{U}} = \Omega^{ind} \cup \Omega^{error}$

Exercise: Guess the meaning of the domains above.

$(K, \mathcal{U})$ staircase laminates

In the following it would be important that the steps of the laminate are supported in $K \cup \mathcal{U}$ for $\mathcal{U}$ relatively open in $\mathcal{M}$.
Thus we define $(K, \mathcal{U})$ steps

$$\omega_n = (1 - \gamma_n)\mu_n + \gamma_n \delta_{A_n}$$

with γ_n, μ_m as before but importantly

$$A_n \in \mathcal{U}.$$

These type of steps ω_n yield the corresponding staircase laminates that we denote by

$$\mathcal{SL}(K; \mathcal{U})$$

and by

$$\mathcal{SL}^p(K; \mathcal{U})$$

if we want to prescribe the integrability properties.

Definition of $\mathcal{SL}(K;\mathcal{U})$

Let us be more precise

Definition

We say that $\nu \in \mathcal{SL}^p(K;\mathcal{U})$ if $\nu \in \mathcal{SL}(K,\mathcal{U})$ and there exists constants c_0, c_1 such that

$$c_0 dist_K(A)|A|^{p-1} \leq \nu(X : |X| \geq t)t^p \leq c_1(1 + |A|^p)$$

Here $dist(\overline{\nu}, K)^p$ stands for the euclidean distance. As a matter of fact, we could replace the $dist(\cdot, K), (1 + |\cdot|^p)$ by another functions Φ_1, Φ_2. For example when
$\mathcal{U} = \mathbb{R}^{n\times m}$ one can replace $dist(\cdot, K)$ by the Euclidean norm.

Approximation by piece wise affine maps

In what follows $\mathcal{M}$ is any of the constraint sets such that if $\nu \in \mathcal{PL}(\mathcal{M})$, there exists an approximating gradient distribution of a piecewise affine map u, ν_u, with ν_u supported in $\mathcal{M}$

Proposition (Approximation of $(K,\mathcal{U})$ steps laminates.)

Let ω_n be an step-laminate supported in $K \cup \mathcal{U}$, where $\mathcal{U}$ is relatively open in $\mathcal{M}$. Then we can find u such that its gradient distribution approximate ω_n and more over u restricted to $\Omega^{\mathcal{U}}$ is piecewise affine.

Partial In-Approximation

The proof uses the idea of in-approximation. For the sake of clarity we provide the argument for a simply laminate ν supported in $B \in \mathcal{K}$ and $A \in \mathcal{U}U \setminus K$, with $C = \lambda B + (1-\lambda)A$.
We will consider a sequence $0 < \delta_k < 1$ converging to 1 and for each such δ_k the corresponding

$$B_k = \delta_k \lambda B + (1 - \delta_k \lambda)A.$$

Step 1 We apply first the roof construction with δ_1 as close to 1 as needed and apply the roof construction to $\nu^{\delta_1} = \lambda_{\delta_1} B_1 + (1 - \lambda_{\delta_1})A$ with λ_{δ_1} chosen so that

$$\bar{\nu_\delta} = \bar{\nu}$$

Pushing the mass to K

Step 2 Next we will apply the roof construction in the region Ω^{B_k} (where $Du_k = B_k$) to "push" B_k to B. Observe that for $\eta_k = \frac{\delta_k}{\delta_{k+1}}$

$$B_k = \eta_k B_{k+1} + (1 - \eta_k)A$$

Notice that if declare $\delta_k = (1 - 2^{-k})$ it holds that

$$(1 - \eta_k) = 1 - \frac{\delta_k}{\delta_{k+1}} = \frac{2^{-k} - 2^{-k-1}}{1 - 2^{-k-1}} \leq 2^{-k}$$

We construct a sequence u^k piecewise affine such that off Ω^{B_k}, $u_{k+m} = u_k$. In particular $u_{k+m} = u_1$ off Ω^{B_1} and thus

$$|x : \Omega : Du_k = A| \geq (1 - \epsilon)\lambda$$

for every k

In order to obtain u^{k+1} from u^k we apply the roof construction on Ω^{B_k} for the measure ν^k. The roof constructions yields a Lipschitz map $v^k : \Omega^{B_k} \to \mathbb{R}^m$ such that $|\Omega^{B_k} \setminus \Omega^{B_{k+1}}| \leq 2(1-\eta_k)$ we declare

$$u^{k+1} = u_k(1 - \chi_{\Omega^{B_k}}) + v^k \Omega^{B_k}$$

Observe now that for $m > m$, u^m satisfies that for a.e $x \in \Omega^{B_m}$, it holds that

$$Du^k(x) - Du^m(x) = B_k - B_m = (\delta_k - \delta_m)\lambda B$$

Thus

$$\int_\Omega |Du^k - Du^m| dx \leq C(\delta_k - \delta_m) + \sum_{i=k}^{m} (1 - \eta_i) \leq C 2^{-k}$$

and thus u_k is a Cauchy sequence in $W^{1,1}$. Moreover if we set $\Omega^B = \cap_{k=1}^{\infty} \Omega^{B_k}$, $Du\chi_{\Omega^B} = B$. On the other hand $\Omega \setminus \Omega_B = \cup_{k=1}^{\infty} \Omega \setminus \Omega_{B_k}$ is a countable union of domains where u_k (and therefore u) are piecewise affine.

From $\mathcal{SL}^p(K,\mathcal{U})$ to solutions

Theorem

Let $\mathcal{U}$ be set open relatively to $\mathcal{M}$. Suppose that for every $A \in \mathcal{U}$, there is $\nu_A \in \mathcal{SL}^p(K,\mathcal{U})$. Then there exists $u \in W^{1,1}(\Omega)$ such that $Du \in L^{p,\infty} \setminus L^p$

The proof follows verbatim the case $\mathcal{U} = \mathbb{R}^{m\times n}$ using the approximation of $(K:\mathcal{U})$ steps.
Namely recall that the proof has two steps.
In the first one approximates ν by a corresponding gradient distribution ν_u. The map ν_u is obtained from a sequence of piecewise affine maps u^k.
If we use the $(K,\mathcal{U})$ Lemma, the maps u^k would be piecewise affine if Du^k does not belong to K and thus we can define $\Omega_k^{ind}, \Omega_k^{error}$ in the same way. Then we would get an approximation of $\mathcal{SL}(K,\mathcal{U})$ as $\mathcal{SL}(K,\mathcal{U})$ but with u piecewise affine only in Ω^{error}
In the second step, we create a new sequence of maps u_k, and corresponding nested sequence of Ω_k^{error}. Again in the case of $\mathcal{SL}(K,\mathcal{U})$, The sequence u_k will just be piecewise affine on Ω_k^{error}

The argument here works for real $\mathcal{H}$. Thus the entire construction lives in the set

$$\mathcal{M} = \{A : \Im(a_-) = 0\} = s\bar{y}m$$

The set K As discussed in the previous section, we for simplicity we will restrict ourselves to $\mathcal{H}(a_+) = k|a_+|$. Therefore for every $A \in \mathcal{U}$ with

$$\mathcal{U} : \{a_- \geq k|a_+|\}$$

we need to find $\nu_A \in \mathcal{SL}^{\frac{2K}{K+1}}(\mathcal{K},\mathcal{U})$ with the correct integrability Indeed ν^{lower} from the section of linear Beltrami equations does the job for $A = (0,1)$ in conformal coordinates. Since $\mathcal{K},\mathcal{U}$ are invariant under dilation by a positive T_t, $(T_t)_{\#}\nu^{lower} = \nu_{(0,t)}$ for positive t. That takes care of the line

$$E_\infty = \{(0,t), t \geq 0\} \subset \mathcal{U}$$

Rank-one lines behave better expect to the norm

$$\|A\|_1 = |a_+| + |a_-|$$

Proposition

Every $A \in \mathcal{U}$ $A = (1-\lambda)P + \lambda Q$, *where* $Q \in E_\infty$, $P \in E_{k|\cdot|}$, *and* $\det(P - Q) = 0$, $\|A\|_1 = \|P\|_1 = \|Q\|_1$ *and* $\lambda = \Phi(\frac{A}{\|A\|_1})$ *for* $\Phi(A) = \frac{1}{1+k}(a_- - k|a_+|)$ *We denote by* ν_A^1 *the simple laminate*

$$(1-\lambda)\delta_P + \lambda\delta_Q$$

Notice that $\Phi(\frac{A}{\|A\|_1})$ is comparable to $dist_{E_{k|\zeta|}}(\frac{A}{\|A\|_1})$

Proof 1: Finding P

Let A be as in the statement of the lemma and write $a_+ = |a_+|e^{i\theta}$ for a real angle θ. We wish to pick P, Q so that $A \in [P, Q]$, where

$$P = (p_+, \mathcal{H}(p_+)) \in E_{\mathcal{H}}, \quad p_+ > 0, \quad Q = (0, |a_+| + a_-) \in E_\infty.$$

We wish to find p_+ using the rank-one direction $(e^{i\theta}, -1)$ and by solving the following equation for t

$$(a_+, a_-) + t(e^{i\theta}, -1) \in E_{\mathcal{H}} \quad \Leftrightarrow \quad \mathcal{H}(a_+ + te^{i\theta}) = a_- - t.$$

Explicitly,

$$\mathcal{H}(a_+) = k|a_+|, \qquad t = \frac{a_- - k|a_+|}{1 + k}$$

and

$$p_+ := (|a_+| + t)e^{i\theta}$$

Proof 2: $\| \|_1$ is constant along diagonal rank-one lines

Notice that for positive $0 < t < a_-$

$$|a_+ + te^{i\theta}| + |a_- - t| = |a_+| + t - t + a_- = |a_+| + a_-$$

that is the norm $\|A\|_1 = |a_+| + |a_-|$ is constant. Therefore $\|A|_1 = \|P\|_1 = \|Q\|_1$ and,

$$|p_+| = \frac{1}{1+k}\|A\|_1$$

The weight:

$$A = (1-\lambda)P + \lambda Q$$

where $\lambda = \frac{t}{|p_+|} = \frac{\Phi(A)}{\|A\|_1}$ for $\Phi(A) = \frac{1}{1+k}(a_- - k|a_+|)$

Lemma

Every A such that $a_- \geq k|a_+|$ is the center of mass of $\nu_A \in \mathcal{SL}^{\frac{2K}{K+1}}(E_{k|\zeta|}, E^{co}_{k|\zeta|})$

We claim that for $A \in \mathcal{U}$,

$$\nu_A = (1-\lambda)\delta_P + (1-\lambda)\nu_Q \in \mathcal{SL}^p(K,\mathcal{U})$$

Indeed let $\{\omega_n\}_{n=1}^{\infty}$ be the steps forming ν_Q and notice (see (5)) that after the dilation by $\|Q\|_1 = \|A\|_1$, the sequence in the definition of the staircase laminate is $A_n = \|A|(0,n)$ and the weights β_N satisfy that

$$\beta_N |N|^{\frac{2K}{K+1}} \to c(K)$$

with $c(K) = (K+1)\frac{\Gamma(\frac{K}{K+1})}{-\Gamma(\frac{-K}{K+1})}$. In other words

$$\beta_N |A_N|^{\frac{2K}{K+1}} \to |A|^{\frac{2K}{K+1}} c(K)$$

The integrability

On the other hand, ν_A is formed by the steps $\{\tilde{\omega_n}\} = \nu_A^1 \cup \{\omega_n\}_{n=1}^{\infty}$, with weights $\tilde{\gamma}_n = \lambda \cup \{\gamma_n\}$. Thus for the corresponding $\tilde{\beta}_n$

$$\tilde{\beta_N}|A_N|^{\frac{2K}{K+1}} \to |A|^{\frac{2K}{K+1}} c'(K)\Phi(\frac{A}{\|A\|_1})$$

where we recall that there exists $C = C(\mathcal{H})$

$$\frac{1}{C} dist_{E_{k|\zeta|}}(\frac{A}{\|A\|_1}) \leq \frac{\Phi(A)}{\|A\|_1} \leq C dist_{E_{k|\zeta|}}(\frac{A}{\|A\|_1})$$

Theorem

Given, $0 \leq \alpha \leq 1$, $\epsilon > 0$, $A = (a_+, a_-)$ *such that* $a_-\mathbb{R}, a_- \geq k|a_+|$. *Thenm For any domain* $\Omega \in \mathbb{R}^2$ *there exists* u $u - l_{A,b} \in W_0^{1,1}(\Omega)$, *with* $[u - l_{A,b}]_{C^\alpha(\Omega)} \leq \epsilon$ *with* $Du \in L^{\frac{2K}{K-1}} \setminus L^{\frac{2K}{K-1},\infty}$ *and such that*

$$\partial_{\bar{z}} u = k|\partial_z u| \;\; a.e \;\; z \in \Omega$$

P-Laplacian

Theorem

Let $p \in (1, \infty) \setminus \{2\}$. *Then, there exists* $q \in \max\{(1, p-1), p\}$ *and a continuous solution* $v \in W^{1,q}(B) \cap C(\tilde{B})$ *of the p-Laplace equation* $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$ *with affine boundary such that:*

$$\int_{B'} |\nabla u|^p \, dx = \infty$$

Proof.

Recall that for $K_p = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{p-1} \end{pmatrix} R : \lambda \geq 0, R \in SO(2) \right\}$. We assume $1 < p < 2$ and conclude by duality. □

Theorem

$\forall\ 1 < p < 2\ \exists\ q_p, M > 1$. $\forall A \in M^{2\times 2}, \alpha, \delta \in (0,1)$ and Ω a domain $\exists$ p.a map with

- $u(x) = Ax$ *on* $\partial\Omega$
- $\|u - Ax\|_{\mathbb{C}^\alpha(\tilde{\Omega})} < \delta$
- $\nabla u \in K_p$ *a.e*
- *For* $t \geq 1 + |A| : M^{-1}(1+|A|)^{q_p} \leq \frac{|x \in \Omega| : |\nabla u(x)| \geq t|}{|\Omega|} \leq Mt^{-q_p}|A|^{q_p}$

We rely on the laminate constructed in Example and fix the parameter b and the corresponding $\nu \in \mathcal{SL}^q(K_p)$.The example yields the laminate for A = I

Step 1 By considering :

$T(X) = -X$ we get $A = (b, 1)$

Step 2

Then $v(x,y) = \alpha_1\delta_{(1,1)} + \alpha_2\delta_{(1,-1)} + \alpha_3\delta_{(\beta,-1)} + \alpha_4\delta_{(-\beta,1)}$

For different values, we notice that the set K_p has another invariance.

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{p-1} \end{pmatrix} \begin{pmatrix} \mu & 0 \\ 0 & \mu^{p-1} \end{pmatrix} = \begin{pmatrix} \lambda\mu & 0 \\ 0 & (\lambda\mu)^{p-1} \end{pmatrix}$$

In fact K_p is not a group respect matrix composition but it is respect to $A \star B = A^T \circ B$

Finally we use the invariance under $\mathbb{R}$ and the conformal anticonformal invariance.

Non conservation of Magnetic helicity

We have stablished theory for gradient distributions and laminates respect to gradients.

- Similar theory holds for Faraday distribution and Faraday Laminates and Faraday-$\mathcal{SL}$
- We need to deal with the anisotropic norm" $|(B, E)|_p = |B|^p + |E|^{p'}$
- We need to show that magnetic helicity is conserved in the corresponding roof construction. This is not automatic for generic Faraday Laminates, but it holds because the elements of the Faraday cone we are using are also the form $|B||E| = 0$
- The Faraday Staircase laminate needs to be transform into an $MHD - \mathcal{SL}$

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Exercises

1. Prove that for a bounded domain $L^p(\Omega) \subset L^{p,\infty}(\Omega) \subset L^q(\Omega)$ for $1 < q < p$
2. Calculate the determinant and inverse of a matrix using conformal coordinates.
3. Show that the Cayley transform relates coefficients to equations in divergence form and Beltrami equations.
4. Relate the p Laplacian equation in 2D with a differential inclusion for a map $f : \mathbb{C} \to \mathbb{C}$
5. Integrate by parts, for E_1, E_2 smooth vector fields on the thorus $\int_\Pi E_1 \cdot \nabla \times E_2 dx = \int_\Pi \nabla \times E_2 \cdot E_1 dx$
6. Show that the Ideal Ohm Faraday equations are equivalent to find an space time two form ω such that

$$d\omega = 0, \omega \wedge \omega = 0$$

More exercises

1. Prove that the glueing lemma yields indeed a Sobolev function.
2. Prove that the wave cone for the curl is the rank-one cone.
3. Give the proof the roof construction for Faraday forms.
4. Prove that
$$\lim_{n\to\infty} \Pi_{n=1}^{N} \frac{(1+\frac{1}{n})^q - 1}{a + (1+\frac{1}{n})^q} N^{\frac{q}{b+1}} = C$$
5. Show that for the clasical Γ function
$$(\Pi_{n=1}^{N}(1-\frac{z}{n+z})(1-\frac{z}{n+1}))N^z N + 1^z \to (1-z)\frac{\Gamma(z)}{-\Gamma(-z)}$$
   Hint: Use Weirtrass representation of the Γ functions.
6. Approximate laminates of finite order by gradient distributions.

Appendix on infinite products

In order to have a more friendly approach to the weight, we review some basic facts on infinite products. We recall the Euler-Mascheroni constant γ.

$$\gamma = H_N - \log(N+1)$$

where $H_N = \sum_{j=1}^{N} \frac{1}{j}$ is the harmonic series.

Definition

We say that a product of complex numbers $\{z_n\}$ converges if:

1. Only a finite number are zero.
2. $\lim \prod_{z_n \neq 0} z_n = P \neq 0, \neq \infty$.
3. $\prod_{n=1}^{\infty} z_n$ converges absolutely to c if and only if $\sum_{j=1}^{\infty} |z_n - 1| < \infty$.

Weierstrass representation of the Γ function

$$\forall z \notin \{0,-1,-2,...,-n\}, \Gamma(z) = \tfrac{1}{z} e^{-\gamma(z)} \prod_{n=1}^{\infty} (\tfrac{z}{z+n}) e^{\frac{z}{n}}$$

We also recall that convergence of products can be related to that of functions.

Lemma (1)

Let $-1 < x_j < \infty$*. Then,* $\prod_{j=1}^{\infty}(1+x_j)$ *converges absolutely* $\iff \sum x_j < \infty$

Proposition

Let $z \notin \{0,-1,\dots,-n\}$ *then :*

$$\lim_{N\to\infty} \prod_{k=1}^{N} (1 - \frac{z}{k+z})(N+1)^z = \Gamma(z+1)$$

Proof.

$$v\prod_{k=1}^{N}\left(\frac{k}{k+z}\right)e^{log(N+1)z} = \prod_{k=1}^{N}\frac{k}{k+z}e^{z(log(N+1)-H_N+H_N)} =$$

$$e^{z(log(N+1)-H_N)}\prod_{k=1}^{N}\left(\frac{k}{k+z}e^{kz}\right)$$

$$= \left(e^{-\gamma z}\prod_{k=1}^{N}\frac{k}{k+z}e^{\frac{z}{k}}\right)e^{z(log(N+1))-H_N+\gamma)} \to z\Gamma(z) = \Gamma(z+1)$$

□

Theorem

Let $w_n \in \mathbb{C}, |w_n| \leq M$, $\exists\ c : \prod(1-\frac{z}{k+w})(N+1)^z = c\Gamma(z)$

Proof We compare with $(1-\frac{z}{n+z})$.Notice that

$$(1-\frac{z}{n+z})^{-1}=\frac{n+z}{n}=1+\frac{z}{n}$$

Thus,

$$\left(1-\frac{z}{n+w}\right)=\left(1-\frac{z}{n+z}\right)c(z,n,w)$$

where

$$c(z,n,w)=\left(1+\frac{z}{n}\right)\left(1-\frac{z}{n+w}\right)=1+\frac{z(w-z)}{n(n+w_n)}$$

Now ,we apply Lemma 1 to obtain that :

$$\sum\frac{z(w_n-z)}{k(k+w_n)}\leq|z|M+k$$

Thus,

$$\prod 1 - \left(\frac{z}{n+z}\right)^{-1} (1 - \frac{z}{n+w_n})$$

converges absolutely and therefore:

$$\lim (N+1)^z \prod_{k=1}^{N} (1 - \tfrac{z}{k+w_k}) = \lim (N+1)^z \prod_{k=1}^{N} \left(1 - \tfrac{z}{k+z}\right) \prod c(k,z,w)$$

$$= \Gamma(z+1) c(z, \{w_n\})$$

Appendix: Wave cone for the Faraday System

We call the states $\bar{B}, \bar{E}, \bar{w}$ to emphasize that they are constant.

Lemma

$$\Lambda^{Faraday} = \{\bar{\omega} \ = (\bar{E}, \bar{B}) : \bar{E} \cdot \bar{B} = 0\}$$

Lemma (Roof for Faraday)

Let $\bar{\omega}^1, \bar{\omega}^2$ such that $\bar{\omega}_1 - \bar{\omega}_1 \in \Lambda$.Exists ω p.a such that:

1. $\omega\big|_{\partial\Omega} = \lambda\omega_1 + (1-\lambda)\omega_2$
2. $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega^{error}, \omega\big|_{\Omega_i} = \bar{\omega}_i$ *for i=1,2*
3. $|\Omega^{error}| \leq \varepsilon, |\omega - \bar{\omega}_0| \leq \varepsilon$ *on* Ω^{error}
4. $\exists\ \tilde{A} \in Lip_0(Q)$ *with* $B_0 + \nabla\tilde{A} \leq \varepsilon$

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4. $\exists\ \tilde{A} \in Lip_0(Q)$ *with* $B_0 + \nabla\tilde{A} \leq \varepsilon$

Scalar Potentials

<u>Obs 1</u> Given any function f and a vector η we can define:

$$\Psi = \eta f$$
$$g = f$$
$$B = \nabla \times \Psi = \eta \times \nabla f$$
$$E = \nabla f - \eta \partial_t f$$

<u>Obs 2</u> If $\bar{B} \cdot \bar{E} = 0 wechoose \xi_t = 0$ we can choose

$$\xi_x = \frac{E}{|E|}$$
$$\bar{B} \cdot \xi_x = 0$$
$$\xi_t \cdot B = 0 = \xi_x \times \bar{E} = 0$$

In particular $B = \eta \times E$ and thus for $\psi = \eta f$ yields a candidate for ω.

Now we can choose the usual saw tooth function s and declare

$$f = \min\{d(t), d(x), \frac{1}{j} s(jx \cdot \frac{E}{|E|})\}$$

Exercise

Prove the lemma

Lemma (Change of Magnetic helicity)

Now we suppose that we have a p.a Faraday form ω_0 *such that :* $\omega_0\big|_{\Omega^0}$ *is constant and* $\alpha\big|_{\Omega^0} d\alpha = \omega^0$ *and consider the change of the helicity.*

$$\int (\tilde{A} + A_0)(\tilde{B} + B_0) - A_0 B_0$$

By construction, $\tilde{A} = \eta f$ *for a vector* η *and* f *compactly supported in* $Q(t)$*. Therefore,*

$$\tilde{A} \cdot \tilde{B} = f\eta \cdot \eta \times \nabla f = 0$$

On the other hand, since $\tilde{A}$ *vanishes we can integrate by parts,*

$$\int A_0 \cdot \nabla \times \tilde{A} = \int \nabla \times A_0 \cdot A = \int B_0 \cdot \tilde{A}$$

where nablaA_0 *is understood distributionally All in all,*

$$\int_{Q(t)} (A_0 + \tilde{A})(B_0 + \tilde{B}) - A_0 B_0 = \int_{Q(t)} B_0 \cdot \tilde{A} = B_0 \cdot \eta$$