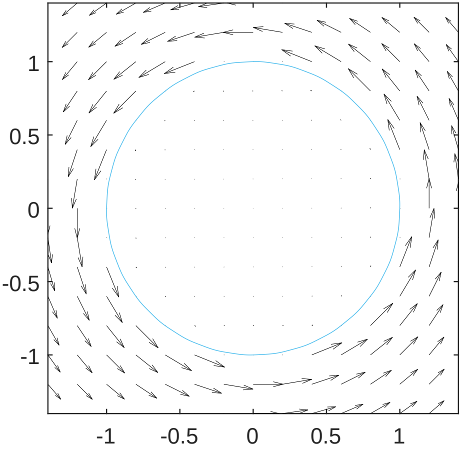
Lecture notes on Convex integration in L^p

Daniel Faraco

Universidad Autónoma de Madrid and ICMAT

Modelización en Fluidos y Estructuras
, Master UAM 2023

European Research Council
Established by the European Commission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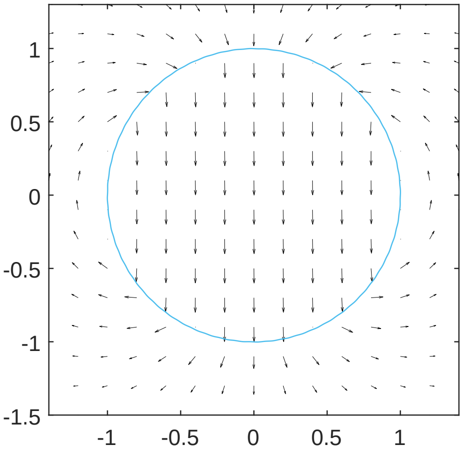
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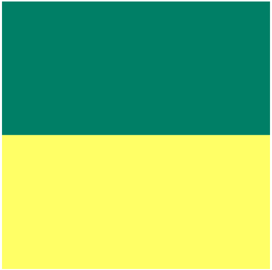
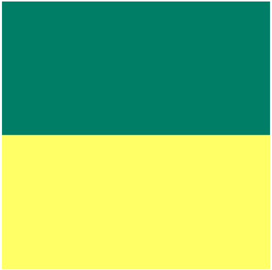
Example: Evolution of microstructures

solution



subsolution



Introduction

Isometric inmmersions versus turbulence.
Two pictures. One of fully developed turbulence, the other of convex integration solutions.

Isommetric inmmerssions

Definition.
Formulation as a partial differential relation.
$Du^T Du = g$

H-Principle

If there exists a short map there exist infinitely many issometric inmmersions.
Philosophy: If there is a solutions to a relaxed version of the problem (being short), there exist a solution of the original problem.
Very important: In differential geometry.

Idea of Nash.
Formula de los steps.

The Euler equations

We consider weak solutions to the incompressible Euler equations with constant density. Thus the evolution of the velocity of a fluid with constant pressure iis given by

Conservation of Volumes.

$$\mathrm{div}(v) = 0,$$

Conservation of momentum (Newton Second Law).

$$\partial_t(v) + \mathrm{div}(v \otimes v) = \nabla p$$

and we consider distributional solutions in $L^2(\mathbb{R}^n)$ with $v(x, 0) = v_0$. The Euler equations are like the isometries in the Nash theorem.

The Relaxed Euler equations (subsolutions)

Conservation of Volumes.

$$\mathrm{div}(v) = 0,$$

Conservation of momentum (Newton Second Law).

$$\partial_t(v) + \mathrm{div}(v \otimes v) = \nabla p + R$$

where R the reynold stress, satisfies that $R \geq 0$ in the sense of symmetric matrices. and we consider distributional solutions in $L^2(\mathbb{R}^n)$ with $v(x,0) = v_0$.
This are like the short maps in the Nash-theory.

Theorem

H principle for the Euler equations If there is a subsolution to the Euler equations, there are infinitely many solutions to the Euler equations with the same initial data.

Theorem (De Lellis-Székelyhidi)

(Non-uniqueness of weak solutions) There exists infinitely many solutions with $v \in L^\infty \cap L^2$ with $v_0 = 0$.

There is no uniqueness of the Euler equations. This is a very powerful machinary in modelling turbulence regimes in fluid mechanics where classical ideas from P.D.E and Calculus do not work.

What we mean by Convex Integration

•. In this course, convex integration is a method to solve an special type of partial differential relation that we call differential inclusions. Abstractly, Given $\mathcal{L}$ a linear differential operator and $\mathcal{D} \subset \mathbb{R}^d, \mathcal{K} \subset \mathbb{R}^N$ a closed set, $z(x) = (z_1(x), ldots, z_i(x), z(x) : \mathcal{D} \to \mathbb{R}^N$ solves an $\mathcal{L}, \mathcal{K}$

$$\mathcal{L}(z) = 0, z(y) \in \mathcal{K}$$

The idea of decomposing a non linear p.d.e as linear system, typically a conservation law and a constitutive relation is the starting point of what is today called as The Tartar Framework and it is intimately associated to the theory of *compensated compactness which will pervade the full course.*

The type of theorems we are seeing stems in Nash-Kuiper theory of irregular isometric inmersions but it has found striking applications in many other fields, as non linear elasticity, regularity of elliptic equations and systems and mostly striking in De Lellis-Székelyhidi theory of weak solutions in hydrodynamics, which will be at the core of the course.

Solutions of the differential inclusion in the course will arise as a consequence of an H-principle. We compare the solutions of (1) with that of

$$\mathcal{L}(z) = 0, z(y) \in \mathcal{U}$$

where $\mathcal{U}$ is called the relaxation, and it is typically an open (or relatively open set) which represents the set of weak limits of the inclusion. Thus during this week our holy grail will be the following type of theorems:

H-Principle (Meta-Theorem).

The set of solutions of the differential inclusion is dense in the set of the solutions to the relaxed system. In particular if there exists a relaxed solution, there exists at least one solution. Even more the solution share many properties with the subsolution as for example the initial data.

Idea fo the Meta-Theorem As Juan Manfredi taught me once, the good thing of Meta-Theorems is that you do not need to prove it but just give a vague "intution" about why they are true. The prototype of a sequence going to zero is a highly oscillating sequence of solutions to $\mathcal{L}$. Therefore one might expect that given a relaxes solution if we substract from it, suitably frozen, and localize in origin and domain and locally highly oscillating sequences

Historical milestones

- 50.Nash-Kuiper Isometric immersion Problem.
- 80 Murat-Tartar invent the compensated compactness theory, which is a theory to understand weak convergence methods in nonlinear P.D.E and it shows that many aspects of it is a geometrical problem.
- 80 H-principle in differential geometry. Largely develop by Gromov in his study of partial differential relation.
- 90 Müller- vSverák discovered that it yields unexpected solutions in the Calculus of Variations in absence of lower semicontinuity. Dacorogna-Marcellini Baire Category. Bernd Kirchheim Baire one maps. Müller- vSverák unexpected counterexamples to regularity of elliptic systems.
- Systematic understanding of the Rank-one Geometry (Kirchheim, Székelyhidi).
- 2008. De Lellis Székelyhidi bring together Tartar compensated compactness and H-principle to deal with the Euler equations.
- 2010-2021 Intense research on the method in hidrodynamics lead to solutions of various open problems.

Menu

- **Appetizers :** Convex integration by hand (Emphasis on The gradient case).
- **First course :**The Baire category approach. H-principles in hidrodynamics.
- **Second Course:** Results in hidrodynamics. Lack of uniqueness and modelling of inestabilities. (Animations!).
- **Dessert:** Intro to MHD equations and convex integration with constraints.

Two States.

A way to get introduced into the topic is as usual by considering the easiest situation. Throughout the course we will focuse of first order differential operators of the type

$$\mathcal{L}(z) = A_{ijk}\partial_j z^k$$

Namely that $\mathcal{K}$ consists only of two costant states $z_1, z_2 \in \mathcal{R}^n$. A basic but important land mark in M-T compensated compactness theory is that there are certain special directions $l \in \mathcal{R}^n$ for which there exists functions z such that

$$\mathcal{L}(z) = 0 \text{ for a.e}(x) \in \Omega, z(x) \in l$$

We have controlled range!

The wave cone

Definition

Let $\mathcal{L}$ be a first order differential operator. Then $l \in \Lambda = \Lambda_L \subset \mathbb{R}^N$ if there exists a direction $\xi \in \mathbb{R}^d$ such that for every $h : \mathbb{R} \to \mathbb{R}$

$$\mathcal{L}(h(\langle x, \xi \rangle) l) = 0$$

• Notice that if $\mathcal{L} = A_{ijk}\partial_j z_k$. Then $\mathcal{L}(h(\langle x, \xi \rangle) l) = h'(\langle x, \xi \rangle) A_{ijk}\xi_j l_k$ Thus $l = (l_1, \ldots, l_k, \ldots, l_N) \in \Lambda$ if there exists ξ such that for every $i, A_{ijk}\xi_j l_k = 0$. • The model example is the gradient case, where the differential operators is $\operatorname{curl}(A_{ij}) = \partial_j A_{ik} - \partial_k A_{ij}$. • Exercise. Show that in the gradient case the wave cone is the cone of rank-one matrices. i.e $A = n \otimes \xi$.

Controlled range

Now supposed that $I \in \Lambda$. Then by definition we have that there exists $\xi \in \mathbb{R}^d$ such that for every 1 dimensional h $z(x) = h(\langle x, \xi \rangle)I$ solves the conservation law. With the new toy, there are at least two thing we can do

- Boundary conditions. For iteration it is convenient that h is 1-periodic and $\int_0^1 h(s)ds = 0$.
- Choose h carefully. For example h could be only take two values $\lambda, \lambda - 1$. $h(x) = \lambda\chi_{(0,1-\lambda)} + (\lambda - 1)\chi_{(0,1-\lambda)}$
- Because of 1, I can make it oscillate faster $h_j(x) = h(jx)$

Then if $z_1 = \lambda I, z_2 = (\lambda - 1)I$ the problem is solved.

Unfortunately the boundary conditions are spoiled and it is very difficult to iterate.

Potentials

The gradient case has an important property which is a Poincare lemma. That is for $\mathcal{L} = \operatorname{curl}, \mathcal{B} = \nabla$ locally it holds that

$$\mathcal{L}(z) = 0 \iff z = \mathcal{B}(\varphi)$$

It turns out that this holds if and only if the operator $\mathcal{L}$ is of constant rank (Raita, Guerra-Raita). However as we will see even if not of constant rank there could exist differential operators $\mathcal{B}$ such that

$$\mathcal{L}\mathcal{B} = 0$$

The importance of the potentials is that the fix range condition associated to the wave cone can negotiated to prescribing the support of the solution at the price of satisfying the inclusion approximately. In the next lemma we show in the gradient case, that the existence of potentials allows for a careful choice of the domains Ω and the function h.

The Roof construction:No boundaries

In the gradient case, the following construction is standard.
Scalar functions as vector value functions. Let $s : \mathbb{R} \to \mathbb{R}$ and $\xi, \in \mathbb{R}^n$.
Then if we set $f(x) = s(\langle x, \xi \rangle)$, $\partial_i f = s'(\langle x, n \rangle)\xi_i$. Moreover if $a \in \mathbb{R}^n$ and we declare $f_a : \mathbb{R}^n \to \mathbb{R}, f_a(x) = f(x)a$

$$\partial_j (f_a)^i = s'(\langle x, \xi \rangle) a_i \otimes \xi_j, i.e\, Df_a = s'(\langle x, \xi \rangle) a \otimes \xi$$

Thus if for example $s' = \{\lambda, (\lambda - 1)\}$, $D(f_a)(x) \in \{\lambda a \otimes \xi, (\lambda - 1) a \otimes \xi\}$ a.e We notice that if s is periodic we can rescale the construction declaring

$$f_{aj}(x) = \frac{1}{j} s(j \langle x, \xi \rangle) a$$

This oscillating wave has the same gradient, but it is very small in L^∞.

Exercise

Correcting the boundary conditions. The problem with this construction is that it can not be iterated. The abstract solution is declaring $g = Ax + \eta_\delta f_j$ where η_δ is a suitable bump function.

However we loose control on what is the values of Dg when $\nabla \eta \neq 0$. This is not terrible and we can guarantee that the values are not far from the rank-one segment but sometimes is useful to keep control on the precise values

The Roof construction: With control on the boundary

We start by prescribing gradients of functions with zero boundary values. It is not possible to have an exact value for two gradients.

Lemma (Roof-Lemma)

Let $\xi \in S^{n-1}$ a direction and let $\lambda \in (0,1)$ and Let $\xi^{(1)}, \ldots, \xi^{(J)} \in \mathbb{R}^n$ be such that $0 \in int\ conv\{\xi, -\xi, \xi^{(1)}, \ldots, \xi^{(J)}\}$. For any open bounded set $\Omega \subset \mathbb{R}^n$ with $|\partial\Omega| = 0$ and any $\delta > 0$ there exists a piecewise affine Lipschitz function $f \in Lip_0(\Omega)$ with

$$\nabla f(x) \in \left\{-\lambda\xi, (1-\lambda)\xi, \xi^{(1)}, \ldots, \xi^{(J)}\right\} \qquad a.e.\ x \in \Omega, \tag{1}$$

and

$$|\{x \in \Omega : \nabla f(x) = -\lambda\xi \ or\ (1-\lambda)\xi\}| \geq (1-\delta)^n |\Omega|. \tag{2}$$

Notice that we could choose the auxiliar vectors $\xi^{(1)}, \ldots, \xi^{(J)} \in \mathbb{R}^n$ as close to zero as we like.

Proof of the Roof-Lemma

Let s be the saw tooth function as before and set

$$P = \left\{ x \in \mathbb{R}^n : \, x \cdot \xi^{(j)} > -1 \text{ for all } j = 1 \dots J \text{ and } |x \cdot \xi| < 1 \right\}.$$

Then P is a convex open set containing 0. Moreover, for any $N \in \mathbb{N}$ the function

$$f_N(x) = \min \left\{ \min_j (1 + x \cdot \xi^{(j)}), \, \frac{1}{N} s(Nx \cdot \xi) \right\}$$

is Lipschitz, satisfies (1), and $f_N = 0$ on ∂P.
Moreover, by choosing N, sufficiently large in terms of δ

$$\frac{1}{N} h(Nx \cdot \xi) < \min_j (1 + x \cdot \xi^{(j)}) \quad \text{on } (1 - \delta)P,$$

and thus, $f_N(x) =, \frac{1}{N} s(Nx \cdot \xi)$ from which (2) follows.

- Exercise 1. For a general Ω we apply a standard rescaling and covering argument
- Exercise 2. If we declare $f_{aN}(x) = f_N a \; Df(x) \in \{\lambda a \dot{\xi}, (1 - \lambda) a \dot{\xi}, a \dot{\xi}^j\}$ a.e

The Tent construction

The tent construction has the same starting point than the roof construction. However now we have a solution which is always in a neighborhood of the desired gradients.

Lemma (Tent-Lemma)

Let $\xi \in S^{n-1}$ a direction and let $\lambda \in (0,1)$ For any open bounded set $\Omega \subset \mathbb{R}^n$ with$|\partial\Omega| = 0$ and any $\delta > 0$ there exists a piecewise affine Lipschitz function $f \in Lip_0(\Omega)$ with

$$\min\{|\nabla f(x) - (-\lambda\xi)|, |\nabla f(x) - (1-\lambda)\xi|\} \leq \delta \quad a.e.\ x \in \Omega, \tag{3}$$

Proof of the Tent construction

Now we choose an orthonormal basis so that $e_n = \xi$ Now we start with the same construction of $f(x) = s(\langle x, \xi\rangle)$ but consider initially the domain

$$V = \{x \in \mathbb{R}^n : (\epsilon(\lambda - 1) \leq \langle x, \xi\rangle \leq \epsilon\lambda, |x_i| \leq 1\}$$

By adding the constant $-\epsilon(\lambda)(1-\lambda)$ the function is zero in the x_n boundaries but not in the rest of ∂V.
Next, we add a function $h(x) = \epsilon\lambda(1-\lambda)\sum_{i=1}^{n-1} |x_i|$. Then h is piecewise linear and

$$\partial_i h = \epsilon\lambda(1-\lambda)\mathrm{sign}(x_i), \langle \nabla h, \xi\rangle = 0.$$

Moreover, $|Dh| = \epsilon\lambda(1-\lambda)\sqrt{n-1}$ and more importantly $\tilde{f} = h + f \geq 0$ on V. Now if declare

$$U = \{x \in V : \tilde{f}^{-1} < 0\}$$

we have that $\tilde{f}$ satisfies the requirements in U. The same covering argument yields the result.

Solution in Balls from the tent constructions

A baby version of convex integration can be devised by solving

$$Du(x) \in B(A,\epsilon) \cup B(B,\epsilon)$$

as an iteration of the tent construction. Namely we construct a sequence $\{u_k\}_{i=1}^{\infty} \in Lip_{Affine}$ such that

$$Du(x) \in B(A,\epsilon) \cup B(B,\epsilon) \cup B(C, \delta(\sum_{i=1}^{k} 2^{-i}),) \| x \in \Omega : Du_k \neq D_{u_{k-1}}) \| \to 0$$

If such a sequence exists, the L^{∞} boundedness and the vanishing measure condition implies that it is a Cauchy sequence in L^p.
The existence of the sequence is a direct corollary from the tent construction: Given u_k let $Du_k \chi_{\Omega_{C+E}} = C + E$ with $|E| \leq \delta(\sum_{i=1}^{k} 2^{-i})$.
Then as $C + E = \lambda(A + E) + (1 - \lambda)(C + E)$ we are entitled to run the tent construction in Ω_{C+E} with error $\delta 2^{-(k+1)}$ and thus u_{k+1} has the required properties.

PreLaminates(Splitting)

The previous game, gets much more interesting when we iterate. In terms of measures code the above process by the notation,

$$\underbrace{\delta_C}_{\mu_1} \rightsquigarrow \underbrace{\lambda\delta_A + (1-\lambda)\delta_B}_{\mu_2}$$

and we say that μ_2 is obtained by splitting μ_1.
Given a discrete probability measure with finite support i.e

$$\mu = \sum_{i=1}^{n} \lambda_i \delta_{A_i}$$

if $A_1 = \lambda B + (1-\lambda)C, A - C \in \Lambda$ such that $A - C \in \Lambda$, we say that

$$\mu_1 \rightsquigarrow \mu_2 = \lambda_1\lambda\delta_B + (1-\lambda)\lambda\delta_C + \sum_{i=2} \lambda_i \delta_{A_i}$$

We say that μ_1 splits in μ_2, that A_1 is in a splitting sequence of μ_2 and that the segment $[B, C]$ is in an skelelton of μ

Laminates:Definition

Definition

The class of Prelaminates $\mathcal{PL}$ is the smallest class of probability measures such that

- *It contains all probability measures.*
- *It is closed after splitting.*

Notice that the splitting sequence or the Skeletons of prelaminates are not unique.

Definition

A measure is called an L^p-laminates if

- *It is the weak star limit of a laminate.*
- *It holds that $\int |\lambda|^p d\nu < \infty$*

Cool examples of laminates not prelaminates. Tartar squares and staircase laminates.

Aproximation of Prelaminates

Lemma

Let $\nu = \sum \lambda_i \delta_{A_i} \in \mathcal{PL}$, and $\epsilon > 0$. Then

- *There exists $f \in Lip_{P_{affine}}(\Omega)$ with $f - Cx \in Lip_0(\Omega)$ such that*

$$|x : Df(x) \in B(A_i, \epsilon)| = \lambda_i$$

- *There exists $f \in Lip_{P_{affine}}(\Omega)$ with $f - Cx \in Lip_0(\Omega)$ such that $Df(x) \in \{A_i\} \cup \{B(C_i, \epsilon)\}$ a.e and*

$$(1-\epsilon)\lambda_i \leq |x : Df(x) \in B(A_i, \epsilon)| \leq \lambda_i$$

  where $\{C_i\}$ is an splitting sequence of ν.

The proof of the lemma consists in reiterative use of the tent construction for the first, and of roof construction and is left as an exercise. We define $\mathcal{PL}(U)$ as those laminates for which there exists an splitting sequence supported in $\mathcal{U}$.
Where instead of the tent construction, we use the roof construction, then we can guarantee that the sequence is in a neighborhood of an Skeleton of μ.

The lamination hull

Given a set $\mathcal{K}$ we define various semiconvex envelopes.

- $\mathcal{K}^{co}$ the convex hull.
- $\mathcal{K}^{\Lambda,lc}$ the lamination convex hull. $K^{1,\Lambda}$. Center of mass of prelaminates supported in $\mathcal{K}$
- $\mathcal{K}^{\Lambda}$ center of mass of laminates. Can be defined with cosets of Λ-convex functions.

In approximation

The concept of in approximation was invented by Gromov in the differential geometric context and extended by Müller and Sverák in the Lipschitz context. We will see various other methods which a priori look easier but in practise, they are as difficult as finding an in approximation.

Definition

Let K be an open set. We say that $\{\mathcal{O}_i\}$ is an in approximation for K. If the following to condition holds

- $\mathcal{O}_i \subset \mathcal{O}_{i+1}^{lc}$
- $\mathcal{O}_i \to K$ *In the sense that if* $F_i \subset \mathcal{O}_i$ *is precompact the limit lies in* K.

Theorem (Müller-Sverak 96)

Let v a piecewise affine map such that $Dv(x) \in \mathcal{O}_1$ a.e $x \in \Omega$. Then there exists a Lipschitz map $u : \Omega \to \mathbb{R}^m$ that satisfies

$$Du \in \mathcal{K}, u = v \ \ on \partial\Omega$$

The proof follows by iteration of the tent-construction. By the approximation lemma we can assume that the original u is piecewise affine.
By the definition of lamination hull, approximating the prelaminates by the tent construction we find sequences u_i such that $Du_i \in U_i$ and $\|u_i - u_{i+1}\|_{L^\infty} \leq \delta_i$.
The arbitrary smallness of δ_i is what ensures strong convergence
Notice that in particular we can solve the differential inclussion given by $\mathcal{K}$ with any affine boundary value belonging to $\mathcal{U} = \mathcal{O}_i$

choice of scales

Let $\Omega_i \subset \Omega$ a nested sequence of open sets $\frac{1}{i}$ from the boundary. Then We choose now a suitable rescaled bump function such that $\|\rho_{\epsilon_i} \star Du_i - Du_1\|_{L^1(\Omega_i)} \leq 2^{-i}$ with $\epsilon_i \|D\rho_{\epsilon_i}\| \leq C$ Then given δ_i, $\delta_{i+1} \leq \epsilon_i(2^{-i} + \frac{\delta_i}{2})$ So that

$$\|D\rho_{\epsilon_i} \star (u - u_i)\|_{L^1(\Omega_i)} \leq \frac{c}{\epsilon_i} \sum_{j=i+1}^{\infty} \delta_i \leq \frac{\delta_i}{\epsilon_i}$$

Since the gradients are uniformly bounded, $|\Omega \setminus \Omega_i$ tend to zero and the mollified gradients also converge the result follow. We arrive to one of the cornerstones of the whole theory
"In general, weak convergence does not imply strong convergence...but sometimes it does".

Tent construction with determinant constraints

As we will see later often we have an additional constraint that $z \in \mathcal{M}$. We discuss the case of equal determinant.
Since A and B have equal determinant it follows that $A - B = \xi^{\perp} \otimes \xi$ where $\xi^{\perp}$ is orthogonal to ξ. Let us assume that $\xi^{\perp} = e_1, \xi = e_2$
In order to have a constant determinant, we will chose $u = \varphi_1$ to be the flow corresponding to a suitable divergence free vector field v.

$$\dot{\varphi} = v(\varphi), \varphi_0 = I$$

Thus by Lipschitz continuity in t

$$|\varphi_t(x) - x| \leq t \sup |v| \tag{4}$$

As we want to prescribe the evolution of the differential, notices that after differentiation $D\varphi$ obeys the nonlinear equation

$$\dot{D\varphi} = Dv(\varphi)D(\varphi)$$

$\bullet$Now we choose $v = \nabla^{\perp} H$ with $H = \epsilon^2 \eta S(\frac{\langle x, \xi \rangle)}{\epsilon}) = \epsilon^2 \eta S(\frac{x_2}{\epsilon})$. Then this divergence free vector field satisfies that,

$$v \approx \epsilon \eta S'(\frac{x_2}{\epsilon}), Dv \approx \eta S''(\frac{x_2}{\epsilon}) e_1 \times e_2$$

Thus we directly estimate that $|\varphi_t - x| \leq \|v\|_\infty \leq \epsilon$ This is not enougth to estimate the term $S''(\frac{x_2}{\epsilon})$ but happily $|(\varphi_t - x)_2| \leq |v_2| \leq C\epsilon^2$. Thus $Dv(\tilde{\varphi})$ is ϵ close to $Dv(x)$ and hence $|\varphi_t - \tilde{\varphi}_t| \leq C$ where $\tilde{\varphi}$ solves the linear system

$$\dot{D\tilde{\varphi}} = Dv(x) D(\tilde{\varphi}), \Rightarrow D\tilde{\varphi} = e^{\eta(x) s'(\frac{x_2}{\epsilon}) e_1 \otimes e_2}$$

but since $(e_1 \otimes e_2)^n = 0$, taylor expanding the last expression is

$$D\tilde{\varphi} = I + t\eta(x) s'(\frac{x_2}{\epsilon})$$

and the result follows.

Approximation via piecewise affine maps

Strategy.

- It is enough to perform the approximation in big piece of the domain.
  - Big piece: $|\Omega_{P_A}| \geq \Theta|\Omega|$
  - The constraint $M(D\tilde{u}) = 1$
  - Approximation and boundary data. $\|u - \tilde{u}\|_{W_0^{1,\infty}} \leq \delta$

  Thus by rescaling and changing variable the situation is reduced to Ω the unit ball and $Du(0) = I$.
- If φ is a bump function, by performing a linear interpolation $\hat{u} = \varphi x + (1-\varphi)u$ we can achieve affinity in $B_{\frac{1}{2}}$ (but we loose the determinant constraint).
- By a result of Dacorogna-Moser we can solve the boundary value problem $\det(D\tilde{u}) = 1, \tilde{u} = \hat{u}$ on the anulus $B_1 \setminus B_{\frac{1}{2}}$ (This last step is more delicate as we have to

Step 1 implies the approximation lemma

From an iterative use of step 1 in we construct a sequence $\{u_i\}$ such that;

- Big piece: $|\Omega \setminus \Omega_{P_A}| \geq (1-\Theta)^i |\Omega|$
- The constrain: $M(D\tilde{u}_i) = 1$
- Approximation and boundary data. $\|u_i - u_{i+1}\|_{W_0^{1,\infty}} \leq 2^{-i-1}\delta$

Proof of Corollary 6.5

The least clear part is the stablishing the proximity between the new and the old maps.
It is crucial that it is assume that $[Du]_\alpha < \delta$. Thus, for every x

$$|Du(0) - Du(x)| \leq \delta$$

and hence $\|u - Id\| \leq \delta$ Moreover by the fact that $\mathcal{L}$ is smooth it holds that

$$\|\mathcal{L}u - \mathcal{L}(Id)\|_{C^{1,\delta}} = \|\mathcal{L}\|\|u - id\|_\delta$$

Since we only need to cover a big piece of the domain we can work in a a compact set which is a finite union of cubes of sizes ρ, which includes small balls of radious ρ. Now the norm $[u]_{C^{1,\alpha}} \leq C$ is uniformly bounded and hence

$$|Du| \leq C \leq \delta\rho^{-\alpha}$$

Easy construction by the roof

Theorem (Kirhcheim 03)

Let $(\mathcal{K} \subset \overline{\mathcal{U}})$ satisfy that for all $A \in \mathcal{U}$ there exist, δ and a prelaminate μ_A such that

- $\langle I, \mu_A \rangle = A$,
- *The splitting sequence of μ_A belongs to $\mathcal{U}$ and there exists $Y_A \in K$ with $\mu(Y_A) \geq \delta$.*

Then for every domain $\Omega, A \in \mathcal{U}$ there exists $f \in Lip_{Ax}(\Omega)$ such that $Df(x) \in K$.

As before it is enough to show the result for a big piece of Ω. Namely the following weaker stament implies the claim There exists $f \in \mathrm{Lip}_{Ax}(\Omega)$ such that

- $Df(x) \in \mathcal{U}, \mathcal{K}$
- $|x \in \Omega : Df(x) \in K| \geq \frac{\delta}{2}$

The second claim is a direct application of the definition of prelaminates and the roof construction (Exercise easy).

The rebirth of the convex integration theory, or the birth of the Lipschitz theory is placed in the non-convex (or quasiconvex vectorial) calculus of variations. There one looks for minimizers of energy functionals $I(f) = \int_\Omega W(Df)dx$ where W is invariant respect to rotations. That is to say $W(RA) = W(A)$ whenever $R \in SO(n)$. The problem was thoroughly studied since the 90s. In our context we consider the case where the zero set of W is therefore

$$\mathcal{K} = SO(2) \cup SO(2)H$$

There is a dichotomy in terms of H. It might be that there is no rank-one conection $\det(A - B) > 0$ for all $A, B \in \mathcal{K}$. It turns out that if $\det(H) = 1$ it can be shown that

$$\mathcal{K}^{rc} = K^{2,lc} = K^c \cap \mathcal{M}$$

where $\mathcal{M} = \{A : \det(A) = 1\}$.
Then it follows that if $\mathcal{U} = \mathrm{int} K^{2,lc}$ and
$\mathcal{O}_i = \{A \in \mathcal{U} : 2^{i-1} < \mathrm{dist}(A, \mathcal{K}) < 2^i\}$.

The space In all the section, we consider X_0 a bounded in L^2 and X will be its closure respect to the weak L^2 topology. Thus the weak topology is described by a metric d and (X, d) is a complete metric space.
The functional We will consider a functional $I : X \to \mathbb{R}_+$ which is continuous respect to the strong topology but typically not continuous respect to d. For a big proportion of the lecture we will consider the following property.
Our goal Our goal is to find zero states of I, which will be the solutions of our differential inclusions, perhaps with additional properties.
The properties of I. Our aim is tot show that some abstracts properties of I in relation with weak convergence suffices to show that actually the zero set of I is dense in X (respect to the weak topology d).

Definition (Properties of (X,I))

- (X, I) *have the approximation property,* $(\mathcal{A})$*, if for every* $u_0 \in X_0$*, there exists* $\{u_j \in X\}$ *such that*

$$\lim_{j \to 0} d(u_j, u) + I(u_j) = 0$$

- (X, I) *have the perturbation property,* $(\mathcal{P})$*, if there exists a continuous function* Φ *such that for every* $u_0 \in X_0$*, with* $I(u_0) > 0$ *there exists* $\{u_j\} \in X_0$ *such that*

$$\lim_{j \to 0} d(u_j, u) = 0, \;\; and \;\; \liminf_{j \to \infty} \|u_j\|^2 \geq \|u\|^2 + \Phi(I(u_0))$$

Theorem (Sz-Lecture notes)

If X has the approximation property, the set $I^{-1}[0]$ is dense,

By hipothesis, given $w \in X$, there exist $u \in X_0$ with $d(u_1, w) < \delta$. We will find a sequence u_j such that

i) $I(u_j) \leq 2^{-j}$

ii) $|\langle u_{j+1} - u_j, u_l \rangle| \leq 2^{-j}$ for $l \geq j$

iii) $d(u_j + 1, u_j) \leq 2^{-j}\delta$

Then we will find a convergence subsequence.

Namely, telescoping property [*ii*)] yields that

$$|\langle u_{j+m} - u_j, u_l \rangle| \leq \sum_{k=j}^{m} 2^{-k}$$

and Hilbert space geometry tells that

$$|\|u_{j+m} - u_j\| - (\|u_{j+m}\|^2 - \|u_j\|^2 = 2| \leq 2|\langle u_{j+m} - u_j, u_l \rangle| \qquad (5)$$

Now by completeness of $\mathbb{R}$ there exists a nonrelabeled subsequence $\|u_j\|_{L^2} \to \alpha$. Thus (5) and property ii) yields that such subsequence is Cauchy in L^2. By completeness of L^2, definition of X follows that $u_j \to u$ in L^2. Finally $I(\omega) = 0$ by continuity of I and property i). Telescoping property iii) yields the the proximity between u and w in (X, d).
Exercise: Do the L^p case, using convolutions.

Definition

Let X be a subset of L^2. Then u is a point of continuity of X

$$u_j \rightharpoonup u \Rightarrow I(u_j) \to I(u)$$

The set of all points of continuity is called S.

Lemma

If X has the approximation property, then $S \subset I^{-1}[0]$.

Let $u \in S$, by definition of X there exists $u_j \rightharpoonup u$. By the approximation property there exists u_{jk} such that $u_{jk} \to u_j, I(u_{jk}) \to 0$. Since the weak topology is metrizable we can choose a diagonal sequence such that

$$\lim_{j\to\infty} d(u_{jk(j)}, u) + I(u_{jk(j)}) = 0$$

and since u is a point of continuity $I(u) = 0$.

Theorem (Baire theorem)

The intersection of open and dense sets is dense

Proof
Let W be an open set U_i the collection of open and dense sets. By density of U_1 and openess of W and U_1 there exists

$$B(x_1, r_1) \subset W \cap U_1$$

.
Now we claim that there exists a sequence $B(x_n, r_n) \subset W \cap_{i=1}^n U_i$. Suppose it holds. Then by density of U_{n+1}, $B(x_n, r_n) \cap U_{n+1}$ is a non empty open set which yields the existence of the corresponding

$$B(x_{n+1}, r_{n+1}) \subset B(x_n, r_n) \cap U_{n+1} \subset W \cap_{i=1}^n U_i \cap U_{n+1}$$

. (The axiom of choice was used here). Hence the induction hipothesis is proven. By completness of X, $x_n \to x_\infty \in W \cap_{i=1}^\infty U_i$ and the proof is finished.

Baire Category

Various versions of Baire category were related to solving partial differential inclussions, as for example described in the treatments of Dacorogna-Marcellini based on earlier work of Cellina in the 90s. However, the theory becomes particularly flexible and transparent with the concepts of Baire one maps and points of continuity.

Definition

Given a metric space X, I is a Baire one map if it the pointwise limit of continuous maps. That is to say

$$I(x) = \lim_{\epsilon \to 0} I_\epsilon(x)$$

We all learn that the pointwise limit of x^n is not continuous. However, a crucial aspect brought to convex integration by B.Kirchheim is that

The set of points of continuity of Baire-1 map is residual

Recall, that residual sets are the large sets of category theory.

Residuality

Fix n and consider the nested sequence,
$E_{n,k} = \cap_{i,j \geq k}\{u \in X : |J_i(u) - J_j(u)| \leq \frac{1}{n}\}$. By continuity of $J_i - J_j$, $E_{n,k}$, as a countable intersection of closed setsis closed. Now for each u $J_i(u) \to J(u)$ and thus for every n

$$X = \cup_{k=1}^{\infty} E_{n,k}$$

Now Baire theorem implies that the open set is $V_n = \cup_{k=1}^{\infty} \mathrm{int} E_{n,k}$ is dense. Indeed let W an open set and applied Baire theorem to $Y = X \cap \overline{B}$. Then $Y = E_{n,k} \cap \overline{B}$ and a direct application of Baire theorem tell as that $\mathrm{int}(\overline{B} \cap E_{n,k}) \neq \emptyset$ for some k. But since $\mathrm{int}(\overline{B} \cap E_{n,k}) \subset B \cap \mathrm{int}(E_{n,k})$ it follows that $V_n \cap B \neq \emptyset$. That is V_n is open an dense. By a second use of Baire theorem

$$\text{Res} = \cap V_n \text{ is open and dense.}$$

Now suppose $u \in \mathrm{Res}$. Fix n, we find $k, \delta_1 = \delta_1(n, k, u)$ such that $B(u, \delta_1) \in E_{n,k}$. Therefore if $d(v, u) \leq \delta$. It holds that

$$|(J - J_k)(v)| + |J - J_k(u)| \leq \frac{1}{n}$$

by continuity of J_k, exists $\delta_2 = \delta_2(k, u)$ such that

$$|J_k(u) - J_k(v)| \leq \frac{1}{n}$$

Thus if $d(u, v) \leq \delta_1 + \delta_2$, triangle inequality yields

$$|J(u) - J(v)| \leq |(J - J_k)(v)| + |J - J_k(u)| + |J_k(u) - J_k(v)| \leq \frac{3}{n}$$

Continuous respect to strong:Baire-one respect to weak

Lemma

Continuous functionals respect to the strong topology are Baire-one maps respect to the weak topology

Let ρ_i a suitable approximation of the identity and define $I_i(u) = I(\rho_i \star u)$. Then we claim that

- $I_i(u) \to I(u)$
- $I_i(u)$ is continuous respect to the weak topology.

The first claim follows from the strong continuity of I since for a fix u it holds that $\rho_i \star u \to u$ strongly, as $i \to \infty$. Furthermore, suppose that $u_j \rightharpoonup u$. Then for a fixed i. $\rho_i \star u \to \rho_i \star u$ by definition of weak convergence and by the strong continuity of I it follow that

$$I_i(u_j) \to I_i(u)$$

Theorem

If (X, I) have the perturbation property and I be continuous respect to the strong topology. Then $I^{-1}[0]$ is residual (i.e dense).

The aim is to show that the stable points belong to $I^{-1}[0]$. Let $u \in S$. Then since X is the weak closure of X_0 and u is a point of continuity there exists a sequence $u_j \in X_0$ such that

$$\lim_{j \to \infty} \|u_j - u\| + |I(u_j) - I(u)| = 0$$

Now we can apply the perturbation argument, together with a diagonal argument using the continuity of Φ to find a sequence such that $u_j \rightharpoonup u$ and

$$\liminf_{j \to \infty} \|u_j\|^2 \geq \|u\|^2 + \Phi(I(u))$$

Since strong convergence implies convergence of the norms, the sequence u_j does not converge strongly and u is not a point of continuity.

The Tartar framework

Now we need to put together the explicit constructions from the first lectures with the abstract Baire category nonsense. In particular we need to relate the abstract space X with subsolutions and $I^{-1}[0]$ with solutions to the differential inclusions. Even if the gradient case is easier, by almost the same effort we can review the Tartar framework invented by Tartar and Murat in the 70, which we will need later. This formalism allows on one hand to to understand a nonlinear p.d.e as as differential inclusion (often a constitutive relation) couple with a linear system (often a conservation law). The prize is to augment the number of variables. Abstractly, suppose that we have variables
$\mathbf{y} = (\mathbf{y_1}, \dots, \mathbf{y_i}, \dots, \mathbf{y_N}) : \mathbf{R^d} \to \mathbf{R^N}$ and we consider the equation

$$\mathcal{L}(\mathbf{y}, \mathbf{F_j}(\mathbf{y}))$$

where $\mathcal{L}$ is a constant coefficient differential operator and $\{F_j\}_{j=1}^m$ are nonlinear functions. Then by declaring new variables
$z_j = F_j(\mathbf{y}), \mathbf{z} = (\mathbf{z_1}, \dots, \mathbf{z_m})$ and $K \subset \mathbf{R}^{N+m}, K = \{\mathbf{y}, \mathbf{z} : \mathbf{z_j} = \mathbf{F_j}(\mathbf{y})$ we can recasted the above situation as the problem

$$\mathcal{L}(\mathbf{y}, \mathbf{z}) = \mathbf{0}, (\mathbf{y}, \mathbf{z})(\mathbf{x}) \in \mathbf{K} \text{ a.e } \mathbf{x} \in \mathbf{\Omega}$$

Note: The gradient case it is a particular instance of the above problem since locally $U = Df \iff \nabla \times U = 0$.

The Tartar structure

The domain and the variables

$$\mathcal{D} \subset \mathbb{R}^d, \qquad z : \mathcal{D} \to \mathcal{R}^N$$

The conservation law Let A_i be constant matrices. We consider the linear system

$$\mathcal{U}(z) = \sum_{i=1}^{d} A_i \partial_i z = 0$$

The pointwise constraint.

$$z(y) \in K, a.ey \in D$$

The relaxed problem A relaxation of K is given a suitable chosen open (or relative open) set $\mathcal{U}$, with $K \subset \overline{\mathcal{U}}$. The corresponding subsolutions are defined by functions such that

$$\mathcal{L}(z) = 0, \ \mathrm{and} z \in \mathcal{U}$$

The set of subsolutions often is described as the set of weak limits of solutions with certain topology. It might be the problem is rigid and the set of subsolutions coincide with that of solutions.
H-principles H-principle state that if there is a subsolutions, there are infinitely many solutions with properties resemble that of the solution.

The wave cone

All versions of convex integration, starts with a subsolution u_0 and form a sequence of subsolutions u_j with u_j converging to the set K. $u_j - u_{j-1}$ is typically an oscillating sequence which would converge to 0.
In the Tartar formalism, the problem becomes very geometric as starting with $z \in \mathcal{U}$ we need to modify it to arrive to a map such that $z \in K$ if we do explicit construction or to show that the perturbation or approximation property holds, which results in finding sequences converging to z but remaining in $\mathcal{U}$ which still it is a pointwise constraint. Therefore we need to control precisely the values of oscillating solutions to the conservation law. This is precisely what Tartar wave cone describes.

Definition (The wave cone)

Let $\mathcal{L}$ as above. We say that $\bar{z} \in \Lambda \subset \mathbb{R}^N$ if there exists ξ such that for every $h : \mathbb{R} \to \mathbb{R}$ $\mathcal{L}(h(\langle x, \xi \rangle)\bar{z}) = 0$.

Exercise: Show that in the case of $\mathcal{L}(U) = \nabla \times U$, Λ consist of rank-one matrices.

Notice that from the point of view of differential inclusions, and our perturbation property we see two advantages.

- Controlled Range.

$$z(\mathcal{D}) \subset (\text{Range of } h)\bar{z}$$

  A naive ansatz for the solution is to find $z(x) = \sum \varphi_i h_i(\langle x, \xi\rangle \bar{z})$ where φ_i are suitable chosen bump functions. Of course introducing the bump function makes the situation complicated and introduces a new error.

- In searching the perturbation property, it is usefull that in principle that if $\{h_j\}$ is a sequence such that $\int |h_j|^2 \geq C$ and $h_j \rightharpoonup 0$. The corresponding $z_j \rightharpoonup 0$ and $\int |z_j|^2 \geq C$. Thus if

$$A + (\text{Range of } h)\bar{z} \subset u$$

  the constant map A satisfies the perturbation property. (Again the support of h_j is an issue)

The relaxation:Definition

The relaxation is given by the set $\mathcal{U}$. As we at the beginning from the H-principle it holds that if $z(x) \in \overline{\mathcal{U}}$ then we aim for z being a weak limit for exact solutions. It turns out that even in the two matrices case for gradients:

$$Df(x) \in \{A_1, A_2\}, \mathrm{rank}(A_1 - A_2) = 0$$

when couple with affine boundary conditions, the relaxation is trivial unless we allow to approximate solutions. That is we want to understand

$$K^{relaxed} = \{z : \text{there exists } z_j \rightharpoonup z \int \mathrm{dist}(z_j, K) \to 0\}$$

The lamination hull is an approximation from inside Therefore it follows from the definition of laminates , that for every n

$$\cup_n K^{n,\Lambda} \subset K^{relaxed}$$

Compensated compactness

A central aspect of the theory of compensated compactness initiated by Murat and Tartar in the seventies, is to identify nonlinear weakly continuous cuantities for solutions of linear p.d.e.s. The prime example is the div-curl lemma which states for instance that if $u_j \rightharpoonup u$, with $div(u_j) = 0$ $\nabla\varphi_j \rightharpoonup \nabla\varphi$ then the nonlinear amount $u_j \cdot \nabla\varphi_j \rightharpoonup u\nabla\varphi$. This follows by compactness of Sobolev embedding after integrating by parts but hightlights the existence of such nonlinear weakly continuous quantities for solutions of linear p.d.e.s
Indeed compensated compactness characterizes such quantities at least in the case of quadratic functions. Recently there has been developments FILLL

The relaxation: Outside Approximations

- On the other hand by Mazur lemma (which follows by Hahn-Banach), if $z_j \rightharpoonup z$, a convex combination of z_j converges strongly. Thus
$$K^{relaxed} \subset K^{co}$$

- Weakly continuous nonlinearities. Indeed let M be weakly continuous. We define the set $K^M \subset \mathbb{R}^{N+1}$
$$K^M = \{(z, M(z)) \in \mathbb{R}^{N+1}\}$$
Again by Mazur, a convex combination of $(z_j, M(z_j))$ converges strongly and thus the weak limit of z_j belongs pointwise to the set
$$K^{M,c} = \{z : (z, M(z)) \in (K^M)^c$$
In the case of gradients, weakly continuous cuantities are subdeterminants and the above notion is equivalent to polyconvexity, central in the theory of nonlinear elasticity.

Now given $\mathcal{L}, K, \mathcal{U}$ we want to give conditions for which our Baire approach works directly. These are three sufficient conditions:

1. Localize plane waves. There exists a set Λ such that if $l \in \Lambda$, for every Ω there exists $z_j \in C_c(\Omega)$ such that $z_j \rightharpoonup 0$ and $\mathrm{dist}(z_j(x), l) \le \epsilon$.
2. Quantitative lamination. For every $z \in \mathcal{U}$, there exists $\nu \in PL(\mathcal{U})$ with $\int \lambda d\nu = z, \int |\lambda|^2 d\nu \ge \Phi(dist(z, K))$
3. The space of subsolutions is not empty. There is a space of functions $X_0 \subset L^2$ uniformly bounded, which is close under perturbation in $\mathcal{U}$. That is if $z(y) \in \mathcal{U}$ and $w \in C_c(\mathcal{D})$ with $z + w \in \mathcal{U}$ then $z + w \in X_0$

These conditions are a prototype of what suffices. Often they need to experiment ad hoc modifications to be tailored to our specific situation. Anyhow next we show why they suffices.

An H-Principle

After $H1 - H3$ we declare (X, d) the closure of X, under L^2 convergence and d the metric.

Theorem

Assuming H1-H3 the set $z \in X : z(y) \in K a.e. \in \mathcal{D}$ is residual.

Remark:
The definition of X_0 is not canonical and indeed often is tailored to the subsolution we are able to construct or to the properties we would like the solution to enjoy. For example we could prescribe initial conditions $z(x, 0) = z_0(x)$ in X_0. In this way a typical H principle read as follows. If there exists $z(x, t)$ such that $\mathcal{L}(z) = 0, z = z_0, z \in \mathcal{U}$ and $\mathcal{U}$ has the required properties there are infinitely many solutions such that $z(x, t) = z_0$. A priori finding a subsolution should be easier as $\mathcal{U}$ is larger but it could be very difficult as in Muskat or the vortex sheet problem.

Proof. Step 1. Constant Subsolutions
Our aim is to show that defining $I : X \to \mathbb{R}$ by $I(z) = \int_{\mathcal{D}} \Phi(z(x))dx$, (X, I) have the perturbation property. Indeed if there was a constant $z \in X_0$, the perturbation property follows directly from the assumption on the corresponding prelaminate ν_z and the approximation theorem.
For a non constant element of X, we discretize it and apply the above. Namely by rescaling for each $B(x_0, r_0)$ there exists z_j such that $z_j \rightharpoonup 0, z(x_0) + z_j \in \mathcal{U}$ and

$$\lim_{j\to\infty} \int_{B(x_0,r_0)} |z_j|^2 \geq \Phi(z(x_0))|B(x_0, r_0)|$$

end of the proof

Proof. Step 2. Discretizing subsolutions Now since $\Phi(z)$ is in L^1 there exist a finite collection of balls $\{B_i\}_{i=1}^N$such that

$$\int \Phi(z)dz \leq 2 \sum |B_i| \Phi(z(y_i))$$

Declaring $z_k = z + \sum_{i=1}^{I} z_k^i dy$ we obtain trivially that $z_k \rightharpoonup 0$ and

$$\lim_{k \to \infty} \int |z_k - z|^2 dy \geq \int \Phi(z) dy$$

Now by openess of $\mathcal{U}$, for each ball B_i, $z_k^i - z(y_0) \in \mathcal{U}$ and therefore again by openness of $\mathcal{U}$ for sufficiently small balls if $y \in B_i$

$$z_k(y) = z_(y) - z_(y_0) + z_(y_0) + z_k^i$$

is also in an open neighborhood.
Q1:Where is the small lie? Q2:Why it can be fixed?

Comments on the Λ-geometry

- On the wave cone condition. For constant rank-operators there exists potencials and property 1 is obvious. However in various natural cases as the Euler equations or separate convexity such property fails. Thus more complicated potentials have to be constructed on an inverse divergence used. Also sometimes we want to restrict to small cones.
- On the distance function. Often it is required to work with another function semiconcave function which acts a a distance, in the sense that it is positive in $\overline{\mathcal{U}}$ and its zero set is in K.
- Condition 2 is not so aestethic as it needs for example the definition of laminate. Often first order laminates are enough and then • There exists a continuous function Φ with $\Phi(0) = 0$ such that for every state $z \in \mathcal{U}$, there exist $\bar{z} \in \Lambda$ such that

$$z \pm \Phi(\mathrm{dist}(z, K))\bar{z} \in K$$

- On the set K.Typically we might want to add some conditions to the set K, the archetyipical choice being e.g prescribing the energy density $|v(x,t)| = e(t)$.

Comments on the functional analytic framework

- The choice of the space X_0 gives a lot of flexibility if we want to add more conditions on the solutions. In practise one finds a subsolution z and defines the space X_0 tailored to z. In this way, one takes care of prescribing the boundary conditions, turbulent regions. In fact it can be shown quantitatively that the solutions will reproduce all properties of solutions which are described by compensated compactness quantities.
- The boundedness of X in some L^p space, or boundedness after some modification is necessary.
- The functional. In the applications to fluid dynamics, typically one deals with space-time domains. By considering functionals of the type

$$\sup_{t \in I} \int \Phi(\mathrm{dist}_K(t)(z(x,t)))dx,$$

  one can achieves results at every (as oppose to almost every) time.

The Euler equations in the Tartar framework. We consider weak solutions to the Euler equations with constant density. Thus the evolution of the fluid is given by

$$\mathrm{div}(v) = 0, \partial(v) + \mathrm{div}(v \otimes v) = \nabla p = \partial v + div(v \circ v) = \nabla(p - |v|^2) = \nabla q$$

and we consider distributional solutions in $L^2(\mathbb{R}^n)$ with $v(x, 0) = v_0$.

Theorem

(Non-uniqueness of weak solutions) There exists infinitely many solutions with $v \in L^\infty \cap L^2$ with $v_0 = 0$.

Results of this type were obtained by Scheffer and Schnirelman before but the non-uniqess above is stronger and much more interesting the method is so robust that allows for a wealth of more advance results that we will review later on.

The Tartar framework for Euler equations

- The conservation law. We swipe the nonlinearity under a new variable. Thus we consider triples $(v, , u, q)$ such that

$$\mathrm{div}(v) = 0, \partial(v) + \mathrm{div}(u) = \nabla q$$

here u is a symmetric trace free part which corresponds to the traceless part of $v \otimes v$.

- The differential inclusion restoring Euler would be that $v \circ v = u$. Since we want to keep control on the L^∞ norm we prescribe $|v|^2 = \bar{e}$. In turns out that both equalities are nicely codified in the inequality

$$v \times v - u = \frac{2}{n}\bar{e}I$$

- The relaxation. It is possible to directly see that the honest convex hull (Sz Lemma 6.4) of the above is given by the inequality

$$v \times v - u \leq \frac{2}{n}\bar{e}$$

The wave cone

By definition $(\hat{v}, \hat{u}, \hat{q})$ belongs to Λ if there exists (ξ_x, ξ_t) such that

$$\xi_t \hat{v} + \hat{u}(\xi_x) + q\xi, \dot{\xi v} = 0$$

By the second equation $\xi_x \in v^{\perp}$. Therefore $\xi_x = P_{\hat{v}}(\xi_x)$. Thus after further applying ξ_x to the first equation it holds that

$$P_{\hat{v}} \hat{u} P_{\hat{v}}(\xi_x) = -\hat{q}\xi_x$$

In other words .
$(\hat{v}, \hat{u}, \hat{q}) \in \Lambda_{Euler}$ if and only if $(-q)$ is an eigenvalue of $(P_{\hat{v}} \hat{u} P_{\hat{v}})$
• It can be also shown, though it does not follow from the general theory that there exists differential operators such that

$$\mathcal{L}[A(\partial)(\phi)] = 0$$

Thus for such operators $A(\partial)(\eta h(\langle (x,t), (\xi_x, \xi_t)\rangle))$

The subsolutions

The Long Λ segments Notice that the condition for the hull only speaks of (u, v). Therefore, we can choose a direction so $(\hat{u}, \hat{v})$ such that $|(\hat{u}, \hat{v})| = \mathrm{dist}((u, v), K)$ and choose $\hat{q}$ any eigenvalue of $(P_{\hat{v}} \hat{u} P_{\hat{v}})$
The boundedness of X_0 Often, the boundedness of X_0 is less trivial than it seems and some elliptic operator is involved. Indeed by taking traces v is L^∞ bounded and similarly u
In order to bound the pressure we take divergence on the equation $\partial_t v + \mathrm{div}(u) = \nabla q$ to arrive to

$$\Delta q = div(divu)$$

which implies that $q = R \times R(u)$ were $R = R_i$ is a matrix of Riesz transforms.

From subsolutions to solutions

In the above theorem, we were using $v \equiv 0$ as a subsolution and using a constant energy. In fact the choice of X_0 can be adjusted to the subsolution ad hoc to obtain various landmarks. This is an example

Theorem

Let $\bar{e} \in L^\infty(\mathbb{T}^n \times (0,T))$ and $(\bar{v}, \bar{u}, \bar{q})$ be a subsolution. Let $\Omega_{turb} \subset \mathbb{T}^n \times (0,T)$ a subdomain such that $(\bar{v}, \bar{u}, \bar{q}), \bar{e}$ are continuous on Ω_{turb} and

$$e(\bar{v}, \bar{u}) < \bar{e} \ on \ \Omega_{turb}, e(\bar{v}, \bar{u}) = \bar{e} \ off \ \Omega_{turb}$$

Then there exists infinitely many weak solutions $v \in L^\infty(0,T; L^2(\mathbb{T}^n))$ such that

$$v = \bar{v} \ off \ \Omega_{turb}, \frac{|v|^2}{2} = \bar{e}$$

Remark: In particular the solution has the same initial condition than the subsolution. Therefore we have that If

LandMarks

As we will see in the other courses, smooth solutions preserve energy and solutions to Navier-Stokes dissipate energy proportional to viscosity. Thus even if in the vanishing viscosity limit some energy is dissipated due to turbulence as predicted by Kolmogorov and Onsager, in no case energy should be created. Thus a weak solution is called admissible if for all times

$$\int_{\mathbb{T}} |v(x,t)^2 dx \leq \int_{\mathbb{T}} |v(x,0)^2 dx$$

- Lack of uniqueness for any divergence free initial data. Wiederman.
- The set of Initial data for which admissible weak solutions exists is dense.
- All time convergence.

IPM

The incompressible porous media equation in short (IPM) investigates the movement of a fluid, through a porous media. There are many models to study the phenomena because of importance in the engineering community. Mathematically it is accepted as a model a simple system in which the density of the porous is transported by a velocity field, which is divergence free.

$$\partial_t \rho + \mathrm{div}(\rho v) = 0, div(v) = 0$$

Here ρ is an scalar. Such an equation are called active scalars and are typically close with a relation between v and ρ. The fact that the media is porous is typically codified in the Darcy´law

$$v = (0, \rho) + \nabla P$$

- The augmented variables

$$(\bar{\rho}, \bar{v}, \bar{m}) \in C([0, T]; L^{\infty}_{w^*}(\mathbb{R}^2; [-1, 1] \times \mathbb{R}^2 \times \mathbb{R}^2))$$

- The conservation law.

$$\partial_t \bar{\rho} + div(\bar{\rho}\bar{m}) = 0, \rho(x, 0) = \rho_0 \tag{6a}$$

$$\mathrm{div}(m) = 0 = 0, \tag{6b}$$

$$\nabla^{\perp}(\bar{v} + \bar{\rho}(0, 1)) = 0, \tag{6c}$$

- The relaxation.

$$\bar{\rho} = \pm 1, \quad \bar{m} = \bar{\rho}\bar{v} \quad \text{on} \quad \Omega_{\pm}, \tag{7a}$$

$$|2(\bar{m} - \bar{\rho}\bar{v}) + (1 - \bar{\rho}^2)i| < (1 - \bar{\rho}^2) \quad \text{on} \quad \Omega_{\mathrm{mix}}. \tag{7b}$$

In addition, it is required that

$$\sup_{0 \le t \le T} \|\bar{v}(t)\|_{L^\infty} < \infty. \tag{8}$$

Wave cone:Exercise.

Firstly the conservation of mass equation says that

$$\hat{\rho}\xi_t + \langle \xi_x, \hat{m} \rangle = 0$$

implies that the wave cone is indepent on m. Since the two other have a div-curl structure we get that

$$\langle \hat{u}, \hat{u} + \hat{\rho}(0, 1) \rangle = 0,$$

which completing the squares yields to $|\hat{u} + (0, \frac{\hat{\rho}}{2})|^2 = \frac{\hat{\rho}^2}{2}$
As notice by Szekelyhidi we can hide the lack of symmetry given by gravity in a new variable $v = 2u + (0, \rho)$ the wave cone is simply $|v|^2 = \rho^2$. Indeed in these coordinates there is a nice Poincare lemma. In the sense that there is a differential operator $\mathcal{B}$such that

$$\mathcal{L}z = 0 \iff z = \mathcal{B}u$$

from which condition one follow with

$$z = B(\eta\psi(\langle(x, t), (\xi_x, \xi_t))$$

as in the gradient case.

The relaxation

- The relaxation is actually the 2-λ hull of the set $K = \{m = \rho u, |\rho| = 1\}$.
- The segment condition is satisfied.
- If $|v| \leq M$. Then there exists $M' = M(M')$ such that $z \in (K \cap M)^{2,\Lambda}$
- The boundedness in the corresponding X_0 is trivial.
- As a matter of fact. It is not necessary to find the full relaxation. In [CFG] it is given a way to solve the problem with a much smaller set $\mathcal{U}$ where the perturbation property holds.

Stationary IPM

The stationary IPM equations

$$\nabla \cdot (\rho v) = 0, \tag{9}$$
$$\nabla \cdot v = 0, \tag{10}$$
$$v + \nabla p = -(0, \rho). \tag{11}$$

are relaxed into the conservation laws

$$\nabla \cdot m = 0, \tag{12}$$
$$\nabla \cdot v = 0, \tag{13}$$
$$\nabla^{\perp} \cdot (v + (0, \rho)) = 0 \tag{14}$$

and the constitutive set

$$K = \{(\rho, v, m) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \colon \ |\rho| = 1, \ m = \rho v\}, \tag{15}$$

Hull has empty interior

The wave cone is

$$\Lambda = \{(\rho, v, m)\colon\ |v + (0, \rho/2)| = |\rho|/2, \quad m\cdot v^{\perp} = 0 \quad \text{and} \quad m\cdot(v+(0,\rho)) = 0\}$$

(see [HL21, Proposition 2.1]).
Three Λ-affine functions.

- $M_1 = (\rho, v, m) := m \cdot v^{\perp}$ vanishes in $K \cap \Lambda$, enforcing $\mathrm{int}(K^{\Lambda}) = \emptyset$.
- $M_2 \, |v|^2 + \rho v_2$ and $M_3 = m \cdot (v + (0, \rho))$– these three functions determine the wave cone.

Easy: If $(\rho, v, m) \in K^{\Lambda}$ with $v \neq 0$, then $m \cdot v^{\perp} = 0$ yields $m = kv$ for some $k \in \mathbb{R}$.

The main challenge in the computation of the lamination convex hull $K^{lc,\Lambda}$ is the determination of the exact range of the constant of proportionality k in $m = kv$ **The lamination convex hull is described in Theorem below** (see [HL21, Theorem 1.1]):

Theorem

$K^{lc,\Lambda} = \cup_{j=1}^{4} X_j$, *where*

$$X_1 := \left\{ \left(\rho, 0, \frac{1-\rho^2}{2}[e-(0,1)]\right) : \, |\rho| \le 1, \, |e| \le 1 \right\},$$

$$X_2 := \left\{ (\rho, v, kv) \colon \, |\rho| \le 1, \, v \ne 0, \, 1 \le k \le \rho - \frac{(1-\rho^2)v_2}{|v|^2} \right\},$$

$$X_3 := \left\{ (\rho, v, kv) \colon \, |\rho| < 1, \, v \ne 0, \, -1 < k = \rho - \frac{(1-\rho^2)v_2}{|v|^2} < 1 \right\},$$

$$X_4 := \left\{ (\rho, v, kv) \colon \, |\rho| \le 1, \, v \ne 0, \, \rho - \frac{(1-\rho^2)v_2}{|v|^2} \le k \le -1 \right\}.$$

The relative interior of $K^{lc,\Lambda}$ with respect to the manifold $\mathscr{M} := \{(\rho, v, m)\colon m \cdot v^{\perp} = 0\}$ is $\mathscr{U} := \mathrm{int}_{\mathscr{M}}(K^{lc,\Lambda}) = Y_1 \cup Y_2$, where

$$Y_1 := \left\{(\rho, v, kv)\colon\ |\rho| < 1,\ v \neq 0,\ 1 < k < \rho - \frac{(1-\rho^2)v_2}{|v|^2}\right\},$$

$$Y_2 := \left\{(\rho, v, kv)\colon\ |\rho| < 1,\ v \neq 0,\ \rho - \frac{(1-\rho^2)v_2}{|v|^2} < k < -1\right\}.$$

Question: Is this an instance where the relaxation is non trivial but c.i fails?

Instabilities in fluid mechanics

A recurrent topic both, in the Euler equation and in the porous media equation is to describe regimes which are mathematically unstable. That is to say typically, is investigated the evolution of two fluids which at time 0 are separated by an smooth interface. Then there is a regime whether there is a global in time analysis and a regime in which linear analysis shows instability and experiments show a turbulent behaviour. Roughly speaking the turbulent behaviour might be cause by a discontinuity of the mass density (Rayleigh-Taylor in the Euler setting, and sometimes called Saffman Taylor in the IPM context) or by a discontinuity of the velocity field which is called Kelvin-Helmoltz.
The method of convex integration has been succesful in producing weak solutions, which agree with the experiments. An interesting feature of these solutions is that they recover the solutions obtained 15 years before by Otto using a Lagrangian relaxation which set the foundation of the mass transport approach for evolution p.d.e.s.

Mixing Solutions: The Muskat Problem

$$\rho^\circ(x) := \begin{cases} \rho_-, & x \in \Omega_-, \\ \rho_+, & x \in \Omega_+, \end{cases} \tag{16}$$

for $x = (x_1, x_2) \in \mathbb{R}^2$. As in the Euler case, long desired solutions of the porous media equation can be obtained from suitable constructed by applying the H-Principle

- In [Sz12], it is solve the case of flat interfase, recovering Otto relaxed solutions obtained from mass transportation.
- In [CCF21] it is solved the fully unstable solutions by constructing a continuous subsolution.
- In [FSz18], [NSz20] It is solve the fully unstable solution by constructing a piecewise constant subsolution with a much handier proof.
- In [CFM21] It is solve the partially unstable case. This allows to find a weak solution after the celebrated breakdown of Rayleigh-Taylor

and smoothness stability.

Classical Muskat

The classical Muskat problem amounts to solve an equation for the curve

$$(\partial_t z - (\rho_+ - \rho_-)B(z)) \cdot \partial_\alpha z^\perp$$

where B is a corrected Birkhoff-Rott operator whose expresion is as follows. ASK FRAN.(SE LLAMA ASI)

$$B(t,\alpha) := \frac{1}{2\pi} \int \left(\frac{1}{z(t,\alpha) - z(t,\beta)} \right)_1 (\partial_\alpha z(t,\alpha) - \partial_\alpha z(t,\beta)) \, \mathrm{d}\beta. \quad (17)$$

Mixing zone

Such an equation is ill-posed in the unstable case and thus we search for a convex integration solution. It turns out that all these problems are solve in a similar way. Firstly one describes Mixing Zone is described as an envelop of a curve evolving in time, that we call the pseudointerfase with a certain speed of opening. That is we declare
At each time slice $0 < t \leq T \ll 1$, the mixing zone is the open set in $\mathbb{R}^2$ given by

$$\Omega_{\mathrm{mix}}(t) := \{z_\lambda(t,\alpha) \,:\, c(\alpha) > 0,\ \lambda \in (-1,1)\}, \tag{18}$$

parametrized by the map

$$z_\lambda(t,\alpha) := z(t,\alpha) - \lambda t c(\alpha)\tau(\alpha)^\perp, \tag{19}$$

where

- $\tau^\perp$ is the direction of opening of the mixing zone.
- $c(\alpha)$ is the speed of opening.
- z is a curve evolving in time which a time 0 coincides with the initial interfase.

partially unstable: Muskat type operators

Depending on our choice of ρ we need to deal with various opertors. The building blocks are the interaction operators, which are analogous to B but consider how the various boundaries interact with each others.

Interaction Operators

$$B_{\lambda,\lambda'}(t,\alpha) := \frac{\rho_+ - \rho_-}{4\pi} \int \left(\frac{1}{z_\lambda(t,\alpha) - z'_\lambda(t,\beta)} \right)_1 (\partial_\alpha z_\lambda(t,\alpha) - \partial_\alpha z_{\lambda'}(t,\beta))\, \mathrm{d}\beta. \tag{20}$$

and we replace ± 1 by $\pm$.

Discrete Average Operators In the c

$$Av(B) := \frac{1}{2} \sum_{a=\pm} B_a, \qquad B_a := \sum_{b=\pm} B_{a,b}, \tag{21}$$

continuous Average Operators For a continuous $\rho = \lambda$ indeed the re operators is

$$\tilde{A}v(B) = \frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} \int B_{\lambda,\lambda'} d\lambda d\lambda'$$

The subsolutions

The current methods to find subsolutions make an ansatz for ρ (and also m though this looks more inocent), then determine the velocity by Bio-Savart and by defining a mixing zone create solutions.
So far in the literature either $\rho = 0$ in the mixing zone, piecewise constant or $\rho = \lambda$.
Then an ansatz for m is natural is suggested by the fact that for $(x, t) \in \Omega_{\mathrm{mix}}$ the triplet $(\rho, v, m) \in \mathcal{U}$. T
Since in the Muskat problem $|2(\bar{m} - \bar{\rho}\bar{v}) + (1 - \bar{\rho}^2)i| < (1 - \bar{\rho}^2)$
Thus, we write

$$m = (\rho v - \frac{1}{2}(1 - \bar{\rho}^2)i) + (1 - \rho^2)\gamma_{error}$$

so that if $|\gamma| < \frac{1}{2}$ we are in the hull (We still need to declare how it is chosen).

In the case of ρ constant the conservation of mass amount to z to solve Muskat type equations in the stable part and in the upper and lower boundaries of the Mixing zone.

$$(\partial_t z - Av(B)) \cdot \partial_\alpha z^\perp = 0 \quad \text{on} \quad tc(\alpha) = 0, \tag{22a}$$

$$(\partial_t z_a - B_a - a(\gamma_a + \tfrac{1}{2} i)) \cdot \partial_\alpha z_a^\perp = 0 \quad \text{on} \quad tc(\alpha) > 0, \tag{22b}$$

and the following conditions on Ω_{mix}

$$\nabla \cdot \gamma = 0, \tag{23a}$$

$$|\gamma| < \tfrac{1}{2}. \tag{23b}$$

Notice that (22b) it is indeed a boundary condition for γ.

On the speed of opening

$$|2c(\alpha) + \partial_\alpha z_1^\circ(\alpha)| < 1, \tag{24}$$

On the curve z In order to deal with the boundary conditions, we prescribe the error in terms of the curve.
Ansatz for $\gamma = \nabla^\perp g, G = g(z(x, \lambda))$

$$G(t, \alpha, \lambda) := \int_{\alpha_1}^{\alpha} \left(\sum_{a=\pm} \frac{\lambda + a}{2} (\partial_t z - B_a) \cdot \partial_\alpha z_a^\perp - (c\tau + \tfrac{1}{2}) \cdot \partial_\alpha z_\lambda \right) \mathrm{d}\alpha'. \tag{25}$$

Then **The above** G **satisfies the conservation law, the boundary condition and** $|\gamma| < \frac{1}{2}$ **for small times if,**

$$\partial_t z - B_a = o(1), \tag{26a}$$

$$\frac{1}{tc(\alpha)} \int_{\alpha_1}^{\alpha} ((\partial_t z - B) \cdot \partial_\alpha z^\perp + tD \cdot \partial_\alpha (c\tau)) \,\mathrm{d}\alpha' = o(1), \tag{26b}$$

Now it remains to find a curve z which satisfies all the above properties and and the same time

$$z_t = F(z, t)$$

has a solution, at least for small times. We will refrain for given more details and do a couple of loose but powerful remarks.

- In the mixing region it is enough to solve (**??**) approximately

$$\partial_t z = Av(B) + \text{error} \quad \text{on} \quad c(\alpha) > 0,$$

- Instead of the full $Av(B)$ we can have the Taylor expansion. error $= B^{1)} - Av(B) +$ error, where $B^{1)}$ denotes the first order expansion in time of $Av(B)$. This choice yields the following well-defined evolution for z

$$\partial_t z = B^{1)} + \text{error} \quad \text{on} \quad c(\alpha) > 0. \tag{27}$$

- Gluing with a cutoff.

$$\partial_t z = \psi_0 B + \psi_1 B^{1)} + \text{error} \quad \text{on} \quad \mathbb{T},$$

where the error is supported on $\{c(\alpha) > 0\}$.

Figure 3.

Getting rid of the up down interactions

Yet the energy inequalities that we obtain for the operator $\psi_0 Av(B)$ (or other modifications) yields a factor $1/c$ which blows up in the region where $c(\alpha)$ tends to zero because of the terms $B_{+,-}$. The way out of this vicious circle is to treat the interaction between separate boundaries as a perturbation. In this way, one can write $B = E + \text{error}$ in such a way that $E = B_{++} + B_{--}$ yields good energy inequalities and the error is small in the supremum norm and supported on $\{c(\alpha) > 0\}$. Thus, the perturbation can be absorbed in the relaxation even if its derivatives are badly behaving. $Av(B) = E + \text{error}$ in such a way that E

$$\partial_t z = \psi_0 E + \psi_1 E^{1)} + \text{error} \quad \text{on} \quad \mathbb{T},$$

which can be solved using ideas from the classical Muskat problem. It remains to show that $B_{+,-}$ after correction is small in L^∞ (very technical and abuses of the argument principle).

Theorem

For every closed chord-arc curve $z^\circ \in H^6(\mathbb{T};\mathbb{R}^2)$ there exist infinitely many mixing solutions to IPM starting from (16)(**??**) *with $\rho_\pm = \pm 1$.*

Theorem

For every open chord-arc curve z°, either x_1-periodic $z^\circ - (\alpha, 0) \in H^6(\mathbb{T};\mathbb{R}^2)$ or asymptotically flat $z^\circ - (\alpha, 0) \in H^6(\mathbb{R};\mathbb{R}^2)$, whose turned region $\{\partial_\alpha z_1^\circ(\alpha) \leq 0\}$ has positive measure there exist infinitely many mixing solutions to IPM starting from (16)(**??**) *with $\rho_\pm = \pm 1$.*

Theorem 26 is the first result proving the continuation of the evolution of IPM after the breakdown exhibited in the works of Castro-Cordoba.Gancedo-Fefferman-

On the admissible regime

In the above calculations we were using arc length parametrization. Thus in another parametrization the admissible regime for the growth-rate $c(\alpha)$ of the mixing zone compatible with the relaxation of IPM. This is

$$\left| c(\alpha) + \frac{\sigma(\alpha)}{\sqrt{\sigma(\alpha)^2 + \varpi(\alpha)^2}} \right| < 1 \quad \text{on} \quad c(\alpha) > 0, \tag{28}$$

which is characterized by the **Rayleigh-Taylor** function $\sigma := (\rho_+ - \rho_-)\partial_\alpha z_1^\circ$ and the **vorticity** strength $\varpi := -(\rho_+ - \rho_-)\partial_\alpha z_2^\circ$ along z° Observe that (28) prevents the two fluids from mixing wherever the initial interface is stable ($\sigma(\alpha) > 0$) and there is not vorticity ($\varpi(\alpha) = 0$).

Vortex Sheet Initial data

Let z_0 be an arc-length parametrization of sufficiently curve (the vortex sheet) and ω_0 be regular enough (but allowed to change sign as opposed to previous work of Delort!), the vorticity strength so that

$$v_0(x)^* = \int_{\mathbb{T}} \frac{\omega_0(\beta)}{x - z^0(\beta)} d\beta$$

Then, after initial work of Székelyhidi for flat interfases

Theorem (Mengual-Székelyhidi CPAM 2019)

There exist infinitely many solutions to the Euler system starting with vortex initial data.

Proof: Find a subsolution and apply previous H-Principle Q.E.D.

The global rate of dissipation and expansion of the turbulence zone are related via ($N \in \mathbb{N}$ fixed)

$$\frac{E(t_2) - E(t_1)}{t_2 - t_1} = -\tfrac{1}{3}\tfrac{2N+1}{2N-1} \int_{\mathbb{T}} c|\varpi^\circ| \left(\tfrac{2N-1}{4N}|\varpi^\circ| - c\right) \mathrm{d}\alpha + O(t_2),$$

for all $0 \leq t_1 < t_2 \leq T$. Hence, the dissipation rate is maximized at $t = 0$

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t)\Big|_{t=0} = -\frac{1}{48}\|\varpi^\circ\|^3_{L^3(\mathbb{R}^2)}$$

as

$$c(\alpha) \to \frac{1}{4}|\varpi^\circ(\alpha)|$$

and $N \to \infty$.

Atwood numbers (Mengual 2020)

In the case of fluids, where not only the densities but the viscosities μ^+, μ^- are different. After rescaling we can arrive to a modified, Darcy's law can be written in terms of the phase θ as

$$v + \theta(\mathrm{At} + i) = -\nabla p, \tag{29}$$

where the **Atwood number** is defined by

$$\mathrm{At} := \frac{\mu^+ - \mu^-}{\mu^+ + \mu^-} \in (-1, 1).$$

$$|2(1 - \theta \mathrm{At})(m - \theta v) + (1 - \theta^2)(\mathrm{At} v + i)| < (1 - \theta^2)|\mathrm{At} v + i|. \tag{30}$$

Observe that (30) generalizes (**??**).
Notice that each slice $\mathcal{U}_{\mathrm{At}}(\theta, v)$ is an (open) disc of radius proportional to $(1 - \theta^2)|\mathrm{At} v + i|$. Thus, while for $\mathrm{At} = 0$ the relaxation $\mathcal{U}_0$ only narrows as $|\theta| \uparrow 1$ (i.e. z tends to $\mathcal{K}$), for $0 < |\mathrm{At}| < 1$ a pinch singularity arises at $\mathrm{At} v + i = 0$ far away from $\mathcal{K}$.

In the Muskat problem the pinch singularity can be interpreted as lack of both Rayleigh-Taylor and Kelvin-Helmholtz inestability. Namely if

$$[\nabla p] = -\mathrm{i}\frac{\sigma}{\partial_s \mathbf{z}^*}, \qquad [\nabla \psi] = -\mathrm{i}\frac{\varpi}{\partial_s \mathbf{z}^*},$$

$$\varpi + \sigma i = -[\theta](\mathrm{At}\check{v} + \mathrm{i})^* \partial_s \mathbf{z}, \tag{31}$$

The question

In my opinion, the biggest open question is the lack of uniqueness.

- There are many subsolutions with the same initial data.
- For each subsolution there are infinitely many solutions.

The second is understood in the coarse-grained language. It is known that subsolutions arise essentially (Szekelyhidi-Wiederman for Euler) as weak limits of solutions.
In [CFM19] we quantify the fact that subsolutions are weakly limits of solutions. Rougly speaking we prove:
Compensated compactness quantities P of solutions agree with subsolutions at every time slice

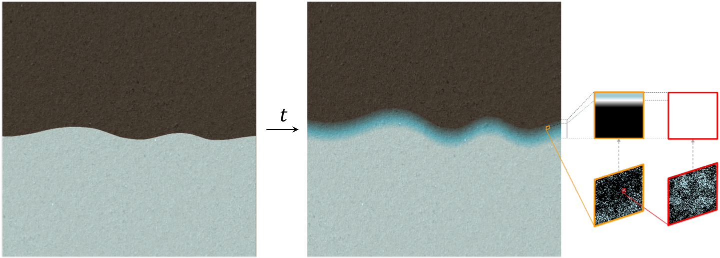
$$\lim_{Q\downarrow\bullet} \fint_{\mathbf{x}(Q,t)} \theta(x,t)\,\mathrm{d}x = \pm 1 \text{ wildly} \quad \lim_{|Q|\to\infty} \fint_{\mathbf{x}(Q,t)} P(\theta(x,t)) - P(\tilde{\theta}(x,t))\,\mathrm{d}x$$

where $\tilde{\theta}$ is a subsolution.
$X : \mathbb{R} \times (-1,1) \times (0,T) : x(\alpha,\lambda,t) = z(\alpha) + \lambda c(\alpha) t.$

For $\lambda \in (-1,1)$ and $0 < \delta < 1$ consider the rectangle $R_N^\delta(\lambda) = (-N, N) \times (\lambda - \frac{1}{N^\delta}, \lambda + \frac{1}{N^\delta})$. This fits the contour line $\mathbb{R} \times \{\lambda\}$ when $N \to \infty$. Then, such degraded mixing solutions display a **perfect linearly degraded macroscopic behaviour on contour lines** $\mathbf{x}(\mathbb{R}, \lambda, t)$

$$\lim_{N\to\infty} \fint_{\mathbf{x}(R_N^\delta(\lambda),t)} \rho(x,t)\,\mathrm{d}x = \lambda \tag{32}$$

uniformly in $\lambda \in (-1,1)$ and $t \in (0,T]$.



MHD

Magneto hydrodynamics, combine Maxwell equations, Navier Stokes and Ohm law to describe the evolution of a charged fluid. It is more or less accepted as model for the evolution of plasma but It is particularly relevant for astrophysics sun flares and tokamaks.
Mathematically, it is very interesting because, there are three integral quantities preserved in time.

Basic Principles:Maxwell equation

The unknowns in *MHD* are the velocity of the fluid v and the magnetic field $\mathcal{B}$ but it will be also important the electric field $\mathbf{E}$, the electric current $\vec{j}$. The evolution of the magnetic field is given by the homogenous Maxwell equations, that we call the Faraday system.

The Faraday system

- $div(\mathcal{B}) = 0$ (No magnetic monoles)
- $\partial_t \mathcal{B} = \nabla \times \mathbf{E}$ (Faraday law of induction)

Ampere law

$$\nabla \times B = \mu_0 \vec{j} \underbrace{(+\frac{1}{c_2}\frac{\partial \mathbf{E}}{\partial_t})}_{\approx = 0}$$

$\mathrm{div}(\vec{j}) \approx 0$. We study non relativistic plasmas and at those speeds we can neglect the displacement current.

Basic electromagnetics

The Lorentz force The electrostatic and magnetic forces on a single particle is $F_L = q(\mathbf{E} + v \times \mathcal{B})$ and thus on on charge body with electric density ρ_e

$$\boxed{F_L = \rho_e \mathbf{E} + \vec{j} \times \mathcal{B}}$$

Ohm law Lorentz force implies that in a frame at rest the force acting on a particle is $F = qE_r$. However if the particle is moving respect to laboratory frame the forces is $F = q(\mathbf{E}_L + v \times \mathcal{B})$. Thus $\mathbf{E}_r = \mathbf{E}_L + v \times \mathcal{B}$. On the other hand in electrostatics $\vec{j} = \sigma \mathbf{E}_r$. Thus we arrive to Ohm law for a moving particle:

$$\boxed{\vec{j} = \sigma(\mathbf{E} + v \times \mathcal{B}) \underbrace{\Rightarrow \mathbf{E}} = -v \times \mathcal{B}}$$

Thus in a in anearly perfectly conducting plasma $\frac{1}{\sigma} \approx 0$, thus what the Electric field is considered a secondary quantity(that we will restore in the relaxation).

Back to Lorentz Finally, by dimensional analysis it can be shown that if $\tau_\epsilon = \frac{\epsilon_0}{\sigma}$ is very small $\rho_e E$ is negligible in comparison with $\vec{j} \times \mathcal{B}$ and thus the forces is summarized into

$$F = \vec{j} \times \mathcal{B}$$

The momentum equation and incompressibility

- $div(v) = 0$ (Incompressibility)
- $\partial_t v + \mathrm{div}(v \times v) + \nu \Delta v = -\nabla p + F$ (Navier Stokes)

We will consider the high Reynolds numbers, $\nu = 0$ situation.
Then if in Ampere law we neglect the displacement current, we solve $\vec{j} = \frac{1}{\mu_0} \nabla \times \mathcal{B}$ where μ_0 (the permitivity of vacuum) can be taken 1 by changing scales. Furthermore since $\mathcal{B}$ is divergence free, it holds $\nabla \times \mathcal{B} \times \mathcal{B} = \mathrm{div}(\mathcal{B} \times \mathcal{B})$. Hence the momentum equation becomes,

$$\boxed{\partial_t v + \mathrm{div}(v \times v) + \nu \Delta v = -\nabla p + \frac{1}{\mu} \mathrm{div}(\mathcal{B} \times \mathcal{B})}$$

MHD equations

We are just left to assuming σ as large as to consider $\mathbf{E} = v \times \mathcal{B}$ to write the ideal *MHD* equation respect to some dimensionless parameters as

- $\partial_t B = \nabla \times (v \times \mathcal{B})$
- $\mathrm{div}(\mathcal{B}) = 0$
- $\partial_t v + \mathrm{div}(v \otimes v) - \mathcal{B} \otimes \mathcal{B}) = -\nabla \pi$
- $\mathrm{div}(v) = 0$

Which is begging to be framed in the Tartar framework!.

Tartar Framework in MHD

$$\nabla \cdot v = \nabla \cdot \mathcal{B} = 0, \tag{33}$$

$$\partial_t v + \nabla \cdot S = 0, \tag{34}$$

$$\partial_t \mathcal{B} + \nabla \times \mathbf{E} = 0 \quad \text{(in 3D)}, \qquad \partial_t \mathcal{B} - \nabla^{\perp} a = 0 \quad \text{(in 2D)}, \tag{35}$$

where in 3D, $S \in \mathbb{R}^{3\times 3}_{\text{sym}}, E \in \mathbb{R}^3$, $S \in \mathbb{R}^{2\times 2}_{\text{sym}}, \in \mathbb{R}$.
We call equations (33)–(35) *Relaxed MHD*

$$K := \{(v, S, \mathcal{B}, \mathbf{E})\colon S = v \otimes v - \mathcal{B} \otimes \mathcal{B} + \Pi I,\ \Pi \in \mathbb{R},\ \mathbf{E} = \mathcal{B} \times v\}.$$

Note that if (u, S, b, a) satisfies (33)–(35) and takes values in K a.e. (x, t), then $(v, \mathcal{B}, \Pi)$ satisfies the MHD equations
As we did with Euler, or IPM, it is needed to consider normalizes suitably the set $\mathcal{K}$. We will do it later.
(v, S) is called the fluid part of the subsolucion and $(\mathcal{B}, \mathbf{E})$ is called the magnetic part.

Wave Cone

There exists (ξ_x, ξ_t)

$$\xi_x \cdot v = \xi_x \cdot \mathcal{B} = 0, \tag{36}$$

$$\xi_t v + S\xi_x = 0, \tag{37}$$

$$\xi_t \mathcal{B} + \xi_x \times \mathbf{E} = 0 \quad \text{(in 3D)}, \qquad \xi_t \mathcal{B} + \mathbf{E}\xi_x^{\perp} = 0 \quad \text{(in 2D)}. \tag{38}$$

Abusing notation, we write (36)–(38) in the concise form $V\xi = 0$.
In 2D the two first equations amount to

$$v\mathcal{B}^{\perp} = 0, \text{ and } Sv^{\perp} = \frac{|v|\mathbf{E}}{|\mathcal{B}|}v. \tag{39}$$

The wave cone for the Maxwell Faraday parts amount to

$$\mathcal{B} \cdot \mathbf{E} = 0$$

In 3D, additional computation yields,

$$S(\mathcal{B} \times v) + (\mathbf{E} \cdot v)v = 0, \tag{40}$$

Rigidity: The Λ hull has empty interior in 2D

- The function $f(v, \mathcal{B}, \mathbf{E}) = \mathbf{E} - v \times \mathbf{E}$ is Λ affine and vanishes on K.
- If $(v, \mathcal{B}, S, \mathbf{E}) \in K^\Lambda$ then $\mathbf{E} = v \times \mathcal{B}$.
- K^Λ has empty interior.
- $K^\Lambda \setminus K$ is not trivial.

An obstruction for the direct application of the H-principle but there exists compactly supported weak solutions?

Stream function in 2D

Let $T > 0$ and suppose $u, b \in C([0, T[; L^2_w(\mathbb{T}^2, \mathbb{R}^2))$

$$\begin{aligned}
&\partial_t u + u \cdot \nabla u - b \cdot \nabla b + \nabla P = 0, \\
&\partial_t b - \nabla^\perp (b \times u) = 0, \\
&\text{div } u = \text{div } b = 0, \\
&\int_{\mathbb{T}^2} u(x,t)\, dx = \int_{\mathbb{T}^2} b(x,t)\, dx = 0 \quad \forall t \in [0, T].
\end{aligned}$$

Then u and b have unique **stream functions** $\phi, \psi \in C_w([0, T[, W^{1,2}(\mathbb{T}^2, \mathbb{R}^2)$ (that is, $-\nabla^\perp \phi = (\partial_2 \phi, -\partial_1 \phi) = u$ and $-\nabla^\perp \psi = (\partial_2 \psi, -\partial_1 \psi) = b$) with $\int_{\mathbb{T}^2} \phi(x,t) dx = \int_{\mathbb{T}^2} \psi(x,t) dx = 0 \, \forall t \in [0, T[$.

Theorem

The mean-square magnetic potential $\int_{\mathbb{T}^2} |\psi(x,t)|^2 \, dx$ is conserved in time.

Corollary

Either $b \equiv 0$ or $\int_{\mathbb{T}^2} |b(x,t)|^2 dx \gtrsim \int_{\mathbb{T}^2} |\psi(x,t)|^2 \, dx = C > 0$ for every $t \in [0, T[$.

Thus, there does not exists compactly spptd solutions to 2D MHD with $b \neq 0$.

Stream function in 2D

Let $T > 0$ and suppose $u, b \in C([0, T[; L^2_w(\mathbb{T}^2, \mathbb{R}^2))$

$$\begin{aligned}
&\partial_t u + u \cdot \nabla u - b \cdot \nabla b + \nabla P = 0, \\
&\partial_t b - \nabla^\perp (b \times u) = 0, \\
&\operatorname{div}\ u = \operatorname{div}\ b = 0, \\
&\int_{\mathbb{T}^2} u(x,t)\,dx = \int_{\mathbb{T}^2} b(x,t)\,dx = 0 \quad \forall t \in [0, T].
\end{aligned}$$

Then u and b have unique **stream functions** $\phi, \psi \in C_w([0, T[, W^{1,2}(\mathbb{T}^2, \mathbb{R}^2)$ (that is, $-\nabla^\perp \phi = (\partial_2 \phi, -\partial_1 \phi) = u$ and $-\nabla^\perp \psi = (\partial_2 \psi, -\partial_1 \psi) = b$) with $\int_{\mathbb{T}^2} \phi(x,t) dx = \int_{\mathbb{T}^2} \psi(x,t) dx = 0\, \forall t \in [0, T[$.

Theorem

The mean-square magnetic potential $\int_{\mathbb{T}^2} |\psi(x,t)|^2\, dx$ *is conserved in time.*

Corollary

Either $b \equiv 0$ *or* $\int_{\mathbb{T}^2} |b(x,t)|^2 dx \gtrsim \int_{\mathbb{T}^2} |\psi(x,t)|^2\, dx = C > 0$ *for every* $t \in [0, T[$.

Thus, there does not exists compactly spptd solutions to 2D MHD with $b \neq 0$.

Basic idea of the proof

$$\partial_t b - \nabla^\perp(b \times u) = \nabla^\perp(-\partial_t \psi - J_{(\psi,\phi)}) = 0. \rightsquigarrow \partial_t \psi + J_{(\psi,\phi)} = 0.$$

If $b = -\nabla^\perp \psi$ and $u = -\nabla^\perp \phi$ are smooth, we compute

$$\begin{aligned}
\partial_t \int_{\mathbb{T}^2} |\psi(x,t)|^2 dx &= -2 \int_{\mathbb{T}^2} \psi(x,t) J_{(\psi,\phi)}(x,t)\, dx \\
&= 2 \int_{\mathbb{T}^2} \phi(x,t) J_{(\psi,\psi)}(x,t)\, dx \\
&= 0.
\end{aligned}$$

We are able to get the general case $u, b \in C_w([0,T), L^2(\mathbb{T}^2, \mathbb{R}^2))$ because of the **Wente inequality**:

$$\begin{aligned}
\int_{\mathbb{T}^2} \psi(x,t) J_{(\psi,\phi)}(x,t)\, dx &\lesssim \|\psi(\cdot,t)\|_{\mathrm{BMO}(\mathbb{T}^2)} \|J_{(\psi,\phi)}(\cdot,t)\|_{\mathcal{H}^1(\mathbb{T}^2)} \\
&\lesssim \|\nabla\psi(\cdot,t)\|_{L^2(\mathbb{T}^2)}^2 \|\nabla\phi(\cdot,t)\|_{L^2(\mathbb{T}^2)} \\
&= \|b(\cdot,t)\|_{L^2(\mathbb{T}^2)}^2 \|u(\cdot,t)\|_{L^2(\mathbb{T}^2)} \\
&\leq \|b\|_{L_t^\infty L_x^2}^2 \|u\|_{L_t^\infty L_x^2}.
\end{aligned}$$

- An approximation argument is needed which uses that Jacobians of $W^{1,2}$ maps are in the Hardy Space
- The key issue is that the equation for b is weakly compact. Thus the result also holds for subsolutions and weak limits of solutions. No subsolutions with $b \neq 0$ compactly supported exists.
- Lack of existence of square integrable stream functions in $\mathbb{R}^2$(Similar to not every Hardy function is the Jacobian of a Sobolev map by Lindberg ARMA 2016)
- Open Question: Is b determined uniquely by the initial data?

A Λ affine function in 3D

Let

$$P(v, S, \mathcal{B}, \mathbf{E}) := \mathbf{E} \cdot B,$$

- P is Λ affine. Since P is quadratic, it suffices to show that $\mathbf{E} \cdot \mathcal{B} = 0$ for all $(\mathcal{B}, \mathbf{E}, v, \mathcal{S}) \in \Lambda$.
- $(u, \mathcal{B}, S, \mathbf{E}) \in K^{\Lambda}$, then $P(\mathbf{E}, \mathcal{B}) = 0$ (Non linear constraint).
- P is weakly compact.
- Formally $\partial_t(MH) = \int Pdx$ i.e $\partial_t(\int \psi\dot{\mathcal{B}}dx) = \int \mathbf{E} \cdot \mathcal{B}dx$
- Magnetic helicity is conserved by subsolutions and weak limits of solutions for $\mathbf{E}, \mathcal{B} \in L^3$.º
- The solutions and subsolutions of Bronzi et all automatically satisfies $P(\mathbf{E}, \mathcal{B}) = 0$.

Lemma

Any quadratic function Q satisfies that

$$Q(tA+(1-t)B) = tQ(A)+(1-t)Q(B)-t(1-t)Q(A-B) \quad (41)$$

Proof.

Recall that

$$(ta+(1-t)b)^2 = ta^2+(1-t)b^2-t(1-t)(a-b)^2$$

Indeed the right hand side is
$t^2a^2+(1-t)^2b^2+2(t)(1-t)ab+t(1-t)(a^2+b^2-2ab) =$
$t^2a^2+t(1-t)a^2+(1-t)^2b^2+t(1-t)b^2 =$
$(t+(1-t)ta^2+((1-t)+t)b^2 = ta^2+(1-t)b^2$
Now, a quadratic form $Q:\mathbb{R}^m \to \mathbb{R}$ by definition, can be expressed as $[Q(x) = \langle Mx, x\rangle$ for a symmetric matrix M and therefore obeys that $Q(A+B) = Q(A)+Q(B)+2\langle MA, B\rangle$ Then exactly as above when dealing with $(ta+(1-t)b)^2$. (41) follows

$\square$

Lemma

Any quadratic function Q which vanishes on Λ is Λ affine, that is if $A - B \in \Lambda$

$$Q(tA + (1-t)B) = tQ(A) + (1-t)Q(B)$$

Lemma

If Q is λ affine and vanishes on Λ and on K, then it vanishes on $K^{\Lambda,lc}$

When we apply this to MHD and to P we discover that the Λ hull has empty interior (does not have non-empty interior). This is a problem for all known versions of c.i in fluid mechanics.
Is this an artificial problem? Is there any interpretation of this product. (this question already appear in Tartar studies of the Maxwell equation).

The electromagnetic potentials

Potentials. Since $\mathcal{B}$ is divergence free there exist many potentials such that

$$\nabla \times \psi = \mathcal{B}$$

ψ is the magnetic potential (vectorial) and thus exchanging $\nabla \times$ and ∂_t in the induction equation

$$\nabla \times (\partial \psi - \mathbf{E}) = 0, i.e \partial_t \psi = E + \nabla g$$

g is the electric potential

Magnetic Helicity

Given any solution to the Faraday system $\mathrm{div}(B) = 0$ and $\partial_t B = \nabla \times E$ the following quantity has shown to be very important as it describes (to some extent) the evolution of the magnetic lines

$$MH(E, B)(t) = \int_{\mathbb{T}^n} \psi \cdot B dx$$

Apart from its physical interpretation magnetic helictiy is preserved even by weak solutions as oppose to energy. This makes MHD very special and the main example of what is called the self-organization conjecture. Let us look at its conservation from the view point of our weakly continuous quantity.

$$\partial_t MH(t) = \int \partial_t \psi \cdot B + \psi \cdot \partial_t B = E + \nabla g \cdot B + \psi \cdot \nabla timesE$$

Now we integrate by parts. For the the curl using that

$$div(F_1 \times F_2) = F_1 \cdot \nabla \times F_2 - F_2 \cdot \nabla \times F_1$$

thus by the divergence theorem (fill details) we get that

$$\partial_t MH(t) = \int E \cdot B + \int \nabla \psi \cdot E = 2 \int E \cdot Bdx$$

Lemma

Let $E(t), B(t)$ vector fields solving the Faraday system such that $E \cdot B = 0$ then Magnetic helicity is conserved

Since for solutions of ideal MHD $E = u \times B$, solutions of ideal MHD preserve helicity. Moreoever subsolutions also preserve MHD.

Conservation of magnetic helicity and c.compactness

Proof.

Suppose $\mathbf{E}, \mathcal{B} \in L^3(\mathbb{T}^3 \times]0,T[;\mathbb{R}^3)$, $S \in L^1_{loc}(\mathbb{T}^3 \times]0,T[;\mathcal{S}^{3\times 3})$ and $\mathbf{E} \in L^{3/2}(\mathbb{T}^3 \times]0,T[;\mathbb{R}^3)$ solves linearized and that takes values in K^Λ a.e.

$$
\begin{aligned}
& \int_\epsilon^{T-\epsilon} \partial_t \eta(t) \int_{\mathbb{T}^3} \Psi(x,t) \cdot \mathcal{B}(x,t)\, dx\, dt \\
= & \underbrace{\left[\int_\epsilon^{T-\epsilon} \eta(t) \int_{\mathbb{T}^3} \left(\mathbf{E}(x,t) - \int_{\mathbb{T}^3} \mathbf{E}(y,t)\, dy - \nabla g(x,t) \right) \cdot \mathcal{B}(x,t)\, dx\, dt \right.}_{div(\mathcal{B})=0} \\
+ & \underbrace{\left. \int_\epsilon^{T-\epsilon} \eta(t) \int_{\mathbb{T}^3} \Psi(x,t) \cdot \nabla \times \mathbf{E}(x,t)\, dx\, dt \right]}_{\text{Integrate by parts}} \\
= & 2 \int_\epsilon^{T-\epsilon} \eta(t) \int_{\mathbb{T}^3} \mathbf{E} \cdot \mathcal{B}\, dx\, dt \\
= & 2 \int_\epsilon^{T-\epsilon} \eta(t) \int_{\mathbb{T}^3} \mathbf{E}(x,t) \cdot \mathcal{B}(x,t)\, dx\, dt = 0,
\end{aligned}
$$

□

Thus, we have a situation where even if the hull is very large, the interior is empty. However there is hope.

Two forms formalism

$$\begin{aligned} F := &B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2 \\ &+ E_1 dx^1 \wedge dt + E_2 dx^2 \wedge dt + a_3 dx^3 \wedge dt \end{aligned} \tag{42}$$

We write $F \cong (B, E)$. Then, Gauss' law and Maxwell-Faraday law are written concisely via differential forms:

$$\nabla \cdot B = 0 \quad \text{and} \quad \partial_t B + \nabla \times E = 0 \qquad \Longleftrightarrow \qquad dF = 0, \tag{43}$$

i.e., F is an exact two-form called *Maxwell two-form* or *electromagnetic two-form*. This readily yields a 4 dimensional potential α such that

$$d\alpha = F$$

Indeed $\alpha = A_i dx_i + gdt$ where A is a magnetic potential $\nabla \times A = B$ and g is an electric one $\nabla g = \partial_t A - E$.

Compensated compactness and simple forms

Recall that in addition to (43), we also need E and B to satisfy $E \cdot B = 0$. We express the latter condition in the language of bivectors:

$$
\begin{aligned}
B \cdot E = 0 \qquad &\Longleftrightarrow \qquad F \wedge F = 2b \cdot a \, dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 = 0 \\
&\Longleftrightarrow \qquad F = v \wedge w \quad \text{for some } v, w \in \mathbb{R}^4
\end{aligned}
$$

The Faraday 2 form is simple.

Lemma

It turns out that $\xi = (\xi_x, \xi_t)$ is a Λ direction for (E, B) if and only if $F \wedge \xi = 0$. Suppose that F_0 and $F \neq 0$ are simple bivectors and that $F \wedge \xi = 0$, where $\xi \in (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}$. The following conditions are equivalent:

(i) *$F_0 + tF$ is simple for all $t \in \mathbb{R}$.*

(ii) *$F_0 \wedge F = 0$.*

(iii) *We can write $F = v \wedge \xi$ and either $F_0 = v_0 \wedge \xi$ or $F_0 = v \wedge w_0$.*

We will see next that in order to find potential, $F_0 = v_0 \wedge \xi$ is a bad case, and $F_0 = v \wedge w_0$ is a good case.

Clebsch variables

$F = d\alpha$. Here the so-called *electromagnetic four-potential* α is of course not unique. We specify a choice of α below. Recall from (43) that our potential α is required to satisfy

$$d\alpha \wedge d\alpha = 0.$$

set

$$\alpha = \varphi\, d\psi, F = d\alpha = d\varphi \wedge d\psi;$$

here $\phi, \psi \in C^\infty(\mathbb{R}^4)$ are called in the literature *Clebsch variables* or *Euler potentials*.

Remark: It is not a Poincare lemma. Thus we need to find specific potentials for the planar waves This we can not do for all lines $F_0 + tF$.
Namely if $F_0 = v_0 \wedge w_0$ and $\Phi_\ell(x,t) = x + \ell^{-1} h'(\ell x \cdot \xi) a$
$\Phi^*(v_0 \wedge w) = \Phi^* v_0 \wedge \Phi^* w_0 = d\psi \wedge d\varphi$

$$d\varphi_\ell(x,t) \wedge d\psi_\ell(x,t) = v_0 \wedge w_0 + \chi(x,t) h''(\ell(x,t) \cdot \xi) \underbrace{(c_2 v_0 - c_1 w_0)}_{=w} \wedge \xi + O\left(\frac{1}{\ell}\right),$$

Λ good segments

- Unfortunately this does not work for all Λ lines (not directions). If ξ is the direction of oscillation of F and $F_0 = w \wedge \xi$,we can not construct potentials for the plane wave taking values in the line $F_0 + tF$
- Thus, segments (F_0, F) are divided in Λ_g (segments) and Λ_b (segments). Similarly we can speak of Λ_g laminates and of Λ_g lamination hull, where only good segments are involved.

Lemma

Let $\mathcal{O}$ an $\mathcal{M}$-open set and Let ν be a Λ_g Laminate supported in $\mathcal{U}$, with an $Skel(\nu) \subset \mathcal{O}$ and mass centered at V_0. Then there is V_j such that: $\mathcal{L}(V_j) = 0$, $V(x,t) \in \mathcal{O}$ and V_j converging weakly to V_0.

A heart breaking discovery K^{Λ_g} is rigid. $E = u \times B$.
Two good news.

- For open sets $\mathcal{O}^{lc,\Lambda} = \mathcal{O}^{\Lambda_g,lc}$
- If $\mathcal{O}$ is $\mathcal{M}$-open then $\mathcal{O}^{\Lambda,lc}$ is $\mathcal{M}$-open

Lamination hulls of open sets in $\mathcal{M}$

We neglect the fluid part as the non-linearity is on the magnetic side. By induction is suffices to show the statement for the first lamination hull. Let

$$\mathrm{seg} = [\omega_0 - \lambda\omega, \omega_0 + (1-\lambda)\omega]$$

be a Λ_g segment.
We consider the case $\omega \wedge \xi = 0 \neq \omega \wedge \xi$. Since $\omega \wedge \omega_0 = 0$ because $\omega_0 + t\omega \in \mathcal{M}$ and ω is a Λ direction we can further assume that there are 4-vectors such that

$$\omega_0 = v \wedge w_0, \omega = v \wedge \xi$$

Now if $|\tilde{\omega}_0 - \omega| \leq \delta$ is close to ω. Then it can be shown that $\tilde{\omega}_0 = \tilde{v} \wedge \tilde{w_0}$ with $|v - \tilde{v}| + |w_0 - \tilde{w}_0| \leq \delta$. Thus, if we declare $\tilde{\omega} = \tilde{v} \wedge \xi$, it satisfies that

$$\tilde{\omega} \wedge \tilde{\omega_0} = 0, \tilde{\omega} \wedge \xi = 0$$

Therefore the Λ_g segment

$$\tilde{\mathrm{seg}} = [\tilde{\omega_0} - \lambda\tilde{\omega}, \tilde{\omega_0} + (1-\lambda)\tilde{\omega}]$$

belongs to $\mathcal{M}$ and is δ close to seg

For open sets, the Lamination hull is its good lamination hull

Suppose that we are in the canonical bad situation: Let

$$\mathrm{seg} = [\omega_0 - \lambda\omega, \omega_0 + (1-\lambda)\omega]$$

be a Λ_b segment, bad segment because $\omega = v \wedge \xi, \omega_0 = v_0 \wedge \xi$ and assume further that ω, ω_0 are not parallel. This results into $\xi \notin \mathrm{span}\{v_0, v\}$, and therefore $v_0 \wedge v \wedge \neq 0$.
Choose then $\omega_\epsilon = \epsilon v_0 \wedge v$ and declare $\tilde{\omega_0} = \omega_0 + \omega_\epsilon$. Then we a segment with the same direction but slightly shifted to pass through $\tilde{\omega_0}$. Then,

$$\mathrm{seg}_\epsilon = [\tilde{\omega_0} - \lambda\Omega, \omega_0 + (1-\lambda)\omega]$$

Thus we still have $\omega \wedge \xi = 0$ but now $(\tilde{\omega} \wedge \epsilon = v_0 \wedge v \wedge \xi \neq 0 =$ and thus $\mathrm{seg}_\epsilon \in \Lambda_g$.
The other bad cases are dealt with similarly.

An strategy which do not require explicit computations

In order to prescribe the energy density and cross helicity densities. It is convenient to use what are called Elssaser variables.
$z_+ = v + \mathcal{B}, z_- = v - \mathcal{B}$ And thus we declare

$$\mathcal{K}_{r,s} = \{|z_+| = r, |z_-| = s, \mathcal{U}_{r,s} = \mathrm{int}\mathcal{K}_{r,s}^{\Lambda,lc}$$

An interesting remark is that we do not compute the Λ hull of a constraint set $K_{r,s}$ but we show that for any $0 < \tau_0 < 1$

$$\mathcal{U}_{r,s} = \cup B_{\mathcal{M}}(K_{\tau r,\tau s}, \epsilon_\tau)^{\Lambda_g} = \cup_{\tau_0 < \tau < 1} \cup K_{\tau r,\tau s}^{lc}$$

The construction is indeed closer in spirit very to the in-approximation (think that $\mathcal{O}_\tau = B_{\mathcal{M}K_{\tau r,\tau s},\epsilon_\tau}$) and
that instead of $\mathcal{O}_\tau \to K$ and $\mathcal{O}_\tau \subset \mathcal{O}_{\tau+1}$ we have that
$(\cup_{\tau_0 < \tau < 1}\mathcal{O}_\tau)^{lc} = U_{r,s}$ and $\mathcal{O}_\tau$ converges to K in the sense of a generalize distance D.

$$U_{r,s} \subset \cup_{\tau_0<\tau<1} K^{lc}_{\tau r,\tau s}$$

First we onitce that $V \in \mathcal{U}_{r,s}, \tau V \in \mathcal{U}_{r,s}, V \in (\tau(K_{r,s})^{lc}$
Now by symmetry of K $(z^+, z^-, S, E) \in K$, imply that
$\pm\tau(K_{r,s}) \subset (K_{\sqrt{\tau}r,\sqrt{\tau}s})^{1,lc}$

$$\begin{aligned}(\mu z^+, \mu z^-, \mu z^+ \otimes z^- + \Pi I) &= \lambda(\sqrt{\mu} z^+, \sqrt{\mu} z^-, \mu z^+ \otimes z^- + \Pi I) \\ &\quad + (1-\lambda)(\sqrt{\mu} z^+, \sqrt{\mu} z^-, \mu z^+ \otimes z^- + \Pi I)\end{aligned}$$

which shows the claim.

Very difficult

$$0 \in B_{\mathcal{M}}(K_{\tau r,\tau s}, \epsilon_\tau) \subset \mathcal{U}_{r,s}$$

$(u, b, S_{u,b} + \Pi I, b \times u) \in K^{lc,\Lambda}_{r,s}$ whenever $|z^+| < \tau r + \varepsilon_\tau$, $|z^-| < \tau s + \varepsilon_\tau$ and $|\Pi| < rs$. (i) $V = (u, b, S_{u,b} + \Pi I, b \times u) \in K^{lc,\Lambda}_{r,s}$.

(ii) $V = (u, b, S_{u,b} + e \otimes e + \Pi I, b \times u) \in K^{lc,\Lambda}$.

(iii) $V = (u, b, S_{u,b} + S + \Pi I, b \times u) \in K^{lc,\Lambda}_{r,s}$.

(iv) $V = (u, b, S_{u,b} + S + \Pi I, b \times u + b \times g) \in K^{lc,\Lambda}_{r,s}$.

(v) $V = (u, 0, S_{u,0} + S + \Pi I, e \times f) \in K^{lc,\Lambda}_{r,s}$.

Easy+Very difficult imply that

$$U_{r,s} \subset \cup_{\tau_0<\tau<1} K^{\Lambda}_{\tau r,\tau s} \subset \cup_{\tau_0<\tau<1} B_{\mathcal{M}}(K_{\tau r,\tau s},\epsilon_\tau)^{\Lambda} \subset U_{r,s}$$

$$U_{r,s} = \cup B_{\mathcal{M}}(K_{\tau r,\tau s},\epsilon_\tau)^{\Lambda} = \cup_{\tau_0<\tau<1} K^{\Lambda}_{\tau r,\tau s}$$

Now it holds that for every $V \in \mathcal{O}_\tau$ $|v|^2 + |\mathcal{B}|^2 \geq \tau\frac{(r^2+s^2)}{2} - \epsilon_\tau$ and by convexity of the norm for every $V_0 \in \mathcal{U}_{r,s}$ $(|v|^2 + |\mathcal{B}|^2) \leq \eta\frac{(^2+s^2}{2}$ for $\eta < 1$. Therefore for all such V essentially holds that

$$D(V_0,\mathcal{K}) \leq 2(|V|^2 - V_0)$$

which give us the required perturbation argument.

Thus we have every ingredient to use our recipe and obtain the following beautiful theorem by choosing for example $r = 1, s = 2$ and noticing that $0 \in \mathcal{U}_{r,s}$.

Theorem

There exists bounded MHD solutions compactly supported in space and time

The proof is an interesting last exercise. A couple of interesting issues at the level of the convex integration schemes. We run convex integration at the level of the factors of the bivectors and do a inner variation argument to assume that we work with locally constant maps to avoid the bad situation $v \wedge w = 0$ with $|v| + |w| \neq 0$