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Bounded Solutions of Ideal MHD with Compact Support in Space-Time

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Abstract

We show that in 3-dimensional ideal magnetohydrodynamics there exist infinitely many bounded solutions that are compactly supported in space-time and have non-trivial velocity and magnetic fields. The solutions violate conservation of total energy and cross helicity, but preserve magnetic helicity. For the 2-dimensional case we show that, in contrast, no nontrivial compactly supported solutions exist in the energy space.

1. Introduction

Ideal magnetohydrodynamics (MHD for short) couples Maxwell equations with Euler equations to study the macroscopic behaviour of electrically conducting fluids such as plasmas and liquid metals (see [31,50]). The corresponding system of partial differential equations governs the simultaneous evolution of a velocity field u and a magnetic field B which are divergence free. The evolution of u is described by the Cauchy momentum equation with an external force given by the Lorentz force induced by B. The evolution of B is, in turn, described by the induction equation which couples Maxwell–Faraday law with Ohm's law.

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The ideal MHD equations give a wealth of structure to smooth solutions and several integral quantities are preserved. In 3D, smooth solutions conserve the *total energy*, but also two other quantities related to the topological invariants of the system are constant functions of time: the *cross helicity* measures the entanglement of vorticity and magnetic field, and the *magnetic helicity* measures the linkage and twist of magnetic field lines. Magnetic helicity was first studied by Woltjer [59] and interpreted topologically in the highly influential work of Moffatt [42], see also [4]. In fact, it was recently been proved in [36] that cross helicity and magnetic helicity characterise all regular integral invariants of ideal MHD.

In this paper we are interested in weak solutions of the ideal MHD system, which in some sense describe the infinite Reynolds number limit. As pointed out in [12] such weak solutions should reflect two properties:

(i) anomalous dissipation of energy;
(ii) conservation of magnetic helicity.

Indeed, just as in the hydrodynamic situation, in MHD turbulence the rate of total energy dissipation in viscous, resistive MHD seems not to tend to zero when the Reynolds number and magnetic Reynolds number tend to infinity. This has been recently verified numerically in 3D; see [20,39,41]. On the other hand simulations, and theoretical results have shown that magnetic helicity is a rather robust conserved quantity even in turbulent regimes, and J.B. Taylor conjectured that magnetic helicity is approximately conserved for small resistivities [57] (unlike subhelicities along Lagrangian subdomains that are magnetically closed at the initial time). Taylor's conjecture is at the core of Woltjer–Taylor relaxation theory which predicts that after an initial turbulent state, various laboratory plasmas relax towards a quiescent state which minimises magnetic energy subject to the constraint of magnetic helicity conservation (see [4,46]).

The conservation of magnetic helicity for weak solutions of ideal MHD was first addressed in [12], and subsequently it was shown in [1,35] that it is conserved if $u, B \in L^3_{x,t}$, that is, in contrast with energy conservation, no smoothess is required. Moreover, the first and second author recently proved that if a solution in the energy space $L^\infty_t L^2_x$ arises as an inviscid limit, then it conserves magnetic helicity (see [29]). In this context, Theorem 2.2 below extends [29, Corollary 1.3], from ideal (that is inviscid, non-resistive) limits of Leray–Hopf solutions to a larger class of possible approximation schemes.

Our main purpose in this paper is to show the existence of nontrivial weak solutions to ideal 3D MHD compatible with both requirements (i)–(ii) above.

Theorem 1.1. *There exist bounded, compactly supported weak solutions of ideal MHD in $\mathbb{R}^3$, with both u, B nontrivial, such that neither total energy nor cross helicity is conserved in time, but magnetic helicity vanishes identically.*

We note that bounded solutions in particular fall into the subcritical regime of [35] for magnetic helicity, so that for the solutions above magnetic helicity must vanish at all times even though the magnetic field B is not identically zero. Moreover, as a corollary of Theorem 2.2 below, it also holds that for bounded solutions on $\mathbb{T}^3$, either the initial data has vanishing magnetic helicity or B cannot have compact support in time. Indeed, as noted by ARNOLD [3], $\int_{\mathbb{T}^3} |B|^2 \,\mathrm{d}x \geq C \left| \int_{\mathbb{T}^3} A \cdot B \,\mathrm{d}x \right|$ at every $t \in [0, T[$, where A is the magnetic potential. It is also worth pointing out that the solutions in Theorem 1.1 have nontrivial cross-helicity.

MHD turbulence in 2D seems to have many similarities with the 3D case (in stark contrast with hydrodynamic turbulence), in particular there is plenty of numerical evidence for anomalous dissipation of energy [6,7]. Nevertheless, we will show in Section 2.2 that in 2D, under very mild conditions, weak solutions with nontrivial magnetic field cannot decay to zero in finite time, in particular solutions as in Theorem 1.1 do not exist in 2D.

Our construction is based on the framework developed in [21] by C. DE LELLIS and the third author for the construction of weak solutions to the Euler equations. This framework is based on convex integration, which was developed by GROMOV [32] following the work of NASH [44], and—in a nutshell—amounts to an iteration procedure whereby one approximates weak solutions via a sequence of subsolutions, in each iteration adding highly oscillatory perturbations designed to cancel the low wavenumber part of the error. In [21] convex integration was used in connection with TARTAR's framework to obtain bounded nontrivial weak solutions of the Euler equations which have compact support and violate energy conservation. Such pathological weak solutions were known to exist [49,51] but the method of [21] turned out to be very robust and many equations in hydrodynamics are amenable to it and its ramifications.

Roughly speaking, the development of the theory followed two strands: concerning the Euler equations and in connection with Onsager's conjecture [27,45], an important problem was to push the regularity of such weak solutions beyond mere boundedness to the Onsager-critical regime. This programme, started in [24], finally culminated in ISETT's work [33], see also [9]. For a thorough report of these developments and connections to Nash's work on isometric embeddings, we refer to [25]. Another, somewhat independent strand, was to adapt the techniques to other systems of equations, such as compressible Euler system [13], active scalar equations [10,18,52,53] and others [8,14,15,37]. A key point in the technique is a study of the phase-space geometry of the underlying system, to understand the interaction of high-frequency perturbations with the nonlinearity in the equations in the spirit of L. Tartar's compensated compactness. A particularly relevant example to this discussion is the case of 2D active scalar equations, where there seems to be a dichotomy between systems closed under weak convergence such as 2D Euler in vorticity form or SQG, and those with a large weak closure such as IPM [18,52]—see the discussion in [23, Section 8] and [34] of [23,34] in this regard.

Concerning the ideal MHD system, setting the magnetic field $b \equiv 0$ obviously reduces to the incompressible Euler equations, and thus [21] applies. More generally, in [8] BRONZI, LOPES FILHO and NUSSENZVEIG LOPES constructed bounded weak solutions of the symmetry reduced form $u(x_1, x_2, x_3, t) = (u_1(x_1, x_2, t), u_2(x_1, x_2, t), 0)$ and $B(x, t) = (0, 0, b(x_1, x_2, t))$, compactly supported in time and not identically zero. These "$2\frac{1}{2}$-dimensional" solutions were obtained by reducing the symmetry reduced 3D MHD to 2D Euler with a passive tracer, where a modification of the strategy of [21] applied. Nevertheless, such reductions to the Euler system do not seem to be able to capture generic, truly 3-dimensional weak solutions, which—with the simultaneous requirement of properties (i) and (ii) above—seem to lie on the borderline between weakly closed (for example SQG) and non-closed (for example IPM) systems.

The remark above will be explained in more detail in Section 2—for now let us merely point out that whilst the Cauchy momentum equation for the evolution of the velocity u has a large relaxation (the main observation behind all results involving convex integration for the Euler equations), the Maxwell system for the evolution of the magnetic field B is weakly closed (an observation going back to the pioneering work of TARTAR [55]). Indeed, our whole philosophy in this paper is to emphasise the role of compensated compactness in connection with conserved quantities—in Section 2 we revisit Taylor's conjecture and conservation of mean-square magnetic potential in this light.

The additional rigidity due to conservation of magnetic helicity is a severe obstruction to applying the available versions of convex integration to MHD: The nonlinear constraint $E \cdot B = 0$, where E is the electric and B the magnetic field, has to be satisfied not just by the weak solutions in Theorem 1.1 but also along any approximating sequence in the convex integration scheme. In order to ensure this constraint, we use nonlinear potentials and, inspired by MÜLLER and ŠVERÁK [43], develop a substantially new version of convex integration directly on differential two-forms (the Maxwell 2-form), consistent with the geometry of full 3D MHD. Indeed, this is the main innovation in this paper, and hence we dedicate Section 4 below to explain the differences to previous schemes in technical terms. We remark in passing that for the special solutions in [8], $E \cdot B = 0$ is automatic.

We close the introduction by commenting on the recent preprint [5] which we learned about after this paper was completed. In [5], $L^\infty_t L^2_x$ weak solutions are constructed which do not preserve magnetic helicity or total energy. The space $L^\infty_t L^2_x$ is super-critical with respect to magnetic helicity, c.f. [29,35] and Theorem 2.2 below, and thus the solutions constructed in [5] seem closer in spirit to unbounded, so-called *very weak* solutions constructed in [2]. The construction in [5] is based on the convex integration scheme developed in [11] for the 3D Navier–Stokes equations, and indeed, we remind the reader that the weak solutions in [11] are super-critical not just in terms of Navier–Stokes regularity, but also in terms of the minimal regularity required for compactness (for example the Leray–Hopf energy space). Thus, the methods of these two papers are completely different, and it would be a very interesting question whether they can be combined—for instance to explore the limiting integrability and smoothness of convex integration solutions to MHD.

2. The Ideal MHD System

We recall that the ideal MHD equations in three space dimensions are written as

$$\partial_t u + u \cdot \nabla u - B \cdot \nabla B + \nabla \Pi = 0, \tag{2.1}$$

$$\partial_t B + \nabla \times (B \times u) = 0, \tag{2.2}$$

$$\nabla \cdot u = \nabla \cdot B = 0, \tag{2.3}$$

for a velocity field u, magnetic field B and total pressure Π. In this paper we consider both the full space case $\mathbb{R}^3$ and the periodic setting $\mathbb{T}^3$. In the latter case the zero-mean condition

$$\langle u \rangle = 0, \quad \langle B \rangle = 0 \qquad \text{for a.e } t \tag{2.4}$$

is added to (2.1)–(2.3), where for notational convenience we write $\langle u \rangle$ for the spatial average on $\mathbb{T}^3$.

As usual, weak solutions of (2.1)–(2.3) can be defined in the sense of distributions for $u, B \in L^2_{loc}$, using the identities $u \cdot \nabla u - B \cdot \nabla B = \nabla \cdot (u \otimes u - B \otimes B)$ and $\nabla \times (B \times u) = \nabla \cdot (B \otimes u - u \otimes B)$ for divergence-free fields. That is,

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3} [u \cdot \partial_t \varphi + (u \otimes u - B \otimes B) : D\varphi] + \int_{\mathbb{R}^3} u_0 \cdot \varphi(\cdot, 0) = 0,\\
&\int_0^T \int_{\mathbb{R}^3} [B \cdot \partial_t \varphi + (B \otimes u - u \otimes B) : D\varphi] + \int_{\mathbb{R}^3} B_0 \cdot \varphi(\cdot, 0) = 0,\\
&\int_0^T \int_{\mathbb{R}^3} u \cdot \nabla \varphi = \int_0^T \int_{\mathbb{R}^3} B \cdot \nabla \varphi = 0
\end{aligned}$$

for appropriate Cauchy data u_0, B_0 for all $\varphi \in C_c^\infty(\mathbb{R}^3 \times [0, T[)$ with $\nabla \cdot \varphi = 0$. An analogous definition is given in the periodic setting on the torus $\mathbb{T}^3$.

2.1. Conserved Quantities

It is well known that there are three classically conserved quantities of ideal 3D MHD on the torus $\mathbb{T}^3$. For the first two, analogous definitions are available in $\mathbb{R}^3$.

Definition 2.1. Let (u, B) be a smooth solution of (2.1)–(2.3) and let A be a vector potential for B, that is $\nabla \times A = B$. The *total energy*, *cross helicity* and *magnetic helicity* of (u, B) are defined as

$$\begin{aligned}
&\frac{1}{2} \int_{\mathbb{T}^3} (|u(x,t)|^2 + |B(x,t)|^2) \, \mathrm{d}x,\\
&\int_{\mathbb{T}^3} u(x,t) \cdot B(x,t) \, \mathrm{d}x,\\
&\int_{\mathbb{T}^3} A(x,t) \cdot B(x,t) \, \mathrm{d}x.
\end{aligned}$$

All three quantities defined above are conserved in time by smooth solutions. The conservation of total energy and cross helicity conservation was studied in [12,35, 58,60]. Conservation of the magnetic helicity was shown in [12] for solutions $u \in C([0, T]; B^{\alpha_1}_{3,\infty})$ and $B \in C([0, T]; B^{\alpha_2}_{3,\infty})$ with $\alpha_1 + 2\alpha_2 > 0$. In [1,35], magnetic helicity conservation is shown under the assumption that $u, B \in L^3(\mathbb{T}^3 \times]0, T[)$.

We note in passing that on the whole space $\mathbb{R}^3$ the analogous definitions of total energy and cross helicity lead to conserved quantities for square integrable solutions, but magnetic helicity is not well-defined. This boils down to the scaling properties of the function spaces in question; see Appendix A. However, for square integrable magnetic fields that are compactly supported in space, magnetic helicity is well-defined. Indeed, every $B \in L^1(\mathbb{R}^3)$ with $\nabla \cdot B = 0$ satisfies $\int_{\mathbb{R}^3} B(x)\,\mathrm{d}x = \hat{B}(0) = 0$ since $\hat{B}(\xi) \cdot \xi / |\xi| = 0$ for all $\xi \neq 0$ and $\hat{B}$ is continuous.

Following L. TARTAR's pioneering work [55] one can understand the system (2.1)–(2.3) as a coupling between linear conservation laws and a set of constitutive laws in form of pointwise constraints. The conservation laws are

$$\partial_t u + \nabla \cdot S = 0, \tag{2.5}$$

$$\partial_t B + \nabla \times E = 0, \tag{2.6}$$

$$\nabla \cdot u = \nabla \cdot B = 0, \tag{2.7}$$

where in 3D, S is a symmetric 3×3 tensor (the Cauchy stress tensor) and E is a vector field (the electric field). Indeed, (2.6) is simply the Maxwell–Faraday law for the electric field. In the periodic case (2.4) is added to (2.5)–(2.7). The *constitutive set* is then obtained by relating the stress tensor S and the electric field E to velocity, magnetic field and pressure, for example via the ideal Ohm's law:

$$K := \{(u, S, B, E)\colon S = u \otimes u - B \otimes B + \Pi I,\ \Pi \in \mathbb{R},\ E = B \times u\}. \tag{2.8}$$

It is easy to verify that the system (2.1)–(2.3) can be equivalently formulated for the state variables (u, S, B, E) as (2.5)–(2.7) together with $(u, S, B, E)(x, t) \in K$ a.e. (x, t).

Using this formulation one can easily identify the conservation of magnetic helicity as an instance of compensated compactness following the work of TARTAR [55] when applied to the Maxwell system

$$\begin{aligned} \partial_t B + \nabla \times E &= 0, \\ \nabla \cdot B &= 0. \end{aligned} \tag{2.9}$$

To explain this, we recall the following generalisation of the div-curl lemma from Example 4 in [55]: suppose we have a sequence of magnetic and electric fields $(B_j, E_j) \rightharpoonup (B, E)$ converging weakly in $L^2_{x,t}$ and such that $\{\partial_t B_j + \nabla \times E_j\}$ and $\{\nabla \cdot B_j\}$ are in a compact subset of H^{-1}. Then $B_j \cdot E_j \overset{*}{\rightharpoonup} B \cdot E$ in the space of measures. In view of the constitutive law $E = B \times u$ we deduce that any reasonable approximation of *bounded* weak solutions of ideal MHD leads in the limit to a solution (u, S, B, E) of (2.5)–(2.7) with $B \cdot E = 0$. That is, the state variables are constrained to the relaxed constitutive set

$$\mathscr{M} = \{(u, S, B, E) : B \cdot E = 0\}. \tag{2.10}$$

In turn, perpendicularity of the electric E and magnetic B fields is closely related to conservation of magnetic helicity. Indeed, if A is a magnetic potential (so that $\nabla \times A = B$), adapting the classical computation (for example [6]) shows that

$$\frac{d}{\mathrm{d}t} \int_{\mathbb{T}^3} A \cdot B \,\mathrm{d}x = -2 \int_{\mathbb{T}^3} B \cdot E \,\mathrm{d}x. \tag{2.11}$$

More generally, we have the following theorem, establishing the connection between compensated compactness and conservation of magnetic helicity, an issue that has been emphasised by L. Tartar [55,56]:

Theorem 2.2. (a) *Suppose that* $(B, E) \in L^p \times L^{p'}(\mathbb{T}^3\times]0,T[)$, $\frac{1}{p} + \frac{1}{p'} = 1$, $1 < p < \infty$, *with* $\langle B\rangle = 0$, *is a solution of* (2.9) *and assume* $B \cdot E = 0$ *a.e. Then magnetic helicity is conserved.*

(b) *Suppose that* (B_j, E_j) *is a sequence of solutions of* (2.9) *as in (a), and in addition*

$$B_j \rightharpoonup B \text{ in } L^2(\mathbb{T}^3\times]0,T[) \text{ and } \sup_{j\in\mathbb{N}} \left\| E_j \right\|_{L^1(\mathbb{T}^3\times]0,T[)} < \infty.$$

Then magnetic helicity is conserved.

Part (a) extends in particular the L^3 result of [35]. Indeed, for weak solutions of ideal MHD with $u, B \in L^3$ we have $E = B \times u \in L^{3/2}$. On the other hand our proof does not rely on a specific regularisation technique as in [17], and merely relies on a weak version of formula (2.11). Part (b) shows that conservation of magnetic helicity holds even beyond the setting of weak continuity of the quantity $B \cdot E$. As a matter of fact this line of argument furnishes a proof of Taylor conjecture for simply connected domains [29].

We begin the proof of Theorem 2.2 by recalling the following L^p Poincaré-type lemma for the Maxwell system (2.9):

Lemma 2.3. *Let* $1 < p < \infty$, $1/p + 1/p' = 1$ *and suppose* $(B, E) \in L^p \times L^{p'}(\mathbb{T}^3\times]0,T[)$ *is a solution of* (2.9). *Then there exist a unique* $A \in L^p_t W^{1,p}_x(\mathbb{T}^3\times]0,T[)$ *and* $\varphi \in L^{p'}_t W^{1,p'}_x(\mathbb{T}^3\times]0,T[)$ *such that*

$$B = \nabla \times A \quad \text{and} \quad \partial_t A + E - \langle E\rangle = \nabla\varphi$$

with $\langle A\rangle = 0$, $\langle\varphi\rangle = 0$ *for a.e.* t *and* $\nabla \cdot A = 0$. *Furthermore,*

$$\|\nabla A\|_{L^p} \lesssim \|B\|_{L^p} \quad \text{and} \quad \|\partial_t A\|_{L^{p'}} + \|\nabla\varphi\|_{L^{p'}} \lesssim \|E\|_{L^{p'}}.$$

Indeed, we set $A = -\Delta^{-1}(\nabla \times B)$ and $\varphi = \Delta^{-1}\nabla \cdot (\partial_t A + E)$ and apply standard Calderón-Zygmund estimates for the Laplacian.

Proof of Theorem 2.2. For part (a) suppose $(B, E) \in L^p \times L^{p'}(\mathbb{T}^3\times]0,T[)$ is a solution of (2.9) with $B \cdot E = 0$. Let $\eta \in C^\infty_c(]0,T[)$, so that for $\varepsilon > 0$ small enough, $\mathrm{supp}(\eta) \subset]\varepsilon, T-\varepsilon[$. Furthermore, let $B_\delta = B * \chi_\delta$ be a standard space-time mollification of B. By using Lemma 2.3 and integrating by parts a few times we get

$$\begin{aligned}
&\int_\varepsilon^{T-\varepsilon} \partial_t\eta(t) \int_{\mathbb{T}^3} A(x,t)\cdot B(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&= \lim_{\delta\searrow 0} \int_\varepsilon^{T-\varepsilon} \partial_t\eta(t) \int_{\mathbb{T}^3} A_\delta(x,t)\cdot B_\delta(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&= \lim_{\delta\searrow 0} \Bigg[\int_\varepsilon^{T-\varepsilon} \eta(t) \int_{\mathbb{T}^3} \big(E_\delta(x,t) - \langle E\rangle - \nabla\varphi_\delta(x,t)\big)\cdot B_\delta(x,t)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

$$+\int_{\varepsilon}^{T-\varepsilon}\eta(t)\int_{\mathbb{T}^3}A_\delta(x,t)\cdot\nabla\times E_\delta(x,t)\,\mathrm{d}x\,\mathrm{d}t\Bigg]$$
$$=2\lim_{\delta\searrow 0}\int_{\varepsilon}^{T-\varepsilon}\eta(t)\int_{\mathbb{T}^3}E_\delta(x,t)\cdot B_\delta(x,t)\,\mathrm{d}x\,\mathrm{d}t$$
$$=2\int_{\varepsilon}^{T-\varepsilon}\eta(t)\int_{\mathbb{T}^3}B(x,t)\cdot E(x,t)\,\mathrm{d}x\,\mathrm{d}t=0,$$

since $|B||E|\in L^1(\mathbb{T}^3\times]0,T[)$ and $B\cdot E=0$.

For part (b) suppose $(B_j,E_j)\in L^p\times L^{p'}(\mathbb{T}^3\times]0,T[)$ is a sequence of solutions of (2.9) with $B_j\cdot E_j=0$ a.e. and assume $B_j\rightharpoonup B$ in $L^2(\mathbb{T}^3\times]0,T[)$ and $\sup_{j\in\mathbb{N}}\|E_j\|_{L^1}<\infty$. We intend to use to the Aubin–Lions Lemma (see for example [48, Lemma 7.7]) to get, up to a subsequence, $A_j\to A$ in $L^2_tL^2_x(\mathbb{T}^3\times]0,T[;\mathbb{R}^3)$; then $\nabla\times A=B$, $\nabla\cdot A=0$ and $\langle A\rangle$ a.e. t, and furthermore $\int_{\mathbb{T}^3}A\cdot B\,\mathrm{d}x$ is constant in t.

First note that $\sup_{j\in\mathbb{N}}\|A_j\|_{L^2_tW^{1,2}_x}<\infty$. Let us denote $W^{1,4}_\sigma(\mathbb{T}^3;\mathbb{R}^3):=\{w\in W^{1,4}(\mathbb{T}^3;\mathbb{R}^3)\colon\nabla\cdot w=0\}$. By using the embedding $L^1(\mathbb{T}^3,\mathbb{R}^3)\hookrightarrow(W^{1,4}_\sigma(\mathbb{T}^3,\mathbb{R}^3))^*$ and the formula $\partial_tA_j=-E_j+\langle E\rangle+\nabla\varphi_j$ we obtain

$$\sup_{j\in\mathbb{N}}\|\partial_tA_j\|_{L^1_t(W^{1,4}_\sigma)^*_x}\lesssim\sup_{j\in\mathbb{N}}\|E_j\|_{L^1}<\infty,$$

which verifies the assumptions of the Aubin–Lions Lemma. □

2.2. The 2-Dimensional Case

In comparison to the above analysis, let us briefly look at the 2-dimensional case. Here (2.2) reduces to

$$\partial_tB+\nabla^\perp(u\cdot B^\perp)=0,\tag{2.12}$$

where we write $B^\perp=(-B_2,B_1)$ for the vector $B=(B_1,B_2)$, and similarly $\nabla^\perp=(-\partial_2,\partial_1)$. The magnetic potential (stream function) of B is a scalar field ψ such that $\nabla^\perp\psi=B$. As for conserved quantities, total energy and cross-helicity has analogous expressions, but magnetic helicity is replaced by the *mean-square magnetic potential*, defined as

$$\int_{\mathbb{T}^2}|\psi|^2\,\mathrm{d}x.$$

Mean-square magnetic potential is conserved by smooth solutions, and in [12] the conservation was shown for weak solutions (u,B) with the regularity $u\in C([0,T];B^{\alpha_1}_{3,\infty})$ and $B\in C([0,T];B^{\alpha_2}_{3,\infty})$ for $\alpha_1+2\alpha_2>1$.

Next, observe that (2.3) implies that $u\cdot B^\perp$ is a div-curl product. Consequently, if we have a sequence of velocity and magnetic fields $(u_j,B_j)\rightharpoonup(u,B)$ converging weakly in L^2 and such that $\{\nabla\cdot u_j\}$ and $\{\nabla\cdot B_j\}$ are in a compact subset of H^{-1}, then $u_j\cdot B_j^\perp\overset{*}{\rightharpoonup}u\cdot B^\perp$ in the space of measures. In other words (2.12) is stable under weak convergence in L^2. Another way of writing (2.12) is by using the stream functions of u and B. Indeed, if we write $u = \nabla^\perp \phi$ and $B = \nabla^\perp \psi$, with $\langle \phi \rangle = \langle \psi \rangle = 0$, then (2.12) becomes

$$\partial_t \psi + J(\phi, \psi) = 0, \tag{2.13}$$

where we write, as usual, $J(\phi, \psi) = \nabla\phi \cdot \nabla^\perp \psi$ for the Jacobian determinant of the mapping $(\phi, \psi)\colon \mathbb{T}^2 \to \mathbb{R}^2$. Observe that the same equation appears also for the 2D Euler equations, where we replace ψ by the vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$ and ϕ by the velocity potential $v = \nabla^\perp \phi$. However, here we do not assume any coupling between ϕ and ψ, and treat (2.13) as a passive scalar equation.

The form (2.13) allows us to prove conservation of the mean-square magnetic potential under very mild conditions as follows:

Theorem 2.4. *Suppose that* $(u, B) \in C_w([0, T[; L^2(\mathbb{T}^2))$ *is a weak solution of* (2.3) *and* (2.12)*. Then the mean-square magnetic potential is conserved.*

We point out that the analogous result for the 2D Euler equations, namely the conservation of enstrophy $\frac{1}{2}\int |\omega|^2 \,\mathrm{d}x$ is well-known [28,40], and the proof is based on the theory of renormalised solutions. Here we give an alternative, short proof, again emphasising that compensated compactness lies at the heart of the matter. For the proof we first recall the $\mathcal{H}^1$ regularity theory of Coifman, Lions, Meyer and Semmes from [16], more precisely the following adaptation of the classical Wente inequality to the torus $\mathbb{T}^2$ (see [30, Theorem A.1]):

Lemma 2.5. *When* $(f_1, f_2, f_3) \in W^{1,2}(\mathbb{T}^2, \mathbb{R}^3)$*, we have*

$$\begin{aligned}\int_{\mathbb{T}^2} f_1(x) J_{(f_2,f_3)}(x)\,\mathrm{d}x &\lesssim \|f_1\|_{\mathrm{BMO}(\mathbb{T}^2)} \left\|J_{(f_2,f_3)}\right\|_{\mathcal{H}^1(\mathbb{T}^2)} \\ &\lesssim \|\nabla f_1\|_{L^2(\mathbb{T}^2)} \|\nabla f_2\|_{L^2(\mathbb{T}^2)} \|\nabla f_3\|_{L^2(\mathbb{T}^2)}. \end{aligned} \tag{2.14}$$

The left-hand side of (2.14) can be understood in terms of $\mathcal{H}^1$–BMO duality, but we in fact only require (2.14) where the left-hand side is Lebesgue integrable.

Proof of Theorem 2.4. First let us assume that u and B are smooth. Then, using (2.13), we obtain after integration by parts

$$\frac{d}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{T}^2} |\psi|^2\,\mathrm{d}x = \int_{\mathbb{T}^2} \psi J(\psi, \phi)\,\mathrm{d}x = -\int_{\mathbb{T}^2} \phi J(\psi, \psi)\,\mathrm{d}x = 0. \tag{2.15}$$

For the general case note first that under the assumption on (u, B), using the compact embedding $W^{1,2} \hookrightarrow L^2$ the stream functions ϕ, ψ belong to $C([0, T]; L^2(\mathbb{T}^2))$ with $\nabla\phi, \nabla\psi \in C_w([0, T]; L^2(\mathbb{T}^2))$. Then the computation (2.15) can be carried out using standard regularisation of ψ, ϕ and the uniform bound in (2.14). When ψ_δ is a standard space-time mollification of ψ, the function $\partial_t |\psi_\delta|^2 = -2\psi_\delta [J(\psi, \phi)]_\delta$ is handled via the formula

$$[J(\psi, \phi)]_\delta = [J(\psi - \psi_\delta, \phi)]_\delta + ([J(\psi_\delta, \phi)]_\delta - J(\psi_\delta, \phi)) + J(\psi_\delta, \phi).$$

□

Theorem 2.4 implies the following corollary:

Corollary 2.6. *Suppose* $(u, B) \in C_w([0, T[; L^2(\mathbb{T}^2))$ *is a weak solution of* (2.3) *and* (2.12). *Then either* $B \equiv 0$ *or there exists a constant* $c > 0$ *such that* $\int_{\mathbb{T}^2} |B|^2 \,\mathrm{d}x \geq c$ *for every* $t \in [0, T[$.

Proof. The proof follows by using the Poincaré inequality at every $t \in [0, T[$ to estimate $\int_{\mathbb{T}^2} |B(x,t)|^2 \,\mathrm{d}x = \int_{\mathbb{T}^2} |\nabla\psi(x,t)|^2 \,\mathrm{d}x \geq C \int_{\mathbb{T}^2} |\psi(x,t)|^2 \,\mathrm{d}x$. □

Thus, in 2D even if the kinetic and magnetic energies $\int_{\mathbb{T}^2} |u|^2 \,\mathrm{d}x$ and $\int_{\mathbb{T}^2} |B|^2 \,\mathrm{d}x$ may fluctuate (and indeed, numerical experiments indicate anomalous dissipation of the total energy even in 2D [7]), by Corollary 2.6 it is impossible for the magnetic energy to dissipate to zero.

Finally, we remark that although it is natural to ask whether an analogue of Theorem 2.4 holds in the whole space $\mathbb{R}^2$, in fact square integrable divergence-free vector fields do not in general have a square integrable stream function in $\mathbb{R}^2$. This is shown in Appendix A.

3. Plane-Wave Analysis

Recall that the ideal MHD system in 3D can be written for a *state variable* (u, S, B, E) in terms of the conservation laws (2.5)–(2.7) with the constitutive set K, defined in (2.8). The framework introduced by Tartar amounts to an analysis of one-dimensional oscillations compatible with (2.5)–(2.7)—the wave-cone—and then the interaction of the wave-cone with the constitutive set; we carry out this analysis in this section.

3.1. The Wave Cone and the Lamination Convex Hull

Plane waves are one-dimensional oscillations of the form $(x, t) \mapsto h((x, t) \cdot \xi)V$ with

$$V = (u, S, B, E) \in \mathbb{R}^{15},$$

$\xi = (\xi_x, \xi_t) \in (\mathbb{R}^3 \times \mathbb{R}) \setminus \{0\}$ and $h \colon \mathbb{R} \to \mathbb{R}$. For a plane wave solution, (2.5)–(2.7) become

$$\xi_x \cdot u = \xi_x \cdot B = 0, \tag{3.1}$$

$$\xi_t u + S\xi_x = 0, \tag{3.2}$$

$$\xi_t B + \xi_x \times E = 0. \tag{3.3}$$

In what follows, we will write, with a slight abuse of notation, (3.1)–(3.3) in the concise form $V\xi = 0$.

Definition 3.1. The *wave cone* for ideal MHD is

$$\Lambda_0 = \{V = (u, S, B, E) \in \mathbb{R}^{15} \colon \exists \xi \in \mathbb{R}^4 \setminus \{0\} \text{ such that (3.1)–(3.3) hold}\}.$$

We also denote

$$\Lambda = \{V = (u, S, B, E) \in \mathbb{R}^{15} : \exists \xi \in (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R} \text{ such that (3.1)–(3.3) hold hold}\}.$$

If $V_1, V_2 \in \mathbb{R}^{15}$ satisfy $V_1 - V_2 \in \Lambda$, then $[V_1, V_2] \subset \mathbb{R}^{15}$ is called a Λ-*segment*.

In the convex integration process we will use Λ instead of Λ_0, as the requirement $\xi_x \neq 0$ is crucial to many of the arguments. We next define lamination convex and Λ-convex hulls.

Given a set $Y \subset \mathbb{R}^{15}$ we denote $Y^{0,\Lambda} := Y$ and define, inductively

$$Y^{N+1,\Lambda} := Y^{N,\Lambda} \cup \{\lambda V + (1-\lambda)W : \lambda \in [0,1],\ V, W \in Y^{N,\Lambda},\ V - W \in \Lambda\}$$

for all $N \in \mathbb{N}_0$.

Definition 3.2. When $Y \subset \mathbb{R}^{15}$, the *lamination convex hull* of Y (with respect to Λ) is

$$Y^{lc,\Lambda} := \bigcup_{N \geq 0} Y^{N,\Lambda}.$$

It is well-known that semiconvex hulls can be expressed by duality in terms of measures, see for example [15,38,47].

Definition 3.3. Let $Y \subset \mathbb{R}^{15}$. The set of *laminates of finite order* (with respect to Λ), denoted $\mathcal{L}(Y)$, is the smallest class of atomic probability measures supported on Y with the following properties:

(i) $\mathcal{L}(Y)$ contains all the Dirac masses with support in Y.
(ii) $\mathcal{L}(Y)$ is closed under splitting along Λ-segments inside Y.

Condition (ii) means that if $\nu = \sum_{i=1}^{M} \nu_i \delta_{V_i} \in \mathcal{L}(Y)$ and $V_M \in [Z_1, Z_2] \subset Y$ with $Z_1 - Z_2 \in \Lambda$, then

$$\sum_{i=1}^{M-1} \nu_i \delta_{V_i} + \nu_M(\lambda \delta_{Z_1} + (1-\lambda)\delta_{Z_2}) \in \mathcal{L}(Y),$$

where $\lambda \in [0,1]$ such that $V_M = \lambda Z_1 + (1-\lambda) Z_2$.

Remark 3.4. Given $V \in Y^{N,\Lambda}$, we may write $V = \lambda_1 V_1 + \lambda_2 V_2$, where

$$V_1, V_2 \in Y^{N-1,\Lambda}, \qquad 0 \leq \lambda_1 \leq 1, \qquad \lambda_1 + \lambda_2 = 1, \qquad V_1 - V_2 \in \Lambda.$$

Similarly, we write $V_1 = \lambda_{1,1} V_{1,1} + \lambda_{1,2} V_{1,2}$. Repeating this process, by induction we arrive at a finite-order laminate with support in Y and barycentre V:

$$\nu = \sum_{\mathbf{j} \in \{1,2\}^N} \mu_{\mathbf{j}} \delta_{V_{\mathbf{j}}}, \qquad \operatorname{supp}(\nu) \subset Y, \qquad \bar{\nu} = V,$$

where $\mu_{\mathbf{j}} = \mu_{j_1,\ldots,j_N} = \lambda_{j_1} \ldots \lambda_{j_1,\ldots,j_N} \in [0,1]$.

In addition to the lamination convex hull, another, potentially larger, hull is used in convex integration theory. In order to define it we recall the notion of Λ-convex functions.

Definition 3.5. A function $f : \mathbb{R}^{15} \to \mathbb{R}$ is said to be Λ-*convex* if the function $t \mapsto f(V + tW) : \mathbb{R} \to \mathbb{R}$ is convex for every $V \in \mathbb{R}^{15}$ and every $W \in \Lambda$.

While the lamination convex hull is defined by taking convex combinations, the Λ-convex hull Y^Λ of $Y \subset \mathbb{R}^{15}$ is defined as the set of points that cannot be separated from Y by Λ-convex functions.

Definition 3.6. When $Y \subset \mathbb{R}^{15}$ is compact, the Λ-*convex hull* Y^Λ consists of points $W \in \mathbb{R}^{15}$ with the following property: if $f : \mathbb{R}^{15} \to \mathbb{R}$ is Λ-convex and $f|_Y \leq 0$, then $f(W) \leq 0$.

3.2. Normalisations of the Constitutive Set

In order to produce bounded solutions of 3D MHD we consider normalised versions of the constitutive set K. We wish to prescribe both the total energy density $(|u|^2 + |B|^2)/2$ and the cross helicity density $u \cdot B$, but for this aim it is obviously not enough to prescribe $|u|$ and $|B|$. However, by using the *Elsässer variables*

$$z^\pm := u \pm B$$

we can write $(|u|^2 + |B|^2)/2 = (\left|z^+\right|^2 + \left|z^-\right|^2)/4$ and $u \cdot B = (\left|z^+\right|^2 - \left|z^-\right|^2)/4$, and thus it suffices to prescribe $\left|z^\pm\right|$. This motivates the normalisation given below; recall that $K := \{(u, S, B, E) : S = u \otimes u - B \otimes B + \Pi I,\ \Pi \in \mathbb{R},\ E = B \times u\}$.

Definition 3.7. Whenever $r, s > 0$, we denote

$$K_{r,s} := \{(u, S, B, E) \in K : |u + B| = r,\ |u - B| = s,\ |\Pi| \leq rs\}. \tag{3.4}$$

As pointed out in Section 2, the Maxwell system is essentially closed under weak convergence; the scalar product $B \cdot E$ is weakly continuous. As an immediate consequence the Λ-convex hull $K_{r,s}^\Lambda$ has empty interior (in 3D, and also in 2D). Indeed, Tartar's result in Example 4 [55] is based on the fact that the quadratic expression $Q(u, S, B, E) := B \cdot E$ satisfies

$$Q(V) = 0 \quad \text{for all } V \in \Lambda_0,$$

and consequently, Q is Λ_0-affine. Then we deduce that

$$K_{r,s}^{\Lambda_0} \subset \mathscr{M},$$

where $\mathscr{M}$ is the set in (2.10).

In 3D, assume now $B \times u \neq 0$. Then (3.1) implies, up to normalisation, that $\xi_x = B \times u$, and then (3.3) yields $\xi_t B = -(B \times u) \times E = (E \cdot u)B - (E \cdot B)u = (E \cdot u)B$. Thus, whenever $B \times u \neq 0$, (3.1)–(3.3) reduce to the conditions

$$S(B \times u) + (E \cdot u)u = 0, \qquad B \cdot E = 0 \tag{3.5}$$

which is an easier condition to check in the sequel.

4. Discussion of the Convex Integration Scheme in 3D

The standard way of finding nontrivial compactly supported solutions for equations of fluid dynamics was first presented in [21] and axiomatised in [53]. We describe it briefly in the case of Theorem 1.1.

With a bounded domain $\Omega \subset \mathbb{R}^4$ fixed, it suffices to find a solution V of the relaxed MHD equations $\mathcal{L}(V) = 0$ such that $V(x, t) \in K_{2,1}$ a.e. in Ω ($K_{2,1}$ defined in (3.4)) and $V(x, t) = 0$ a.e. outside Ω. One intends to construct V as a limit of subsolutions, that is, mappings V_ℓ solving $\mathcal{L}(V_\ell) = 0$ and taking values in $K_{2,1}^{lc,\Lambda}$.

The basic building blocks of the construction are plane waves which oscillate in directions of Λ. In order to prevent harmful interference of the waves and to make the eventual solutions compactly supported, one needs to localise the plane waves. The localisation is customarily carried out by constructing potentials. This causes small error terms, and in order for each V_ℓ to take values in the lamination convex hull, one hopes to prove that the hull has non-empty interior. The specifics of the convex integration scheme vary (see for example [15,18,21] for three different approaches in fluid dynamics and [38,53] for a more general discussion).

In the case of 3D MHD, the process is more subtle, as $K_{2,1}^{lc,\Lambda}$ has an empty interior, more precisely $K_{2,1}^{lc,\Lambda} \subset \mathscr{M}$. Therefore, although we may proceed with the 'symmetric (fluid) part' u and S, the 'anti-symmetric (electromagnetic) part' B and E needs special attention.

As a first step towards overcoming the emptiness of $\operatorname{int}(K_{2,1}^{lc,\Lambda})$, we construct a pair of non-linear potential operators P_B and P_E that satisfy $\nabla \cdot P_B[\varphi, \psi] = 0$, $\partial_t P_B[\varphi, \psi) + \nabla \times P_E[\varphi, \psi] = 0$ and $P_E[\varphi, \psi] \cdot P_E[\varphi, \psi] = 0$ for all $\varphi, \psi \in C^\infty(\mathbb{R}^4)$. (For u and S we simply use the potentials in [21] for the Euler equations.) We add the localised plane waves *within* P_B and P_E; despite their non-linearity, P_B and P_E have cancellation properties which allow them to map suitable sums of localised plane waves to sums of localised plane waves (up to a small error term).

As a drawback, P_B and P_E do not allow oscillating plane waves for every Λ-segment—their applicability depends not only on the direction but also on the location of the segment. We consider Λ-segments for which P_B and P_E give plane waves and call them *good* Λ*-segments* or Λ_g-segments. This leads us to study $K_{2,1}^{lc,\Lambda_g}$, the restricted lamination convex hull of $K_{2,1}$ in terms of Λ_g.

A priori, Λ_g is a rather large subset of Λ-segments. However, even though $K_{2,1}^{lc,\Lambda}$ has non-empty interior relative to $\mathscr{M}$, the electromagnetic part of $K_{2,1}^{lc,\Lambda_g}$ is rigid: the constraint $E = B \times u$ holds for all $(u, S, B, E) \in K_{2,1}^{lc,\Lambda_g}$. Nevertheless, as the in-approximation formulation of convex integration shows, the iterative step happens at relatively open sets and it is a limit procedure which leads to the inclusion in closed sets. Thus, in this case the size of Λ_g saves the day; as it turns out, for *relatively open* subsets $U \subset \mathscr{M}$ we have $U^{lc,\Lambda} = U^{lc,\Lambda_g}$. This eventually allows us to apply the Baire category framework of convex integration in $\mathcal{U}_{2,1} := \operatorname{int}_{\mathscr{M}}(K_{2,1}^{lc,\Lambda})$. We present useful characterisations of $\mathcal{U}_{2,1}$ in Theorem 6.7; in particular, $\operatorname{int}_{\mathscr{M}}(K_{2,1}^{lc,\Lambda}) = \cup_{0 \le \tau < 1} K_{2\tau,\tau}^{lc,\Lambda}$. Theorem 6.7 is the most technically difficult part of the paper and the heart of the convex integration scheme. The proof of Theorem 1.1 is then completed in Section 7.

Notice that actually, we do not compute the exact hull $K_{2,1}^{lc,\Lambda}$. However, the formula $\mathrm{int}_{\mathscr{M}}(K_{2,1}^{lc,\Lambda}) = \cup_{0\leq\tau<1} K_{2\tau,\tau}^{lc,\Lambda}$ turns out to give us enough information about $K_{2,1}^{lc,\Lambda}$. The formula is used in a similar manner as in [15].

5. Potentials in 3D

We wish to find potentials corresponding to Λ-segments. For the fluid variables (u, S), we simply use the potentials of [21,22] for the Euler equations. In the case of the electromagnetic variables (B, E), the question about existence of potentials is more subtle because of the non-linear constraint $B \cdot E = 0$ that the potentials need to obey. This issue is studied in Sections 5.3–5.8.

5.1. Potentials for the Fluid Side

We recall from [21,22] that potentials for the fluid part, that is, the variables u and S, can be obtained as follows. First of all, recall that (2.5)–(2.6) can be written equivalently for the symmetric 4×4 matrix

$$U = \begin{pmatrix} S & u \\ u^T & 0 \end{pmatrix} \tag{5.1}$$

as $\nabla_{x,t} \cdot U = 0$. With this notation (3.1)–(3.2) (that is belonging to the wave-cone) is equivalent to $U\xi = 0$ for some $\xi \in \mathbb{R}^4 \setminus \{0\}$. Let us denote $\mathbb{R}^{4\times 4}_{\mathrm{sym},0} := \{U \in \mathbb{R}^{4\times 4}_{\mathrm{sym}} : U_{4,4} = 0\}$.

Lemma 5.1. *Suppose $U \in \mathbb{R}^{4\times 4}_{\mathrm{sym},0}$ such that $U\xi = 0$ for some $(\xi_x, \xi_t) \in (\mathbb{R}^3\setminus\{0\}) \times \mathbb{R}$. Then there exists $P_U : C^\infty(\mathbb{R}^3 \times \mathbb{R}) \to C^\infty(\mathbb{R}^3 \times \mathbb{R}; \mathbb{R}^{4\times 4}_{\mathrm{sym},0})$ with the following properties:*

- (i) $\nabla \cdot P_U[\phi] = 0$ *for every* $\phi \in C^\infty(\mathbb{R}^3 \times \mathbb{R})$.
- (ii) *If* $\phi(x, t) = h((x, t)\cdot\xi)$ *for some* $h \in C^\infty(\mathbb{R}^3 \times \mathbb{R})$, *then we have* $P_U[\phi](x, t) = h''((x, t) \cdot \xi)U$ *for all* $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$.

This lemma essentially follows from the proof of [21, Proposition 3.2]. For the convenience of the reader we sketch a simplified proof, following the exposition in [54]:

Proof. As noted in [22,54], a matrix-valued quadratic homogeneous polynomial $P : \mathbb{R}^4 \to \mathbb{R}^{4\times 4}$ gives rise to a differential operator $P(\partial)$ as required in the lemma, if $P = P(\eta)$ satisfies

$$P\eta = 0, \ P^T = P, \ Pe_4 = 0, \ P(\xi) = U.$$

Elementary examples satisfying the first 3 conditions above are given by $P(\eta) = \frac{1}{2}(R\eta \otimes Q\eta + Q\eta \otimes R\eta)$ for antisymmetric 4×4 matrices R, Q such that $Re_4 = 0$. In particular, for any $a, b \perp \xi$ with $a \perp e_4$, set $R = a\otimes\xi - \xi\otimes a$ and $Q = b\otimes\xi - \xi\otimes b$, to obtain $P_{a,b}(\eta)$. One quickly verifies that $P_{a,b}(\xi) = \frac{1}{2}(a \otimes b + b \otimes a)$. Since any $U \in \mathbb{R}^{4\times 4}_{sym,0}$ with $U\xi = 0$ can be written as a linear combination

$$U = \sum_i \tfrac{1}{2}(a_i \otimes b_i + b_i \otimes a_i)$$

for vectors $a_i, b_i \in \mathbb{R}^4$ with $a_i \cdot \xi = b_i \cdot \xi = 0$ and $a_i \cdot e_4 = 0$, we obtain P_U as required in the lemma as

$$P_U(\eta) = \sum_i P_{a_i,b_i}(\eta).$$

□

5.2. Wave Cone Conditions on u, B and E

It will turn out that when we choose which Λ-directions to use, we have much more freedom in the choice of S than the three other variables u, B and E. Recall that in 3D, the wave cone conditions are

$$\xi_x \cdot u = \xi_x \cdot B = 0, \tag{5.2}$$

$$\xi_t u + S\xi_x = 0, \tag{5.3}$$

$$\xi_t B + \xi_x \times E = 0. \tag{5.4}$$

We can typically first find u, B, E, ξ satisfying (5.2) and (5.4) and afterwards choose S satisfying (5.3). This motivates the following observation:

Lemma 5.2. *Let $u, B, E \in \mathbb{R}^3$. The following conditions are equivalent.*

(i) (5.2) *and* (5.4) *have a solution* $\xi \in (\mathbb{R}^3\setminus\{0\}) \times \mathbb{R}$.
(ii) $B \cdot E = 0$.

Proof. We first show that (i) ⇒ (ii). Choose a solution $\xi \in (\mathbb{R}^3\setminus\{0\}) \times \mathbb{R}$ of (5.2) and (5.4). If $\xi_t \neq 0$, then (5.4) gives $B \cdot E = -(\xi_x \times E \cdot E)/\xi_t = 0$. If $\xi_t = 0$, then (5.4) gives $E = k\xi_x$ for some $k \in \mathbb{R}$, so that (5.2) gives $B \cdot E = 0$.

We then show that (ii) ⇒ (i). If $B \times u \neq 0$, we choose $\xi_x = B \times u$. Since $B \cdot E = 0$, we may write $E = c_1\, B \times u + c_2 B \times (B \times u)$ for some $c_1, c_2 \in \mathbb{R}$. (The set $\{B, B \times u, B \times (B \times u)\}$ is an orthogonal basis of $\mathbb{R}^3$.) Thus $\xi_x \times E = c_2\, |B \times u|^2\, B$ and we may choose $\xi_t = -c_2\, |B \times u|^2$. If, on the other hand, $B \times u = 0$, we may set $\xi_t = 0$ and choose $\xi_x = a$ if $E \neq 0$ and any $\xi_x \in \{B\}^\perp\setminus\{0\}$ if $E = 0$. □

5.3. Maxwell Two-Forms

Our aim in the rest of this chapter is to find potentials for the variables B and E. We carry out this task using the formalisms of two-forms and bivectors in $\mathbb{R}^4$. In electromagnetics, it is customary to express $(B, E) \in \mathbb{R}^3 \times \mathbb{R}^3$ as a unique bivector $\omega \in \Lambda^2(\mathbb{R}^4)$ via the identification

$$\begin{aligned}\omega := B_1\, \mathrm{d}x^2 \wedge \mathrm{d}x^3 + B_2 \mathrm{d}x^3 \wedge \mathrm{d}x^1 + B_3 \mathrm{d}x^1 \wedge \mathrm{d}x^2 \\ + E_1 \mathrm{d}x^1 \wedge \mathrm{d}x^4 + E_2 \mathrm{d}x^2 \wedge \mathrm{d}x^4 + E_3 \mathrm{d}x^3 \wedge \mathrm{d}x^4\end{aligned} \tag{5.5}$$

(see [26]). We write $\omega \cong (B, E)$. Then, Gauss' law and Maxwell–Faraday law are written concisely via differential forms:

$$\nabla \cdot B = 0 \quad \text{and} \quad \partial_t B + \nabla \times E = 0 \qquad \Longleftrightarrow \qquad \mathrm{d}\omega = 0, \tag{5.6}$$

that is, ω is an exact two-form called *Maxwell two-form* or *electromagnetic two-form*.

Recall that in addition to (5.6), we also need E and B to satisfy $B \cdot E = 0$. We express the latter condition in the language of bivectors:

$$\begin{aligned} B \cdot E = 0 \qquad &\Longleftrightarrow \qquad \omega \wedge \omega = 2B \cdot E\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 = 0 \\ &\Longleftrightarrow \qquad \omega = v \wedge w \quad \text{for some } v, w \in \mathbb{R}^4 \end{aligned}$$

(where the last equality showing that ω is simple will be proved in the forthcoming Proposition 5.3). Here and in the sequel, we identify a vector $v \in \mathbb{R}^4$ and a 1-form $\sum_{i=1}^4 v_i \mathrm{d}x^i$.

Our nonlinear constraint simplifies to

$$\mathscr{M} = \{(u, S, \omega) : \omega \wedge \omega = 0\} \tag{5.7}$$

and the wave cone conditions for (B, E), (5.2) and (5.4), are reduced to

$$\omega \wedge \xi = 0. \tag{5.8}$$

If such a ξ is found, in view of Lemma 5.2 it can be modified to verify $\xi_x \cdot u = 0$ as well. Thus it only remains to verify the condition involving S, that is, (5.3).

It turns out that the interaction of (5.7) and (5.8) is very neat with the forms formalism. This is the content of the next section.

5.4. Λ-Segments in Terms of Simple Bivectors

The following well-known proposition collects characterisations equivalent to the condition $B \cdot E = 0$ (The Plücker identity for the bivector ω):

Proposition 5.3. *Let $\omega \cong (B, E) \in \mathbb{R}^3 \times \mathbb{R}^3$. The following conditions are equivalent:*

(i) *ω is* degenerate*, that is, $\omega \wedge \omega = 0$.*
(ii) *ω is* simple*, that is, $\omega = v \wedge w$ for some $v, w \in \mathbb{R}^4$, called the* factors of *ω.*
(iii) $B \cdot E = 0$

(iv) $\omega \wedge \xi = 0$ *for some* $\xi \in (\mathbb{R}^3\backslash\{0\}) \times \mathbb{R}$.

Proof. The equivalence of (i) and (iii) was already noted, and (ii) clearly implies (i). Suppose then (iii) holds; our aim is to prove (ii). If $E = 0$, choose any $v_x, w_x \in \mathbb{R}^3$ such that $v_x \times w_x = b$. Then $(v_x, 0) \wedge (w_x, 0) \cong (v_x \times w_x, 0) = (B, 0)$. If $E \neq 0$, then $(E, 0) \wedge (B \times E/|E|^2, 1) \cong (B, E)$, giving (ii).

The implication (iii) $\Rightarrow$ (iv) follows from Lemma 5.2, and the proof of Lemma 5.2 also gives (iv) $\Rightarrow$ (iii). Alternatively, (iv) $\Rightarrow$ (ii) follows from Proposition 5.4 below. □

Using Proposition 5.3, we formulate some useful further characterisations of (5.8).

Proposition 5.4. *Suppose* $\omega = v \wedge w \neq 0$ *and* $\xi \in (\mathbb{R}^3\backslash\{0\}) \times \mathbb{R}$. *The following conditions are equivalent:*

(i) $\omega \wedge \xi = 0$.
(ii) $\xi \in \operatorname{span}\{v, w\}$.
(iii) $\omega = \tilde{v} \wedge \xi$ *for some* $\tilde{v} \in \operatorname{span}\{v, w\}\backslash\{0\}$.

Proof. For (i) $\Rightarrow$ (ii) suppose $v \wedge w \wedge \xi = 0$. We may thus write $c_1 v + c_2 w + c_3 \xi = 0$, where $\{c_1, c_2, c_3\} \neq \{0\}$. If $c_3 = 0$, we get a contradiction with $v \wedge w \neq 0$, and therefore $\xi \in \operatorname{span}\{v, w\}$. For (ii) $\Rightarrow$ (iii) choose $\tilde{v} \in \operatorname{span}\{v, w\}\backslash\{0\}$ with $\tilde{v} \cdot \xi = 0$. After normalising $\tilde{v}$ we get $\tilde{v} \wedge \xi = v \wedge w$. The direction (iii) $\Rightarrow$ (i) is clear. □

Recall that every Λ-segment is contained in $\mathscr{M}$. We give equivalent characterisations for this condition.

Proposition 5.5. *Suppose that* ω_0 *and* $\omega \neq 0$ *are simple bivectors and that* $\omega \wedge \xi = 0$, *where* $\xi \in (\mathbb{R}^3\backslash\{0\}) \times \mathbb{R}$. *The following conditions are equivalent:*

(i) $\omega_0 + t\omega$ *is simple for all* $t \in \mathbb{R}$.
(ii) $\omega_0 \wedge \omega = 0$.
(iii) *We can write* $\omega = v \wedge \xi$ *and either* $\omega_0 = v_0 \wedge \xi$ *or* $\omega_0 = v \wedge w_0$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is clear since $\omega_0 \wedge \omega = \omega \wedge \omega_0$. The direction (iii) $\Rightarrow$ (ii) is also clear, and we complete the proof by showing that (ii) $\Rightarrow$ (iii). The case $\omega_0 = 0$ being clear, we assume that $\omega_0 \neq 0$.

Use Proposition 5.4 to write $\omega = \tilde{v} \wedge \xi$ for some $\tilde{v} \in \mathbb{R}^4\backslash\{0\}$. Also write $\omega_0 = \tilde{v}_0 \wedge \tilde{w}_0$. Since $\omega_0 \wedge \omega = 0$ and $\omega_0 \neq 0$ by assumption, we conclude that $\tilde{v}_0 = d_1 \tilde{w}_0 + d_2 \tilde{v} + d_3 \xi$ for some $d_1, d_2, d_3 \in \mathbb{R}$.

If $d_3 = 0$, we set $v = \tilde{v}$ and $w_0 = d_2 \tilde{w}_0$. Next, if $d_3 \neq 0$ and $d_2 \neq 0$, we choose $v = \tilde{v} + (d_3/d_2)\xi$ and $w_0 = d_2 \tilde{w}_0$. Finally, if $d_3 \neq 0$ and $d_2 = 0$, we select $v = \tilde{v}$ and $v_0 = -d_3 \tilde{w}_0$. □

5.5. Clebsch Variables

Now (5.6) means that ω is closed and thus, by Poincaré lemma, exact: $\omega = \mathrm{d}\alpha$. Here the so-called *electromagnetic four-potential* α is of course not unique. We specify a choice of α below. Recall from (5.7) that our potential α is required to satisfy

$$\mathrm{d}\alpha \wedge \mathrm{d}\alpha = 0.$$

This fact, among other things, motivates us to set $\alpha = \varphi\,\mathrm{d}\psi$ which leads to $\omega = \mathrm{d}\alpha = \mathrm{d}\varphi \wedge \mathrm{d}\psi$; here $\phi, \psi \in C^\infty(\mathbb{R}^4)$ are traditionally called *Clebsch variables* or *Euler potentials*.

Definition 5.6. We define $P_B, P_E : C^\infty(\mathbb{R}^4) \times C^\infty(\mathbb{R}^4) \to C^\infty(\mathbb{R}^4; \mathbb{R}^3)$ via

$$\mathrm{d}\varphi \wedge \mathrm{d}\psi \cong (\nabla\varphi \times \nabla\psi, \partial_t\psi\nabla\varphi - \partial_t\varphi\nabla\psi) =: (P_B[\varphi,\psi], P_E[\varphi,\psi]). \tag{5.9}$$

With the Clebsch variables at our disposal we make a natural Ansatz on the electromagnetic side of the localised plane waves. Fix $V_0 = (u_0, S_0, \omega_0) \in \mathscr{M}$, $V = (u, S, \omega) \in \Lambda$ with $\xi \in (\mathbb{R}^3\backslash\{0\}) \times \mathbb{R}$ being a solution to (3.1)–(3.3) and $\omega_0 \wedge \omega = 0$.

Use the simplicity of ω_0 to write $\omega_0 = v_0 \wedge w_0$, and recall the operator P_U given by Lemma 5.1, with U given in (5.1).

Fix a cube $Q \subset \mathbb{R}^4$ and a cutoff function $\chi \in C_c^\infty(Q)$. Given $h \in C^\infty(\mathbb{R})$ and $\ell \in \mathbb{N}$, our aim is to find ϕ_ℓ, φ_ℓ and ψ_ℓ such that

$$V_\ell := ((u_0, S_0) + P_U(\phi_\ell), \mathrm{d}\varphi_\ell \wedge d\psi_\ell) = V_0 + \chi(x,t)h''(\ell(x,t)\cdot\xi)V + O\left(\frac{1}{\ell}\right) \tag{5.10}$$

and $V_\ell \rightharpoonup V_0$ in $L^2(Q; \mathbb{R}^{15})$. The choice of ϕ_ℓ is specified in Lemma 5.1. For the electromagnetic part we define Clebsch variables of the form

$$\varphi_\ell(x,t) := v_0 \cdot (x,t) + \frac{c_1\chi(x,t)h'(\ell(x,t)\cdot\xi)}{\ell}, \tag{5.11}$$

$$\psi_\ell(x,t) := w_0 \cdot (x,t) + \frac{c_2\chi(x,t)h'((x,t)\cdot\xi)}{\ell}. \tag{5.12}$$

In (5.11)–(5.12), we use h' instead of h in order to be consistent with the scaling on the fluid part.

The Ansatz (5.11)–(5.12) yields

$$\mathrm{d}\varphi_\ell(x,t)\wedge\mathrm{d}\psi_\ell(x,t) = v_0\wedge w_0 + \chi(x,t)h''(\ell(x,t)\cdot\xi)(c_2v_0 - c_1w_0)\wedge\xi + O\left(\frac{1}{\ell}\right), \tag{5.13}$$

which is of the form (5.10) if

$$(c_2v_0 - c_1w_0) \wedge \xi = \omega. \tag{5.14}$$

This raises the question whether (5.14) can be solved for $c_1, c_2 \in \mathbb{R}$. Notice that if $\omega_0 \neq 0$, the answer is independent of the factors v_0, w_0 of $\omega_0 = v_0 \wedge w_0$.

It turns out that given general ω_0, ω with $\omega_0 \wedge \omega = 0$, such c_1, c_2 do not always exist. (The canonical bad case is $\omega_0 = v \wedge \xi$, $\omega = w \wedge \xi$, as then $(c_1v + c_2\xi) \wedge \xi = c_1v \wedge \xi = w \wedge \xi$ if and only if v is parallel to w). Essentially, when (5.14) holds, the segment defined by V_0 and V is good (the case $\omega_0 = 0$ yielding some additional cases).

Remark 5.7. In (5.13), we use crucially the cancellation properties of the wedge product $\mathrm{d}\varphi_\ell \wedge \mathrm{d}\psi_\ell$ to overcome the nonlinearity of P_B and P_E. In fact, $\mathrm{d}\varphi_\ell \wedge \mathrm{d}\psi_\ell$ arises, up to a term $O(1/\ell)$, as pullbacks of the bivector $v_0 \wedge w_0$. In other words,

$$(\mathrm{d}\varphi_\ell, \mathrm{d}\psi_\ell) = \Phi_\ell^*(\mathrm{d}\varphi, \mathrm{d}\psi),$$

where $\varphi(x,t) = (x,t)\cdot v_0$, $\psi(x,t) = (x,t)\cdot w_0$ and $\Phi_\ell(x,t) = x + \ell^{-1} h'(\ell x\cdot\xi)\zeta$ with $\zeta \cdot v_0 = c_1$ and $\zeta \cdot w_0 = c_2$. Note that the class of simple two-forms is closed under taking pull-backs with $\Phi \in C^\infty(\mathbb{R}^4; \mathbb{R}^4)$, as a consequence of the formula $\Phi^*(v \wedge w) = \Phi^* v \wedge \Phi^* w$.

5.6. States in Clebsch Variables

As a matter of fact, when we iterate the construction and apply convex integration we will be modifying $\mathrm{d}\varphi$ and $\mathrm{d}\psi$ instead of $\mathrm{d}\varphi \wedge \mathrm{d}\psi$. We will therefore use a separate notation in which we keep track of the factors forming a bivector:

$$W = (u, S, v, w) \in \mathbb{R}^4 \times \mathbb{R}^{3\times 3}_{\mathrm{sym}} \times \mathbb{R}^4 \times \mathbb{R}^4, \qquad V = p(W) := (u, S, v\wedge w) \in \mathscr{M}. \tag{5.15}$$

The case $\omega_0 = 0$ is special as we will be able to construct potentials only when we interpret $0 = 0 \wedge 0$.

5.7. Good and Bad Λ-Segments

To start, we consider simple two-forms $\omega_0 = v_0 \wedge w_0 \neq 0$ and $\omega = v \wedge w \neq 0$ with $\omega_0 \wedge \omega = 0$. Since ω is simple, there exists $\xi \in (\mathbb{R}^3\backslash\{0\}) \times \mathbb{R}$ such that $\omega \wedge \xi = 0$. We study separately the case where ω and ω_0 are parallel and the one in which they are not.

Proposition 5.8. *If $\omega = k\omega_0 \neq 0$ for some $k \in \mathbb{R}$, then* (5.14) *is satisfied for some $c_1, c_2 \in \mathbb{R}$.*

Proof. Since $\omega \wedge \xi = k v_0 \wedge w_0 \wedge \xi = 0$, we may write $d_1 v_0 + d_2 w_0 + d_3 \xi = 0$ for some $d_1, d_2, d_3 \in \mathbb{R}$, not all zero. Since $v_0 \wedge w_0 \neq 0$, we have $d_3 \neq 0$, which implies that $\{d_1, d_2\} \neq \{0\}$ (since $\xi \neq 0$). If $d_2 \neq 0$, set $c_1 = 0$ and $c_2 = -kd_3/d_2$: then $[c_2 v_0 - c_1 w_0] \wedge \xi = k v_0 \wedge w_0 = \omega$. The case $d_1 \neq 0$ is similar. □

Proposition 5.9. *Suppose $\omega_0 \neq 0$ and $\omega \neq 0$ satisfy $\omega_0 \wedge \omega = 0$ but ω is not a multiple of ω_0. The following conditions are equivalent.*

(i) *There exist $c_1, c_2 \in \mathbb{R}$ such that* (5.14) *holds.*
(ii) $\omega \wedge \xi = 0$ *but* $\omega_0 \wedge \xi \neq 0$.
(iii) *There exist $\tilde{v}, \tilde{w}_0 \in \mathbb{R}^4\backslash\{0\}$ such that $\omega_0 = \tilde{v} \wedge \tilde{w}_0$ and $\omega = \tilde{v} \wedge \xi$.*

Proof of (i) $\Longrightarrow$ (ii). Suppose (i) holds and fix c_1 and c_2. Then $\omega \wedge \xi = 0$. Seeking contradiction, assume $\omega_0 \wedge \xi = 0$. Then there exist constants $d_1, d_2, d_3 \in \mathbb{R}$, not all zero, such that $d_1 v_0 + d_2 w_0 + d_3 \xi = 0$. If $d_3 = 0$, then v_0 and w_0 are linearly dependent, which gives a contradiction with $\omega_0 = v_0 \wedge w_0 \neq 0$. On the other hand, if $d_3 \neq 0$, then $\xi \in \operatorname{span}\{v_0, w_0\}$ and thus $\omega = (c_2 v_0 - c_1 w_0) \wedge \xi$ is a multiple of $\omega_0 = v_0 \wedge w_0$, giving a contradiction. □

Proof of (ii) $\implies$ *(iii).* By Proposition 5.5, we can write $\omega = \tilde{v} \wedge \xi$ and either $\omega_0 = \tilde{v_0} \wedge \xi$ or $\omega_0 = \tilde{v} \wedge \tilde{w}_0$. The latter condition must then hold in view of (ii). □

Proof of (iii) $\implies$ *(i).* By assumption, $\omega_0 = v_0 \wedge w_0 = \tilde{v} \wedge \tilde{w}_0$. Thus $\tilde{v} \in \operatorname{span}\{v_0, w_0\}$. Writing $\tilde{v} = c_2 v_0 - c_1 w_0$ we obtain $[c_2 v_0 - c_1 w_0] \wedge \xi = \tilde{v} \wedge \xi = \omega$. □

Thus we are ready to define a class of Λ-segments for which there exist the desired compactly supported plane waves (which are constructed in Proposition 5.13). We then define the corresponding Λ_g-convexity notions needed in the sequel.

Definition 5.10. Suppose $V_0 := (u_0, S_0, \omega_0) \in \mathscr{M}$, $V := (u, S, \omega) \in \Lambda$ and $0 < \lambda < 1$. We say that

$$[V_0 - (1-\lambda)V, V_0 + \lambda V] \quad \text{is a } \textit{good } \Lambda\textit{-segment } (\Lambda_g\textit{-segment})$$

if there exists $\xi \in (\mathbb{R}^3 \backslash \{0\}) \times \mathbb{R}$ such that (3.1)–(3.3) and one of the conditions

$$\omega = 0, \tag{5.16}$$

$$\omega_0 \wedge \xi \neq 0, \tag{5.17}$$

$$\omega = k\omega_0 \neq 0, \quad k \in \mathbb{R} \backslash \{-1/\lambda, 1/(1-\lambda)\}, \tag{5.18}$$

$$u = S = \omega_0 = 0 \tag{5.19}$$

holds. Otherwise we say that $[V_0 - (1-\lambda)V, V_0 + \lambda V]$ is a *bad* Λ*-segment*.

The restriction on $k \in \mathbb{R}$ in (5.18) ensures that the endpoints $V_0 - (1-\lambda)V$ and $V_0 + \lambda V$ have non-vanishing ω-components; this is used in Propositions 5.13 and 7.4.

We define a lamination convex hull in terms of Λ_g-segments.

Definition 5.11. Let $Y \subset \mathscr{M}$. We define the sets Y^{k,Λ_g}, $k \in \mathbb{N}_0$, as follows:

(i) $Y^{0,\Lambda_g} := Y$.
(ii) If $k \geq 1$ and $V_0 \in \mathscr{M}$, the point V_0 belongs to Y^{k,Λ_g} if $V_0 \in Y^{k-1,\Lambda_g}$ or there exist $\lambda \in (0,1)$ and $V \in \mathscr{M}$ such that $[V_0 - (1-\lambda)V, V_0 + \lambda V] \subset \mathscr{M}$ is a good Λ-segment whose endpoints belong to Y^{k-1,Λ_g}.

Furthermore, we denote $Y^{lc,\Lambda_g} := \cup_{k \in \mathbb{N}_0} Y^{k,\Lambda_g}$.

We also give a related notion for finite-order laminates; recall Remark 3.4.

Definition 5.12. Suppose $\nu = \sum_{\mathbf{j} \in \{1,2\}^N} \mu_{\mathbf{j}} \delta_{V_{\mathbf{j}}}$ is a finite-order laminate supported in $Y \subset \mathscr{M}$. We say that ν is a *good finite-order laminate*, and denote $\nu \in \mathcal{L}_g(Y)$, if for all $\mathbf{j}' \in \{1,2\}^k$, $1 \leq k \leq N-1$, the Λ-segment $[V_{\mathbf{j}',1}, V_{\mathbf{j}',2}]$ is good.

5.8. Localised Plane Waves Along Λ_g Segments

To every Λ_g-segment there corresponds a potential, and thus we can localise the plane waves.

Proposition 5.13. *Let $W_0 = (u_0, S_0, v_0, w_0) \in \mathbb{R}^{17}$, and suppose $[V_0 - (1-\lambda)\bar{V}, V_0 + \lambda\bar{V}] \subset \mathscr{M}$ is a Λ_g-segment. If $\omega_0 = 0$, then suppose $v_0 = w_0 = 0$. Fix a cube $Q \subset \mathbb{R}^4$ and let $\varepsilon > 0$.*

There exist $W_\ell := W_0 + (\bar{u}_\ell, \bar{S}_\ell, \mathrm{d}\bar{\varphi}_\ell, \mathrm{d}\bar{\psi}_\ell) \in W_0 + C_c^\infty(Q; \mathbb{R}^{17})$ with the following properties.

(i) $\mathcal{L}(V_\ell) = 0$, *where* $V_\ell = p(W_\ell)$.
(ii) *For every $(x,t) \in Q$ there exists $\tilde{W} = \tilde{W}(x,t) \in \mathbb{R}^{17}$ such that*

$$\tilde{V} = p(\tilde{W}) \in [V_0 - (1-\lambda)\bar{V}, V_0 + \lambda\bar{V}],$$
$$|W_\ell(x,t) - \tilde{W}| < \varepsilon, \quad |V_\ell(x,t) - \tilde{V}| < \varepsilon.$$

(iii) *For every $\ell \in \mathbb{N}$ there exist pairwise disjoint open sets $A_1, A_2 \subset Q$ such that*

$$V_\ell(x,t) = V_0 + \lambda\bar{V} \text{ in } A_1 \text{ with } |A_1| > (1-\varepsilon)(1-\lambda)\,|Q|\,,$$
$$V_\ell(x,t) = V_0 - (1-\lambda)\bar{V} \text{ in } A_2 \text{ with } |A_2| > (1-\varepsilon)\lambda\,|Q|\,.$$

Furthermore, W_ℓ is locally constant in A_1 and A_2. For $j = 1, 2$, writing $W_\ell = (u_j, S_j, v_j \wedge w_j)$ in A_j, we have either $v_j = w_j = 0$ or $v_j \wedge w_j \neq 0$.
(iv) $V_\ell \rightharpoonup V$ *in* $L^2(Q; \mathbb{R}^{15})$.

For the proof we first specify the oscillating functions that we intend to use. Their first derivatives can be chosen to be mollifications of 1-periodic sawtooth functions.

Lemma 5.14. *Suppose $0 < \lambda < 1$ and $\varepsilon > 0$. Then there exists $h \in C^\infty(\mathbb{R})$ with the following properties:*

(i) *h'' is 1-periodic.*
(ii) $-(1-\lambda) \leq h'' \leq \lambda$.
(iii) $\int_0^1 h''(s)\,\mathrm{d}s = 0$. *(Thus, h' is 1-periodic.)*
(iv) $\left|\{s \in [0,1]\colon h''(s) = \lambda\}\right| \geq (1-\varepsilon)(1-\lambda)$.
(v) $\left|\{s \in [0,1]\colon h''(s) = -(1-\lambda)\}\right| \geq (1-\varepsilon)\lambda$.

Proof of Proposition 5.13, *the cases* (5.16)–(5.18). Suppose one of the conditions (5.16)–(5.18) holds. Define the perturbation $(\bar{u}_\ell, \bar{S}_\ell, \mathrm{d}\bar{\varphi}_\ell, \mathrm{d}\bar{\psi}_\ell)$ via Lemma 5.1 and (5.11)–(5.12). Claims (i) and (iv) are clear. In (ii) we choose $\tilde{W} = W_0 + \chi(x,t)h''((\ell(x,t)\cdot\xi)\bar{W}$.

In (iii) let $\varepsilon > 0$, fix a cube $\tilde{Q} \subset Q$ with $|\tilde{Q}| > (1-\varepsilon/3)\,|Q|$ and choose χ such that $\chi = 1$ in $\tilde{Q}$. Cover $\tilde{Q}$, up to a set of measure $\varepsilon\,|Q|\,/3$, by cubes $Q_1, \ldots, Q_N$ with one of the sides parallel to ξ. We wish to show that $\left|\{y \in Q_k\colon h''(\ell y\cdot\xi) = \lambda\}\right| \geq |Q_k|\,(1-\varepsilon/3)(1-\lambda)$ for every large enough $\ell \in \mathbb{N}$; in (iii) we may then choose $A_1 = \cup_{k=1}^N \{y \in Q_k\colon h''(\ell y\cdot\xi) = \lambda\}$. Similarly, $A_2 = \cup_{k=1}^N \{y \in Q_k\colon h''(\ell y\cdot\xi) = 1-\lambda\}$.

Choose an orthonormal basis $\{f_1, f_2, f_3, f_4\}$ of $\mathbb{R}^4$ such that $f_1 = \xi/|\xi|$ and

$$Q_k = \{y \in \mathbb{R}^4 \colon \zeta \cdot f_j \le y \cdot f_j \le \zeta \cdot f_j + l(Q_k)\}$$

for some $\zeta \in \mathbb{R}^4$. In order to switch to coordinates where Q_k has sides parallel to coordinate axes, define $L := \sum_{j=1}^4 e_j \otimes f_j \in \mathbb{R}^{4\times 4}$, so that $Lf_j = e_j$ for $j = 1, \dots, 4$ and therefore $L \in O(4)$. Then, denoting $z = Ly$,

$$\begin{aligned} Q_k &= \{y \in \mathbb{R}^4 \colon L\zeta \cdot e_j \le Ly \cdot e_j \le L\zeta \cdot e_j + l(Q_k)\} \\ &= L^{-1}\{z \in \mathbb{R}^4 \colon L\zeta \cdot e_j \le z \cdot e_j \le L\zeta \cdot e_j + l(Q_k)\} \\ &= L^{-1}\left(\prod_{j=1}^4 [(L\zeta)_j, (L\zeta)_j + l(P)]\right) = L^{-1}(LQ_k). \end{aligned}$$

Thus

$$\begin{aligned} \left|\{y \in Q_k \colon h''(\ell y \cdot \xi) = \lambda\}\right| &= \left|L^{-1}\{z \in LQ_k \colon h''(\ell\,|\xi|\,z_1) = \lambda\}\right| \\ &= l(Q_k)^3 |\{s \in [(L\zeta)_1, (L\zeta)_1 + l(Q_k)] \colon \\ &\quad h''(\ell\,|\xi|\,z_1) = \lambda)\}| \\ &\ge |Q_k|\,(1 - \varepsilon/3)(1 - \lambda) \end{aligned}$$

for all large $\ell \in \mathbb{N}$.

To finish the proof of (iii), write $W_\ell = (u_j, S_j, v_j \wedge w_j)$ in A_j, where $j \in \{1, 2\}$. In the case (5.16), if $\omega_0 = 0$, then $v_j = w_j = 0$ by assumption, and if $\omega_0 \neq 0$, then $\omega_0 - (1-\lambda)\bar{\omega} \neq 0$ and $\omega_0 + \lambda\bar{\omega} \neq 0$. Next, in the case (5.17), by (5.8) we have $(\omega_0 + t\bar{\omega}) \wedge \xi = \omega_0 \wedge \xi \neq 0$, hence in particular $\omega_0 - (1-\lambda)\bar{\omega} \neq 0$ and $\omega_0 + \lambda\bar{\omega} \neq 0$. Finally, the case (5.18) follows from the restriction $k \notin \{1/\lambda, 1/(1-\lambda)\}$. □

The case (5.19) requires a separate argument since in this case, (5.14) has no solutions $c_1, c_2 \in \mathbb{R}$. In fact, if $\lambda = 1/2$, we let $\mathrm{d}\bar{\varphi}_\ell$ and $\mathrm{d}\bar{\psi}_\ell$ oscillate in *different* directions, and thus W_ℓ is not a plane wave. However, $V_\ell = p(W_\ell)$ oscillates along the Λ_g-segment $[-V/2, V/2]$. The general case $\lambda \in (0, 1)$ the follows by combining with the case (5.18).

Proof of Proposition 5.13, *the case* (5.19). The case $\bar{\omega} = 0$ being obvious, assume $\bar{\omega} = \bar{v} \wedge \bar{w} \neq 0$. Suppose first $\lambda = 1/2$.

Without loss of generality, assume $\bar{v} \cdot \bar{w} = 0$. Let $\varepsilon > 0$ and choose $\tilde{Q} \subset Q$ and χ as above. Then

$$\bar{\varphi}_\ell(x,t) := \ell^{-1}\chi(x,t)h'((x,t)\cdot \ell\bar{v}), \qquad \bar{\psi}_\ell(x,t) := 2\ell^{-1}\chi(x,t)h'((x,t)\cdot \ell\bar{w})$$

have the sought properties for all large enough $\ell \in \mathbb{N}$.

Indeed, for (ii) choose $\tilde{W} = (0, 0, \chi(x,t)h''((x,t)\cdot \ell\bar{v})\bar{v}, 2\chi(x,t)h''((x,t)\cdot \ell\bar{w})\bar{w})$. For (iii), note that when $(x,t) \in \tilde{Q}$, we have

$$V_\ell(x,t) = \begin{cases} V_0 + 2^{-1}(0, 0, \bar{v} \wedge \bar{w}) & \text{when } h''((x,t)\cdot \ell\bar{v}) = h''((x,t)\cdot \ell\bar{w}) = \pm 2^{-1}, \\ V_0 - 2^{-1}(0, 0, \bar{v} \wedge \bar{w}) & \text{when } h''((x,t)\cdot \ell\bar{v}) = -h''((x,t)\cdot \ell\bar{w}) = \pm 2^{-1}. \end{cases} \tag{5.20}$$

Cover $\tilde{Q}$ up to a small set by cubes $Q_1, \ldots, Q_N$ with two sides parallel to $\bar{v}_x$ and $\bar{w}_x$; recall that $\bar{v}_x \cdot \bar{w}_x = 0$.

For $k = 1, \ldots, N$ we get $\{(x,t) \in Q_k : h''((x,t) \cdot \ell\bar{v}) = h''((x,t) \cdot \ell\bar{w}) = 2^{-1}\} \geq \left|\{s \in [0,1] : h''(s) = 2^{-1}\}\right|^2 |Q_k| - O(1/\ell) > |Q_k|(1 - \varepsilon/3)(1/2)^2$ as in the previous proof, and a similar inequality holds for the other three cases of (5.20). This completes the proof of the case $\lambda = 1/2$.

We then cover the case $\lambda \neq 1/2$. Let $0 < \delta < \min\{\lambda, 1-\lambda\}$. Using the case above, we choose $\mathrm{d}\bar{\varphi}_\ell, \mathrm{d}\bar{\psi}_\ell$ satisfying claims (i)–(iv) for $[V_0 - \delta\bar{V}, V_0 + \delta\bar{V}]$ and $\varepsilon/2$. Note that $\mathrm{d}\bar{\varphi}_\ell \wedge \mathrm{d}\bar{\psi}_\ell = \delta\bar{\omega} \neq 0$ in A_1 and $\mathrm{d}\bar{\varphi}_\ell \wedge \mathrm{d}\bar{\psi}_\ell = -\delta\bar{\omega} \neq 0$ in A_2. We then cover the sets A_1 and A_2 by cubes up to a small set and apply the case (5.18) in the cubes. (The last claim of (iii) is clear.) □

Remark 5.15. We have looked for solutions of (5.10) of the form (5.11)–(5.12), and in some special cases, a solution does not exist. It is conceivable that another Ansatz would satisfy (5.10) in some of the cases excluded by (5.11)–(5.12). This would essentially require a degenerate Darboux Theorem with a Dirichlet boundary condition—more concretely, solving $\mathrm{d}\varphi_\ell \wedge \mathrm{d}\psi_\ell = v_0 \wedge w_0 + h''(\ell(x,t) \cdot \xi) v \wedge w + O(1/\ell)$ with $(\mathrm{d}\varphi_\ell, \mathrm{d}\psi_\ell) = (v_0, w_0)$ on ∂Q. However, such theorems are remarkably difficult to prove and to the authors' knowledge, a suitable existence result is not available at this point; we refer to [19, Section 14] and the references contained therein.

6. Characterisations of the Relative Interior of the Lamination Convex Hull

Our next task is to find a suitable (relatively open) set $\mathcal{U}_{r,s} \subset \mathrm{int}_{\mathscr{M}}(K_{r,s}^{lc,\Lambda})$ in state space, which will serve the purpose of defining subsolutions—see Section 7.1 below. Since we have only constructed potentials for Λ_g-segments, we would like to produce $\mathcal{U}_{r,s}$ by using Λ_g-segments only. The choice of $\mathcal{U}_{r,s}$ is, however, non-trivial, as discussed in Section 6.1. Nevertheless, eventually the following simple definition turns out to suffice.

Definition 6.1. We denote

$$\mathcal{U}_{r,s} := \mathrm{int}_{\mathscr{M}}(K_{r,s}^{lc,\Lambda}).$$

In the main result of this chapter, Theorem 6.7, we give several characterisations of $\mathcal{U}_{r,s}$ and show, in particular, that $0 \in \mathcal{U}_{r,s}$.

6.1. A Rigidity Result on the Good Λ-Hull

Initially, it appears natural to choose some set $\mathcal{U}_{r,s} \subset K_{r,s}^{lc,\Lambda_g}$ for strict subsolutions. However, $K_{r,s}^{lc,\Lambda_g}$ turns out to be rather small; in fact,

$$E_0 = B_0 \times u_0 \qquad \text{for every } V_0 = (u_0, S_0, B_0, E_0) \in K^{lc,\Lambda_g}.$$

Proposition 6.2. *Suppose $[V_0 - (1-\lambda)V, V_0 + \lambda V] \subset \mathscr{M}$ is a Λ_g-segment, and assume that $V_1 := V_0 + \lambda V$ and $V_2 := V_0 - (1-\lambda)V$ satisfy $E_j = B_j \times u_j$. Then $E_0 = B_0 \times u_0$.*

The proof consists of two parts. First, the Λ_g-conditions and the assumption $E_j = B_j \times u_j$ lead to the conclusion $(B_1 - B_2) \times (u_1 - u_2) = 0$. Then a bit of algebra gives

$$\lambda B_1 \times u_1 + (1-\lambda) B_2 \times u_2 = (\lambda B_1 + (1-\lambda) B_2) \times (\lambda u_1 + (1-\lambda) u_2),$$

that is, $E_0 = B_0 \times u_0$.

At first sight, Proposition 6.2 seems to prevent convex integration unless potentials are found for bad Λ-segments. However, this rigidity disappears once one considers Λ_g-convex hulls of *relatively open* sets. Indeed, whenever $\mathcal{U}$ is bounded and relatively open in $\mathscr{M}$, we have $\mathcal{U}^{lc,\Lambda_g} = \mathcal{U}^{lc,\Lambda}$ (see Proposition 6.6). The basic reason behind this phenomenon is the fact that, loosely speaking, bad Λ-segments become good when translated to almost any direction.

6.2. Laminates of Relatively Open Sets in $\mathscr{M}$

We start the proof of Proposition 6.6 by showing that the class of relatively open sets in $\mathscr{M}$ is closed with respect to taking laminates:

Proposition 6.3. *Suppose $\mathcal{U}$ is relatively open in $\mathscr{M}$. Then U^{lc,Λ_g} is relatively open in $\mathscr{M}$.*

Before beginning the proof of Proposition 6.3 we describe the main difficulty. The proof proceeds by induction. Suppose $V_0 - (1-\lambda)V,\ V_0 + (1-\lambda)V \in \mathcal{U}^{k,\Lambda_g}$, $[V_0 - (1-\lambda)V,\ V_0 + \lambda V] \subset \mathscr{M}$ is a Λ_g-segment and $B_{\mathscr{M}}(V_0 + \lambda V, \delta) \cup B_{\mathscr{M}}(V_0 - (1-\lambda)V, \delta) \subset \mathcal{U}^{lc,\Lambda_g}$. Given $\tilde{V}_0 \in \mathscr{M}$ with $|V_0 - \tilde{V}_0|$ small our aim is to get $\tilde{V}_0 \in \mathcal{U}^{lc,\Lambda_g}$. It is tempting to write $\tilde{V}_0 = \lambda[\tilde{V}_0 - (1-\lambda)V] + (1-\lambda)[\tilde{V}_0 + \lambda V]$. It is, however, not guaranteed that the endpoints $\tilde{V}_0 + \lambda V$, $\tilde{V}_0 - (1-\lambda)V$ lie on the nonlinear manifold $\mathscr{M}$! Therefore, we need to perturb $\tilde{V}_0 + \lambda V$ and $\tilde{V}_0 - (1-\lambda)V$ in order to place an entire Λ_g-segment on $\mathscr{M}$. This is in stark contrast to equations of fluid dynamics where the lamination convex hull has non-empty interior. Again, the two-form formalism comes to the rescue.

We overcome the difficulties via the following lemma which allows us to choose the factors $v, w \in \mathbb{R}^4$ of a simple two-form $v \wedge w$ in a continuous way. Henceforth, we denote $\|\omega\| := \max_{|f|=|g|=1} \omega(f, g)$ for every $\omega \in \Lambda^2(\mathbb{R}^4)$.

Lemma 6.4. *Suppose $v_1, w_1, v_2, w_2 \in S^3$ with $v_1 \cdot w_1 = v_2 \cdot w_2 = 0$, and let $0 < \varepsilon < 1$. If $\|v_1 \wedge w_1 - v_2 \wedge w_2\| < \varepsilon$, then there exist orthogonal $\tilde{v}_2, \tilde{w}_2 \in S^3$ such that*

$$\tilde{v}_2 \wedge \tilde{w}_2 = v_2 \wedge w_2, \quad |v_1 - \tilde{v}_2| < \sqrt{2}\varepsilon \quad \textit{and} \quad |w_1 - \tilde{w}_2| < \sqrt{2}\varepsilon.$$

Proof. First, if $v_2 \cdot v_1 = w_2 \cdot v_1 = 0$, then $(v_1 \wedge w_1 - v_2 \wedge w_2)(v_2, w_2) = -1$, which yields a contradiction. Assume, therefore, that $v_2 \cdot v_1$ and $w_2 \cdot v_1$ are not both zero.

Denote by $\tilde{v}_2$ the normalised projection of v_1 onto $\text{span}\{v_2, w_2\}$ and by $\tilde{w}_2$ its rotation in $\text{span}\{v_2, w_2\}$, that is,

$$\tilde{v}_2 = \frac{(v_1 \cdot v_2)v_2 + (v_1 \cdot w_2)w_2}{|(v_1 \cdot v_2)v_2 + v_1 \cdot w_2)w_2|}, \qquad \tilde{w}_2 = \frac{-(v_1 \cdot w_2)v_2 + (v_1 \cdot v_2)w_2}{|(v_1 \cdot v_2)v_2 + (v_1 \cdot w_2)w_2|}.$$

Thus $\tilde{v}_2 \wedge \tilde{w}_2 = v_2 \wedge w_2$ and $\tilde{w}_2 \cdot v_1 = 0$. Now

$$
\begin{aligned}
&(v_1 \wedge w_1 - \tilde{v}_2 \wedge \tilde{w}_2)\left(\frac{\tilde{v}_2 - (\tilde{v}_2 \cdot v_1)v_1}{|\tilde{v}_2 - (\tilde{v}_2 \cdot v_1)v_1|}, \tilde{w}_2\right) \\
&= -\frac{1 - (\tilde{v}_2 \cdot v_1)^2}{\sqrt{1 - (\tilde{v}_2 \cdot v_1)^2}} = -\sqrt{1 - (\tilde{v}_2 \cdot v_1)^2}.
\end{aligned}
$$

Thus $\sqrt{1 - (\tilde{v}_2 \cdot v_1)^2} \leq \|v_1 \wedge w_1 - \tilde{v}_2 \wedge \tilde{w}_2\| < \varepsilon$. Since clearly $\tilde{v}_2 \cdot v_1 \geq 0$, we conclude that $\tilde{v}_2 \cdot v_1 > \sqrt{1 - \varepsilon^2}$. Hence, $|v_1 - \tilde{v}_2|^2 < 2 - 2\sqrt{1 - \varepsilon^2} < 2\varepsilon^2$.

We then show that $|w_1 - \tilde{w}_2| < \sqrt{2}\varepsilon$. First,

$$
(v_1 \wedge w_1 - \tilde{v}_2 \wedge \tilde{w}_2)\left(v_1, \frac{w_1 - (\tilde{w}_2 \cdot w_1)\tilde{w}_2}{|w_1 - (\tilde{w}_2 \cdot w_1)\tilde{w}_2|}\right) = \sqrt{1 - (\tilde{w}_2 \cdot w_1)^2}
$$

gives $\sqrt{1 - (\tilde{w}_2 \cdot w_1)^2} < \varepsilon$. Next,

$$
(v_1 \wedge w_1 - \tilde{v}_2 \wedge \tilde{w}_2)(v_1, w_1) = 1 - (v_1 \cdot \tilde{v}_2)(\tilde{w}_2 \cdot w_1) < \varepsilon
$$

implies that $\tilde{w}_2 \cdot w_1 > 0$. As above, we conclude that $|w_1 - \tilde{w}_2|^2 < 2\varepsilon^2$. $\square$

We also need a lemma which gives a solution of a matrix equation with a natural norm estimate.

Lemma 6.5. *If* $x \in \mathbb{R}^3 \setminus \{0\}$ *and* $y \in \mathbb{R}^3$, *then*

$$
S := \frac{x \otimes y + y \otimes x - (x \cdot y)I}{|x|^2} \in \mathbb{R}^{3\times 3}_{\text{sym}}
$$

satisfies $Sx = y$ *and* $|S| \leq 3\,|y|\,/\,|x|$.

Proof of Proposition 6.3. We need to show for every $k \in \mathbb{N}_0$ that if $V_0 \in \mathcal{U}^{k,\Lambda_g}$, then there exists $\delta > 0$ such that $B_{\mathscr{M}}(V_0, \delta) \subset \mathcal{U}^{lc,\Lambda_g}$. The claim holds for $k = 0$ by assumption, so assume, by induction, it holds for k.

Let $V_0 \in \mathcal{U}^{k+1,\Lambda_g}$. Write $V_0 = \lambda(V_0 - (1-\lambda)V) + (1-\lambda)(V_0 + \lambda V)$, where $[V_0 - (1-\lambda)V, V_0 + \lambda V]$ is a Λ_g-segment. By assumption, there exists $\delta > 0$ such that

$$
B_{\mathscr{M}}(V_0 - (1-\lambda)V, \delta) \cup B_{\mathscr{M}}(V_0 + \lambda V, \delta) \subset \mathcal{U}^{lc,\Lambda_g}.
$$

We intend to show that whenever $\tilde{\delta} = \tilde{\delta}_{V_0,V,\lambda} > 0$ is small enough, $B_{\mathscr{M}}(V_0, \tilde{\delta}) \subset \mathcal{U}^{lc,\Lambda_g}$. The case (5.16) is clear.

Suppose first (5.17) holds, that is, $\omega \wedge \xi = 0$ but $\omega_0 \wedge \xi \neq 0$. By Proposition 5.9 and scaling, we may write $V_0 = (u_0, S_0, \|\omega_0\|\, v \wedge w_0)$ and $V = (u, S, v \wedge \xi)$, where $|v| = |w_0| = 1$ and $v \cdot w_0 = 0$.

Let now $\tilde{V}_0 = (\tilde{u}_0, \tilde{S}_0, \tilde{\omega}_0) \in \mathscr{M}$ and $\|\tilde{V}_0 - V_0\| < \tilde{\delta}$. By Lemma 6.4, we may write $\tilde{\omega}_0/\|\tilde{\omega}_0\| = \tilde{v} \wedge \tilde{w}_0$, where $|\tilde{v}| = |\tilde{w}_0| = 1$, $\tilde{v} \cdot \tilde{w}_0 = 0$ and $|\tilde{v} - v| + |\tilde{w}_0 - w_0| \lesssim_{V_0} \tilde{\delta}$. In the last estimate we used the inequality $\|\omega_0/\,\|\omega_0\| - \tilde{\omega}_0/\|\tilde{\omega}_0\|\| \leq 2\|\omega_0 - \tilde{\omega}_0\|/\,\|\omega_0\|$.

Now choose $\tilde{V} = (u, S, \tilde{v} \wedge \xi) \in \Lambda$. As long as $\tilde{\delta} > 0$ is small enough, it is ensured that $\tilde{v} \wedge \tilde{w}_0 \wedge \xi \neq 0$, so that $[\tilde{V}_0 - (1-\lambda)\tilde{V}, \tilde{V}_0 + \lambda\tilde{V}]$ satisfies (5.17). Thus $\tilde{V}_0 = \lambda(\tilde{V}_0 - (1-\lambda)\tilde{V}) + (1-\lambda)(\tilde{V}_0 + \lambda\tilde{V}) \in \mathcal{U}^{lc,\Lambda_g}$, as claimed.

Suppose next (5.18) holds, so that $\omega = k\omega_0 \neq 0$ with $k \notin \{-1-\lambda, 1/(1-\lambda)\}$. Write $\omega_0 = \|\omega_0\|\, v_0 \wedge \xi \neq 0$, where $|v_0| = |\xi| = 1$ and $v_0 \cdot \xi = 0$. Again, let $\tilde{V}_0 = (\tilde{u}_0, \tilde{S}_0, \tilde{\omega}_0) \in \mathscr{M}$ and $\|\tilde{V}_0 - V_0\| < \tilde{\delta}$. This time, we may write $\tilde{\omega}_0 = \|\tilde{\omega}_0\| \tilde{v}_0 \wedge \tilde{\xi}$, where $|\tilde{v}_0| = |\tilde{\xi}| = 1$, $\tilde{v}_0 \cdot \tilde{\xi} = 0$ and $|\tilde{v}_0 - v_0| + |\tilde{\xi} - \xi| \lesssim_{V_0} \tilde{\delta}$.

Our aim is to choose $\tilde{u} \approx u$ and $\tilde{S} \approx S$ such that $\tilde{V} = (\tilde{u}, \tilde{S}, k\tilde{\omega}_0)$ satisfies $\tilde{V}\tilde{\xi} = 0$. We select

$$\tilde{u} := u - \frac{u \cdot \tilde{\xi}_x}{|\tilde{\xi}_x|^2} \tilde{\xi}_x$$

so that $\tilde{u} \cdot \tilde{\xi}_x = 0$ and $|\tilde{u} - u| \lesssim_{V_0,V,\lambda} \tilde{\delta}$ as soon as, say, $\delta < |\xi_x|/2$. We then use Lemma 6.5 to choose $\tilde{S} \in \mathbb{R}^{3\times 3}_{\text{sym}}$ satisfying

$$\tilde{S}\tilde{\xi}_x + \tilde{\xi}_t\tilde{u} = (\tilde{S} - S)\tilde{\xi}_x + S(\tilde{\xi}_x - \xi_x) + \tilde{\xi}_t(\tilde{u} - u) + (\tilde{\xi}_t - \xi_t)u = 0$$

with $|\tilde{S} - S| \lesssim_{V_0,V,\lambda} \tilde{\delta}$. Now $[\tilde{V}_0 - (1-\lambda)\tilde{V}, \tilde{V}_0 + \lambda\tilde{V}]$ satisfies (5.18) and the endpoints belong to $\mathcal{U}^{lc,\Lambda_g}$. We conclude that $\tilde{V}_0 \in \mathcal{U}^{lc,\Lambda_g}$.

Last suppose $u = S = \omega_0 = 0 \neq \omega$. Let $\tilde{V}_0 = (\tilde{u}_0, \tilde{S}_0, \tilde{v}_0 \wedge \tilde{w}_0) \in \mathscr{M}$ with $|\tilde{V}_0 - V_0| < \tilde{\delta}$. Suppose first $\tilde{v}_0 \wedge \tilde{w}_0 = 0$. Then $\tilde{V}_0 \in [\tilde{V}_0 - (1-\lambda)V, \tilde{V}_0 + \lambda V]$, the Λ-segment satisfies (5.19) and the endpoints belong to $\mathcal{U}^{lc,\Lambda_g}$, so that $\tilde{V}_0 \in \mathcal{U}^{lc,\Lambda_g}$.

Suppose then $\tilde{v}_0 \wedge \tilde{w}_0 \neq 0$. We write $V = (0, 0, \xi \wedge w)$ and choose

$$\tilde{V} = (0, 0, \tilde{\xi} \wedge (\tilde{w} + \tilde{w}_0)) \in \Lambda,$$

where $\tilde{\xi}_x \neq 0$, $|\tilde{\xi} - \xi| + |\tilde{w} - w| < \tilde{\delta}$ and furthermore $\tilde{v}_0 \wedge \tilde{w}_0 \wedge \tilde{\xi} \neq 0$ and $\tilde{v}_0 \wedge \tilde{w}_0 \wedge \tilde{w} \neq 0$. Thus

$$\tilde{V}_0 + (0, 0, \tilde{v}_0 \wedge \tilde{w}) + \lambda\tilde{V} = (\tilde{u}_0, \tilde{S}_0, (\tilde{v}_0 + \lambda\xi) \wedge (\tilde{w}_0 + \tilde{w})) \in \mathcal{U}^{lc,\Lambda_g}$$

and $\tilde{V}_0 + (0, 0, \tilde{v}_0 \wedge \tilde{w}) - (1-\lambda)\tilde{V} \in \mathcal{U}^{lc,\Lambda_g}$. Now $\tilde{V}_0 + (0, 0, \tilde{v}_0 \wedge \tilde{w}) \in \mathcal{U}^{lc,\Lambda_g}$; the Λ-segment is good because $\tilde{V}\tilde{\xi} = 0$ but $\tilde{v}_0 \wedge \tilde{w}_0 \wedge \tilde{\xi} \neq 0$.

An entirely similar argument gives $\tilde{V}_0 - (0, 0, \tilde{v}_0 \wedge \tilde{w}) \in \mathcal{U}^{lc,\Lambda_g}$. Now $[\tilde{V}_0 - (0, 0, \tilde{v}_0 \wedge \tilde{w}), \tilde{V}_0 + (0, 0, \tilde{v}_0 \wedge \tilde{w})]$ is a Λ_g-segment because we assumed that $\tilde{v}_0 \wedge \tilde{w}_0 \wedge \tilde{w} \neq 0$. Thus $\tilde{V}_0 \in \mathcal{U}^{lc,\Lambda_g}$, as claimed. □

6.3. Equivalence of Hulls of Relatively Open Sets

Proposition 6.6. *Suppose $\mathcal{U}$ is bounded and relatively open in $\mathscr{M}$. Then $\mathcal{U}^{lc,\Lambda} = \mathcal{U}^{lc,\Lambda_g}$.*

Proof. The direction $\mathcal{U}^{lc,\Lambda_g} \subset \mathcal{U}^{lc,\Lambda}$ is obvious. We prove the converse direction by induction, first assuming that $(u, S, 0) \notin \mathcal{U}^{lc,\Lambda}$ for each u and S. Clearly $\mathcal{U} \subset \mathcal{U}^{lc,\Lambda_g}$. Assume, therefore, that $\mathcal{U}^{k,\Lambda} \subset \mathcal{U}^{lc,\Lambda_g}$; our aim is to show that $\mathcal{U}^{k+1,\Lambda} \subset \mathcal{U}^{lc,\Lambda_g}$.

Suppose $[V_0-(1-\lambda)V, V_0+\lambda V] \subset \mathscr{M}$ is a bad Λ-segment and the endpoints $V_0-(1-\lambda)V, V_0+\lambda V \in \mathcal{U}^{k,\Lambda} \subset \mathcal{U}^{lc,\Lambda_g}$. Assume first that $\omega_0 \neq 0$ and that ω_0 and ω are not parallel. Thus $\omega_0-(1-\lambda)\omega, \omega_0+\lambda\omega \neq 0$. Now $\omega_0 = \xi \wedge w_0$ and $\omega = \xi \wedge w$ by Propositions 5.4 and 5.9. Choose $\tilde{\omega} := \varepsilon w_0 \wedge w$ and $W = (0, 0, \tilde{\omega})$, where $\varepsilon \neq 0$ is small. Now, since $\omega_0 \wedge \tilde{\omega} = \omega \wedge \tilde{\omega} = 0$, we have $V_0+W+\lambda V, V_0+W-(1-\lambda)V \in \mathscr{M}$. Proposition 6.3 then gives $V_0+W+\lambda V, V_0+W-(1-\lambda)V \in \mathcal{U}^{lc,\Lambda_g}$. Furthermore, $[V_0+W-(1-\lambda)V, V_0+W+\lambda V]$ is a Λ_g-segment because $\omega \wedge \xi = 0$ but $(\omega_0+\varepsilon w_0 \wedge w) \wedge \xi \neq 0$. Therefore $V_0+W \in \mathcal{U}^{lc,\Lambda_g}$. Similarly, $V_0-W \in \mathcal{U}^{lc,\Lambda_g}$. Finally, $[V_0-W, V_0+W]$ is a Λ_g-segment because $\tilde{\omega} \wedge w = 0$ and yet $\omega_0 \wedge w \neq 0$. Consequently, $V_0 \in \mathcal{U}^{lc,\Lambda_g}$.

Assume next that $\omega_0 = 0$. Since $[V_0-(1-\lambda)V, V_0+\lambda V] \subset \mathscr{M}$ is a bad Λ-segment, we have $\omega = \xi \wedge w \neq 0$. We may assume that $w_x \neq 0$ (by possibly adding a constant multiple of $\xi_x \neq 0$ to w_x). This time select a basis $\{\xi, w, f, g\}$ of $\mathbb{R}^4$ with $f_x \neq 0$ and $w_x \cdot f_x = 0$. Select $W = (0, 0, \varepsilon\, w \wedge f)$ with $\varepsilon \neq 0$ small. Arguing as in the previous paragraph, $V_0 \pm W + \lambda V, V_0 \pm W-(1-\lambda)V \in \mathcal{U}^{lc,\Lambda_g}$. As above, $V_0 \pm W \in \mathcal{U}^{lc,\Lambda_g}$ since $(\omega_0 \pm w \wedge f) \wedge \xi \neq 0$. Now $[V_0-W, V_0+W]$ (with $\lambda = 1/2$) satisfies (5.19); thus $V_0 \in \mathcal{U}^{lc,\Lambda_g}$.

Finally assume $\omega_0 \neq 0$ and $\omega = k\omega_0$ for $k \in \{-1/\lambda, 1/(1-\lambda)\}$. We may thus write $\omega_0 = v_0 \wedge \xi$. Choose $W = (0, 0, v_0 \wedge w)$, where $v_0 \wedge w \wedge \xi \neq 0$; thus, after scaling w, $V_0 \pm W \in \mathcal{U}^{lc,\Lambda_g}$. Indeed, $V_0+\lambda V \pm W \in \mathcal{U}^{lc,\Lambda_g}$ and $V_0-\lambda V \pm W \in \mathcal{U}^{lc,\Lambda_g}$ by Proposition 6.3. The Λ-segment $[V_0+\lambda V \pm W, V_0-(1-\lambda)V \pm W]$ is good since $\omega \wedge \xi = 0$ but $(\omega_0 \pm v_0 \wedge w) \wedge \xi \neq 0$. Now the Λ-segment $[V_0-W, V_0+W]$ is good since $v_0 \wedge w \wedge w = 0$ but $v_0 \wedge \xi \wedge w \neq 0$. Thus, again, $V_0 \in \mathcal{U}^{lc,\Lambda_g}$. □

6.4. Formulation of the Characterisations

Proposition 6.6 allows us to use the whole wave cone Λ in computations on hulls of relatively open sets. In order to exploit this, in Theorem 6.7 we characterise $\mathcal{U}_{r,s} := \operatorname{int}_{\mathscr{M}}(K_{r,s}^{lc,\Lambda})$ via different relatively open sets. Our main aim is twofold: first, $\mathcal{U}_{r,s} = \cup_{\tau \in [0,1)} (B_{\mathscr{M}}(K_{\tau r,\tau s}, \varepsilon_\tau))^{lc,\Lambda_g}$ whenever the constants $\varepsilon_\tau > 0$ are small enough, and secondly, $0 \in \mathcal{U}_{r,s}$. We prove the first one via the (a priori) easier equality $\mathcal{U}_{r,s} = \cup_{\tau \in [0,1)} (B_{\mathscr{M}}(K_{\tau r,\tau s}, \varepsilon_\tau))^{lc,\Lambda}$ and Proposition 6.6.

In order to prove both of our two aims in a unified manner, we introduce some further terminology. For every $u, B \in \mathbb{R}^3$ we denote

$$S_{u,B} := u \otimes u - B \otimes B \in \mathbb{R}^{3\times 3}_{sym},$$

and for every $c > 0$ we define relatively open sets

$$\begin{aligned}\mathcal{V}_{r,s,c} := \{(u, S, B, E) \colon\ & |u+B| < r+c, |u-B| < s+c, \left|S - S_{u,B} - \Pi I\right| < c,\\ & |\Pi| < rs+c, |E - B \times u| < c, B \cdot E = 0\}.\end{aligned}$$

Note that given $c > 0$ we have $0 \in \mathcal{V}_{r,s,c}$ and $B_{\mathscr{M}}(K_{r,s}, \tilde{c}) \subset \mathcal{V}_{r,s,c}$ for every small enough $\tilde{c} > 0$.

Theorem 6.7. *There exist constants $\varepsilon_\tau = \varepsilon_{\tau,r,s} > 0$ such that for any $\tau_0 \in (0, 1)$,*

$$\mathcal{U}_{r,s} = \bigcup_{\tau_0<\tau<1} \mathcal{V}^{lc,\Lambda_g}_{\tau r,\tau s,\varepsilon_\tau} = \bigcup_{\tau_0<\tau<1} (B_{\mathscr{M}}(K_{\tau r,\tau s}, \varepsilon_\tau))^{lc,\Lambda_g} = \bigcup_{\tau_0<\tau<1} K^{lc,\Lambda}_{\tau r,\tau s}.$$

We divide the proof of Theorem 6.7 into two propositions.

Proposition 6.8. *For every* $\tau \in [0, 1)$ *there exists* $\varepsilon_\tau > 0$ *such that* $\mathcal{U}_{r,s} \supset \mathcal{V}_{\tau r,\tau s,\varepsilon_\tau}$.

Proposition 6.9. $\mathcal{U}_{r,s} \subset \cup_{\tau_0<\tau<1} K^{lc,\Lambda}_{\tau r,\tau s}$ *for every* $\tau_0 \in (0, 1)$.

Propositions 6.8–6.9 are proved in the rest of this chapter. Assuming Propositions 6.8–6.9, Theorem 6.7 is obtained as follows:

Proof of Theorem 6.7. Whenever $0 < \tau_0 < 1$ and the constants $\varepsilon_\tau > 0$ are small enough, Propositions 6.6 and 6.8 give $K^{lc,\Lambda}_{r,s} \supset \cup_{\tau_0<\tau<1} \mathcal{V}^{lc,\Lambda}_{\tau r,\tau s,\varepsilon_\tau} = \cup_{\tau_0<\tau<1} \mathcal{V}^{lc,\Lambda_g}_{\tau r,\tau s,\varepsilon_\tau}$. Together with Proposition 6.3, which says that $\cup_{\tau_0<\tau<1} \mathcal{V}^{lc,\Lambda_g}_{\tau r,\tau s,\varepsilon_\tau}$ is relatively open in $\mathscr{M}$, this yields that $\mathcal{U}_{r,s} \supset \cup_{\tau_0<\tau<1} \mathcal{V}^{lc,\Lambda_g}_{\tau r,\tau s,\varepsilon_\tau}$ by the definition of $\mathcal{U}_{r,s}$.

Next, the inclusion $\cup_{\tau_0<\tau<1} \mathcal{V}^{lc,\Lambda_g}_{\tau r,\tau s,\varepsilon_\tau} \supset \cup_{\tau_0<\tau<1} K^{lc,\Lambda}_{\tau r,\tau s}$ follows directly from the fact that $\mathcal{V}_{\tau r,\tau s,\varepsilon_\tau} \supset K_{\tau r,\tau s}$ and Proposition 6.6. Proposition 6.9 then says that $\mathcal{U}_{r,s} \subset \cup_{\tau_0<\tau<1} K^{lc,\Lambda}_{\tau r,\tau s}$.

Given parameters $\varepsilon_\tau > 0$ we choose $\tilde{\varepsilon}_\tau > 0$ such that $\mathcal{V}_{\tau r,\tau s,\varepsilon_\tau} \supset B_{\mathscr{M}}(K_{\tau r,\tau s}, \tilde{\varepsilon}_\tau)$, and then $\mathcal{U}_{r,s} \supset \cup_{\tau_0<\tau<1} B_{\mathscr{M}}(K_{\tau r,\tau s}, \tilde{\varepsilon}_\tau)^{lc,\Lambda_g} \supset \cup_{\tau_0<\tau<1} K^{lc,\Lambda}_{\tau r,\tau s} \supset \mathcal{U}_{r,s}$. Theorem 6.7 holds for these adjusted parameters $\tilde{\varepsilon}_\tau > 0$. □

6.5. Elsässer Variables in Relaxed MHD

In some of the computations on relaxed MHD it will be convenient to replace the variables (u, S, B, E) by Elsässer variables and a matrix component, (z^+, z^-, M), which satisfy

$$z^\pm = u \pm B, \qquad u = \frac{z^+ + z^-}{2}, \qquad B = \frac{z^+ - z^-}{2},$$

$$M = S + A, \qquad M^T = S - A, \qquad S = \frac{M + M^T}{2}, \qquad A = \frac{M - M^T}{2}.$$

The main advantage is that the constraint set obtains the particularly simple form

$$K_{r,s} = \{(z^+, z^-, z^+ \otimes z^- + \Pi I) \colon |z^+| = r,\ |z^-| = s,\ |\Pi| \le rs\}.$$

The wave cone conditions (3.1)–(3.3) are written in Elsässer formalism as

$$\xi_x \cdot z^\pm = 0, \qquad M\xi_x + \xi_t z^+ = 0, \qquad M^T \xi_x + \xi_t z^- = 0. \tag{6.1}$$

6.6. The Proof of Proposition 6.8

Proposition 6.8 gives our first estimation on the hull $K^{lc,\Lambda}_{r,s}$. Below, we further divide the proof of Proposition 6.8 into five steps.

Let $0 \le \tau < 1$. Below, steps (i)–(v) are expressed under the assumption that $V \in \mathcal{V}_{\tau r,\tau s,\varepsilon_\tau}$, that is, $|u + B| < \tau r + \varepsilon_\tau$, $|u - B| < \tau s + \varepsilon_\tau$, $|e \otimes e| < \varepsilon_\tau$, $|S| < \varepsilon_\tau$, $|B \times v| < \varepsilon_\tau$, $|E| < \varepsilon_\tau$ and $|\Pi| < \tau^2 rs + \varepsilon_\tau$. The constant $\varepsilon_\tau > 0$ varies from step to step.

(i) $V = (u, S_{u,B} + \Pi I, B, B \times u) \in K_{r,s}^{lc,\Lambda}$.
(ii) $V = (u, S_{u,B} + e \otimes e + \Pi I, B, B \times u) \in K_{r,s}^{lc,\Lambda}$.
(iii) $V = (u, S_{u,B} + S + \Pi I, B, B \times u) \in K_{r,s}^{lc,\Lambda}$.
(iv) $V = (u, S_{u,B} + S + \Pi I, B, B \times u + B \times v) \in K_{r,s}^{lc,\Lambda}$.
(v) $V = (u, S_{u,0} + S + \Pi I, 0, E) \in K_{r,s}^{lc,\Lambda}$.

Steps (i)–(v) are restated in Lemmas 6.10–6.14.

In the first step we relax the constraints $|z^+| = r$ and $|z^-| = s$ to $|z^+| \le r$ and $|z^-| \le s$. The proof is most conveniently presented in Elsässer variables which facilates the search for Λ combinations. For later use the statement is expressed in terms of the sets $\mathcal{V}_{\tau r, \tau s, \varepsilon_\tau}$.

Lemma 6.10. (Relaxation of the normalisation) $(u, S_{u,B} + \Pi I, B, B \times u) \in K_{r,s}^{lc,\Lambda}$ *whenever* $|z^+| < \tau r + \varepsilon_\tau^{(1)}$, $|z^-| < \tau s + \varepsilon_\tau^{(1)}$ *and* $|\Pi| < rs$, *where* $\varepsilon_\tau^{(1)} = \min\{r - \tau r, s - \tau s\}$.

Proof. Suppose first $z^+, z^- \neq 0$. In terms of Elsässer variables,

$$(z^+, z^-, z^+ \otimes z^- + \Pi I) = \lambda \left(\frac{r}{|z^+|} z^+, z^-, \frac{r}{|z^+|} z^+ \otimes z^- + \Pi I \right) + (1 - \lambda) \left(-\frac{r}{|z^+|} z^+, z^-, -\frac{r}{|z^+|} z^+ \otimes z^- + \Pi I \right)$$

for $2\lambda - 1 = |z^+|/r \in (0, 1)$; here the Λ-direction is $(2r z^+/|z^+|, 0, 2r z^+/|z^+| \otimes z^-)$, so that (6.1) are satisfied with any $\xi_x \in \{z^+, z^-\}^\perp \backslash \{0\}$ and $\xi_t = 0$. Furthermore,

$$\begin{aligned} &\left(\pm \frac{r}{|z^+|} z^+, z^-, \pm \frac{r}{|z^+|} z^+ \otimes z^- + \Pi I \right) \\ &= \tilde{\lambda} \left(\pm \frac{r}{|z^+|} z^+, \frac{s}{|z^-|} z^-, \pm \frac{r}{|z^+|} z^+ \otimes \frac{s}{|z^-|} z^- + \Pi I \right) \\ &\quad + (1 - \tilde{\lambda}) \left(\pm \frac{r}{|z^+|} z^+, -\frac{s}{|z^-|} z^-, \mp \frac{r}{|z^+|} z^+ \otimes \frac{s}{|z^-|} z^- + \Pi I \right) \in K_{r,s}^{1,\Lambda} \end{aligned}$$

for $2\tilde{\lambda} - 1 = |z^-|/s \in (0, 1)$; to show that the corresponding directions belong to Λ we can again take $\xi_x \in \{z^+, z^-\}^\perp \backslash \{0\}$ and $\xi_t = 0$. Thus we have shown that $(z^+, z^-, z^+ \otimes z^- + \Pi I) \in K_{r,s}^{2,\Lambda}$.

Suppose next $z^+ = 0$ and $z^- \neq 0$. Now $(0, z^-, \pi I) = 2^{-1}(z^-, z^-, z^- \otimes z^- + \Pi I) + 2^{-1}(-z^-, z^-, -z^- \otimes z^- + \Pi I) \in K_{r,s}^{3,\Lambda}$, where we may choose ξ with $\xi_t = 0$ and $\xi_x \in \{z^-\}^\perp \backslash \{0\}$. The remaining cases with $z^- = 0$ are similar. □

Steps (ii)–(iii) are covered in the next two lemmas. This time we get rid of the constraint $S = S_{u,b} + \Pi I$. It is easier to deal first with a symmetric rank-one matrix and then iterate.

Lemma 6.11. (Adding a symmetric rank-one matrix) *There exists* $\varepsilon_\tau^{(2)} > 0$*, depending on* $\varepsilon_\tau^{(1)} > 0$ *such that* $(u, S_{u,B} + e \otimes e + \Pi I, B, B \times u) \in K_{r,s}^{lc,\Lambda}$ *whenever* $|z^+| < \tau r + \varepsilon_\tau^{(2)}$, $|z^-| < \tau s + \varepsilon_\tau^{(2)}$, $|e| < \varepsilon_\tau^{(2)}$ *and* $|\Pi| < \tau^2 rs + \varepsilon_\tau^{(2)}$.

Proof. We use the formula $S_{u,B} + e \otimes e = (S_{u+e,B} + S_{u-e,B})/2$ to write

$$
\begin{aligned}
&(u, S_{u,B} + e \otimes e + \Pi I, B, B \times u) \\
&= \frac{1}{2}(u + e, S_{u+e,B} + \Pi I, B, B \times (u + e)) \\
&\quad + \frac{1}{2}(u - e, S_{u-e,B} + \Pi I, B, B \times (u - e)) \\
&=: \frac{1}{2}(V_1 + V_2).
\end{aligned}
$$

Here Lemma 6.10 gives $V_1, V_2 \in K_{r,s}^{lc,\Lambda}$ as long as $\varepsilon_\tau^{(2)} \le \varepsilon_\tau^{(1)}/2$. (We do not track such dependence of $\varepsilon_\tau^{(k)}$ on $\varepsilon_\tau^{(k-1)}$ explicitly in the forthcoming proofs.) The Λ-direction is

$$V_1 - V_2 = (2e, 2(u \otimes e + e \otimes u), 0, 2B \times e).$$

If $B \times e \neq 0$, we choose $\xi_x = B \times e$ and $\xi_t = -u \cdot B \times e$; if $B \times e = 0$, we choose any $\xi_x \in \{u, e\}^\perp \setminus \{0\}$ and $\xi_t = 0$. □

We then take further Λ-convex combinations to replace $e \otimes e$ by more general symmetric matrices.

Lemma 6.12. (Relaxation of the fluid side) *There exists* $\varepsilon_\tau^{(3)} > 0$*, depending on* $\varepsilon_\tau^{(2)} > 0$ *such that* $(u, S_{u,B} + S + \Pi I, B, B \times u) \in K_{r,s}^{lc,\Lambda}$ *whenever* $|z^+| < \tau r + \varepsilon_\tau^{(3)}$, $|z^-| < \tau s + \varepsilon_\tau^{(3)}$, $|S| < \varepsilon_\tau^{(3)}$ *and* $|\Pi| < \tau^2 rs + \varepsilon_\tau^{(3)}$.

Proof. First we cover the case where $S = -e \otimes e$. Choose an orthogonal basis $\{e, f, g\}$ of $\mathbb{R}^3$, where $|e| = |f| = |g|$. Write $I = |e|^{-2}(e \otimes e + f \otimes f + g \otimes g)$ which, in combination with Lemma 6.11, yields

$$
\begin{aligned}
(u, S_{u,B} - e \otimes e + \Pi I, B, B \times u) &= (u, S_{u,B} + f \otimes f + g \otimes g + (\Pi - |e|^2)I, B, B \times u) \\
&= \frac{1}{2}(u, S_{u,B} + 2f \otimes f + (\Pi - |e|^2)I, B, B \times u) \\
&\quad + \frac{1}{2}(u, S_{u,B} + 2g \otimes g + (\Pi - |e|^2)I, B, B \times u) \\
&\in K_{r,s}^{lc,\Lambda};
\end{aligned}
$$

the Λ-direction is $\bar{V} = (0, 2f \otimes f - 2g \otimes g, 0, 0)$ and we may choose $(\xi_x, \xi_t) = (e, 0)$.

By noting that $(0, e \otimes e \pm f \otimes f, 0, 0) \in \Lambda$ for every $e, f \in \mathbb{R}^3$ and iterating, we obtain the case

$$S = \sum_{i=1}^{N} c_i f_i \otimes f_i \tag{6.2}$$

for any unit vectors $f_i \in \mathbb{S}^2$ and $c_i \in \mathbb{R}$ with $\sum_{i=1}^N |c_i| < \varepsilon_\tau$. The proof is finished by noting that every $S \in \mathbb{R}^{3\times 3}_{\text{sym}}$ with $|S| < \varepsilon_\tau$ can be written in the form (6.2). Indeed, whenever f and g are unit vectors, we may write $f \otimes g + g \otimes f = 2^{-1}(f+g)\otimes(f+g) - 2^{-1}(f-g)\otimes(f-g)$. □

We have now covered the case where V differs from an element of K by the perturbation S of the symmetric matrix part. Our next aim, in the following two lemmas, is to allow $E \neq B \times u$ in $V = (u, S, B, E)$. Recall that $K_{r,s}^{lc,\Lambda} \subset \mathscr{M}$. Thus, if $B \neq 0$, $E = B \times f$, $f \in \mathbb{R}^3$ is a necessary condition. We will see next that it is also sufficient with the appropriate size normalisations. We will make use of the formula

$$S_{u,B} = \frac{1}{2}(S_{u+\tilde{u},B+\tilde{B}} - S_{\tilde{u},\tilde{B}}) + \frac{1}{2}(S_{u-\tilde{u},B-\tilde{B}} - S_{\tilde{u},\tilde{B}}) \qquad (u, \tilde{u}, B, \tilde{B} \in \mathbb{R}^3). \quad (6.3)$$

Finally, recall that in view of Proposition 6.2, we are forced to use bad Λ-segments.

Lemma 6.13. (Relaxation of the magnetic side) *There exists* $\varepsilon_\tau^{(4)} > 0$*, depending on* $\varepsilon_\tau^{(3)} > 0$ *such that* $(u, S_{u,B} + S + \Pi I, B, B \times u + B \times v) \in K_{r,s}^{lc,\Lambda}$ *whenever* $|z^+| < \tau r + \varepsilon_\tau^{(4)}$, $|z^-| < \tau s + \varepsilon_\tau^{(4)}$, $|S| < \varepsilon_\tau^{(4)}$, $|B \times v| < \varepsilon_\tau^{(4)}$ *and* $|\Pi| < \tau^2 rs + \varepsilon_\tau^{(4)}$.

Proof. We may assume that $B \times v \neq 0$ and $B \cdot v = 0$. Then $|B \times v| = |B|\,|v|$. The difficulty is that if B is very small, v can be very large.

We denote $c := (|B| \,/\, |v|)^{1/2}$ so that

$$|cv| = |c^{-1}B| = |B \times v|^{1/2} < |\varepsilon_\tau^{(4)}|^{1/2}. \quad (6.4)$$

We then use (6.3) to show that $(u, B, S_{u,B} + S + \Pi I, B \times u + B \times v)$ is the middle point of a suitable Λ segment. Indeed,

$$\begin{aligned}
&(u, B, S_{u,B} + S + \Pi I, B \times (u+v)) \\
&\quad = \frac{1}{2}(u + cv, S_{u+cv,(1+c^{-1})B} - S_{cv,c^{-1}B} + S + \Pi I, B + c^{-1}B, (1+c^{-1})B \times (u+cv)) \\
&\qquad + \frac{1}{2}(u - cv, S_{u-cv,(1-c^{-1})B} - S_{cv,c^{-1}B} + S + \Pi I, B - c^{-1}B, (1-c^{-1})B \times (u-cv)).
\end{aligned}$$

(6.4) we can apply Lemma 6.12 to deduce that the endpoints lie in $K_{r,s}^{lc,\Lambda}$. The direction of the segment is

$$\bar{V} = \left(2cv, 2\left(u \otimes cv + cv \otimes u - \frac{2}{c}B \otimes B\right), 2c^{-1}B, 2\left(B \times cv + c^{-1}B \times u\right)\right),$$

which belongs to Λ since (3.5) is satisfied. □

The case $B = 0$ needs to be dealt with separately, since lying in $\mathscr{M}$ does no longer constrain E. The following lemma proves step (v) and completes the proof of Proposition 6.8:

Lemma 6.14. (The case $B = 0$) *There exists* $\varepsilon_\tau^{(5)} > 0$, *depending on* $\varepsilon_\tau^{(4)}$ *such that* $(u, 0, S_{u,0} + S + \Pi I, E) \in K_{r,s}^{lc,\Lambda}$ *whenever* $|u| < \tau r + \varepsilon_\tau^{(5)}$, $|u| < \tau s + \varepsilon_\tau^{(5)}$, $|S| < \varepsilon_\tau^{(5)}$, $|E| < \varepsilon_\tau^{(5)}$ *and* $|\Pi| < \tau^2 rs + \varepsilon_\tau^{(5)}$.

Proof. We choose orthogonal e, f such that $E = e \times f$, $|e| = |f| = |E|^{1/2} < (\varepsilon_\tau^{(5)})^{1/2}$. Using (6.3), we write

$$
\begin{aligned}
&(u, S_{u,0} + S + \Pi I, 0, e \times f) \\
&\quad = \frac{1}{2}(u + e \times f, S_{u+e\times f,e} - S_{e\times f,e} + S + \Pi I, e, e \times (u + e \times f + f) \\
&\qquad + \frac{1}{2}(u - e \times f, S_{u-e\times f,-e} - S_{e\times f,e} + S + \Pi I, -e, -e \times (u - e \times f + (2e \times f - f)).
\end{aligned}
$$

By Lemma 6.13 the endpoints belong to $K_{r,s}^{lc,\Lambda}$. Now $\bar{V} = (2e \times f, 2(u \otimes e \times f + e \times f \otimes u), 2e, 2e \times (u + e \times f)) \in \Lambda$ since (3.5) is satisfied. □

6.7. *The Proof of Proposition 6.9*

Recall that Proposition 6.9 states the inclusion $\mathcal{U}_{r,s} \subset \cup_{\tau_0<\tau<1} K_{\tau r,\tau s}^{lc,\Lambda}$ and completes the proof of Theorem 6.7.

Proof of Proposition 6.9. Let $V \in \mathrm{int}_{\mathcal{M}}(K_{r,s}^{lc,\Lambda})$ and $0 < \tau_0 < 1$. By relative openness of $\mathrm{int}_{\mathcal{M}}(K_{r,s}^{lc,\Lambda})$, we may choose μ such that $\tau_0 < \sqrt{\mu} < 1$ and $V/\mu \in K_{r,s}^{lc,\Lambda}$. Now $V \in (\mu K_{r,s})^{lc,\Lambda}$ since the conditions $\bar{W} \in \Lambda$ and $\mu\bar{W} \in \Lambda$ are equivalent for all $\bar{W} \in \mathbb{R}^{15}$. It thus suffices to show that $\mu K_{r,s} \subset K_{\sqrt{\mu}r,\sqrt{\mu}s}^{lc,\Lambda}$.

We use Elsässer variables. When $(\mu z^+, \mu z^-, \mu z^+ \otimes z^- + \mu\Pi I) \in \mu K_{r,s}$, we note that $\sqrt{\mu} \in (\mu, 1)$ and write

$$
\begin{aligned}
(\mu z^+, \mu z^-, \mu z^+ \otimes z^- + \Pi I) &= \lambda(\sqrt{\mu} z^+, \sqrt{\mu} z^-, \mu z^+ \otimes z^- + \Pi I) \\
&\quad + (1 - \lambda)(-\sqrt{\mu} z^+, -\sqrt{\mu} z^-, \mu z^+ \otimes z^- + \Pi I) \\
&\in K_{\sqrt{\mu}r,\sqrt{\mu}s}^{1,\Lambda}
\end{aligned}
$$

for $2\lambda - 1 = \sqrt{\mu} \in (0, 1)$; here $\bar{V} = (2\sqrt{\mu} z^+, 2\sqrt{\mu} z^-, 0) \in \Lambda$. Hence $\mu K_{r,s} \subset K_{\sqrt{\mu}r,\sqrt{\mu}s}^{1,\Lambda}$. □

7. The Proof of Theorem 1.1

This chapter is dedicated to proving Theorem 1.1. In Section 7.1 we define the set of subsolutions that we use in the proof, and the main steps of the proof are listed in Section 7.2. The proof itself is carried out in the rest of the chapter.

7.1. Restricted Subsolutions

We intend to prove Theorem 1.1 by using subsolutions that take values in $\mathcal{U}_{r,s}$ and whose B and E components arise via P_B and P_E. For this, recall the notations

$$W = (u, S, \mathrm{d}\varphi, \mathrm{d}\psi), \qquad V = p(W) = (u, S, \mathrm{d}\varphi \wedge \mathrm{d}\psi).$$

Fix a non-empty bounded domain $\Omega \subset \mathbb{R}^3 \times \mathbb{R}$, and let $r, s > 0$, $r \neq s$.

Definition 7.1. The set of *restricted subsolutions* is defined as

$$X_0 := \{V = (u, S, \omega) \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^{15}) : \text{there exists } \varphi, \psi \in C_c^\infty(\mathbb{R}^4) \text{ such that}$$
$$\omega = \mathrm{d}\varphi \wedge \mathrm{d}\psi,\ \mathcal{L}(V) = 0,\ \operatorname{supp}(u, S, \varphi, \psi) \subset \Omega \text{ and } V(x,t) \in \mathcal{U}_{r,s} \forall (x,t) \in \mathbb{R}^4\}.$$

We define X to be the weak sequential closure of X_0 in $L^2(\mathbb{R}^4; \overline{\mathrm{co}}(K_{r,s}))$, where $\overline{\mathrm{co}}(K_{r,s})$ denotes the closed convex hull of the constitutive set $K_{r,s}$ defined in Definition 3.7. Observe that $K_{r,s} \subset \mathbb{R}^{15}$ is compact so that the weak topology of L^2 on X is metrizable.

Now $X \ni \{0\}$ is a compact metrisable space, and we denote a metric by d_X.

7.2. The Main Steps of the Proof

Following [21], our main aim is to prove Proposition 7.2 below. Once Proposition 7.2 is proved, Theorem 1.1 follows rather easily in Section 7.5.

Proposition 7.2. *There exists* $C = C_{r,s} > 0$ *with the following property. If* $V = (u, S, \mathrm{d}\varphi \wedge \mathrm{d}\psi) \in X_0$*, then there exist* $V_\ell = (u_\ell, S_\ell, \mathrm{d}\varphi_\ell \wedge \mathrm{d}\psi_\ell) \in X_0$ *such that* $d_X(V_\ell, V_0) \to 0$ *and*

$$\int_\Omega (|u_\ell(x,t)|^2 + |B_\ell(x,t)|^2 - |u(x,t)|^2 - |B(x,t)|^2)\,\mathrm{d}x\,\mathrm{d}t$$
$$\geq C \int_\Omega \left(\frac{r^2 + s^2}{2} - |u(x,t)|^2 - |B(x,t)|^2 \right) \mathrm{d}x\,\mathrm{d}t.$$

For the proof of Proposition 7.2 we need a so-called *perturbation property*, formulated in our setting in Proposition 7.4. To motivate the formulation of Proposition 7.4, we note that Theorem 6.7 implies the following proposition where we choose any $\varepsilon_\tau = \varepsilon_{\tau,r,s}$ such that

$$\mathscr{O}_\tau := B_{\mathscr{M}}(K_{\tau r, \tau s}, \varepsilon_\tau) \subset \mathcal{U}_{r,s};$$

recall from Theorem 6.7 that for any $\tau_0 \in (0,1)$ we have

$$\mathcal{U}_{r,s} = \bigcup_{\tau_0 < \tau < 1} \mathscr{O}_\tau^{lc, \Lambda_g}.$$

Proposition 7.3. *Let $V_0 \in \mathcal{U}_{r,s}$. Then for every large enough $\tau \in (0,1)$ there exists*

$$\nu = \sum_{\mathbf{j}\in\{1,2\}^N} \mu_{\mathbf{j}} \delta_{V_{\mathbf{j}}} \in \mathcal{L}_g(\mathscr{O}_\tau)$$

with barycentre $\bar{\nu} = V_0$ and $[V_{\mathbf{j}',1}, V_{\mathbf{j}',2}] \subset \mathcal{U}_{r,s}$ for all $\mathbf{j}' \in \{1,2\}^k$, $1 \le k \le N-1$. Furthermore, for each $V_{\mathbf{j}} = (u_{\mathbf{j}}, S_{\mathbf{j}}, v_{\mathbf{j}} \wedge w_{\mathbf{j}})$, $\mathbf{j} \in \{1,2\}^N$, we have

$$\frac{r^2+s^2}{2} - |u_0|^2 - |B_0|^2 \le 2(|u_{\mathbf{j}}|^2 + |B_{\mathbf{j}}|^2 - |u_0|^2 - |B_0|^2). \tag{7.1}$$

Indeed, if $V = (u, S, B, E) \in K_{r,s}$, then $|u|^2 + |B|^2 = (r^2+s^2)/2$ whereas, since $\operatorname{supp}(\nu) \subset \mathscr{O}_\tau$, $|u_{\mathbf{j}}|^2 + |B_{\mathbf{j}}|^2 \geqq \tau^2(r^2+s^2)/2 - \varepsilon_\tau$. Therefore, (7.1) follows by choosing $\tau \in (0,1)$ large enough.

Whereas Proposition 5.13 says, roughly speaking, that every good Λ-segment can be approximated by oscillating mappings with certain properties, Proposition 7.4 makes an analogous claim about good laminates.

Proposition 7.4. *Let $Q \subset \mathbb{R}^4$ be a cube, and let $V_0 = p(W_0) \in \mathcal{U}_{r,s}$. If $\omega_0 = v_0 \wedge w_0 = 0$, then assume that $v_0 = w_0 = 0$. Choose $\nu \in \mathcal{L}_g(\mathscr{O}_\tau)$ with $\bar{\nu} = V_0$ via Proposition 7.3.*

For every $\varepsilon > 0$ there exist

$$W_\ell := W_0 + (\bar{u}_\ell, \bar{S}_\ell, \mathrm{d}\bar{\varphi}_\ell, d\bar{\psi}_\ell) \in W_0 + C_c^\infty(Q; \mathbb{R}^{17})$$

with the following properties:

(i) *$\mathcal{L}(V_\ell) = 0$ and $V_\ell(x,t) \in \mathcal{U}_{r,s}$ for all $(x,t) \in \Omega$.*

(ii) *There exist pairwise disjoint open subsets $A_{\mathbf{j}} \subset Q$ with $\big||A_{\mathbf{j}}| - \mu_{\mathbf{j}}\big| < \varepsilon$ such that*

$$V_\ell(x,t) = V_{\mathbf{j}} \text{ for all } \mathbf{j} \in \{1,2\}^N \text{ and } (x,t) \in A_{\mathbf{j}}$$

and $\mathrm{d}\bar{\varphi}_\ell$ and $\mathrm{d}\bar{\psi}_\ell$ are locally constant in $A_{\mathbf{j}}$.

(iii) *For every $(x,t) \in Q$ there exist $\mathbf{j}' \in \{1,2\}^k$ and $\tilde{W} = \tilde{W}(x,t) \in \mathbb{R}^{17}$ such that*

$$p(\tilde{W}) \in [V_{\mathbf{j}',1}, V_{\mathbf{j}',2}], \qquad |W_\ell(x,t) - \tilde{W}| < \varepsilon, \qquad |V_\ell(x,t) - p(\tilde{W})| < \varepsilon.$$

(iv) *$V_\ell - V_0 \rightharpoonup 0$ in $L^2(Q; \mathbb{R}^{15})$.*

Condition (iii) says, in particular, that at every $(x,t) \in \Omega$, $V_\ell(x,t)$ is close to one of the Λ_g-segments $[V_{\mathbf{j}',1}, V_{\mathbf{j}',2}]$, where $\mathbf{j}' \in \{1,2\}^k$, $1 \le k \le N-1$. We will also need the estimate on $W_\ell(x,t)$. Proposition 7.4 is proved by a standard induction via Proposition 5.13; we sketch the main ideas.

Proof. The proof follows by iteratively modifying the sequence at the sets where it is locally affine (via Proposition 5.13) and using a diagonal argument. Namely, if $N = 1$, the result follows from Proposition 5.13. Suppose now that $1 \le k \le N-1$ and that we have costructed a sequence of mappings $W_{\ell_k} = (u_{\ell_k}, S_{\ell_k}, \mathrm{d}\varphi_{\ell_k}, \mathrm{d}\psi_{\ell_k})$ which satisfies (i), (iii) and (iv) and furthermore (ii) holds with the condition $\mathbf{j} \in \{1,2\}^N$ replaced by $\mathbf{j} \in \{1,2\}^k$.

Fix $\mathbf{j} \in \{1,2\}^k$. We cover $A_{\mathbf{j}}$ by disjoint cubes up to a set of small measure and modify W_{ℓ_k} in each cube via Proposition 5.13. This gives rise to a new sequence which we again modify at the sets $A_{\mathbf{j}}$, $\mathbf{j} \in \{1,2\}^{k+1}$, where it is locally affine. Note that we can use Proposition 5.13 iteratively because in each $A_{\mathbf{j}}$, claim (iii) of Proposition 5.13 implies that either $\mathrm{d}\varphi_{k_\ell} \wedge \mathrm{d}\psi_{k_\ell} \neq 0$ or $\mathrm{d}\varphi_{k_\ell} = \mathrm{d}\psi_{k_\ell} = 0$. Finally a standard diagonal argument provides the norm bounds. □

7.3. Modifications at the Set Where $\mathrm{d}\varphi(x,t) \wedge \mathrm{d}\psi(x,t) = 0$

The following issue needs to be addressed in the proof of Proposition 7.2: on one hand, the mapping $V = (u, S, \mathrm{d}\varphi \wedge \mathrm{d}\psi) \in X_0$ can have a large set where $\mathrm{d}\varphi(x,t) \wedge \mathrm{d}\psi(x,t) = 0$ but $(\mathrm{d}\varphi(x,t), \mathrm{d}\psi(x,t)) \neq 0$, and on the other hand, in Proposition 5.13, in the case $\omega_0 = v_0 \wedge w_0 = 0$, we only constructed potentials when $v_0 = w_0 = 0$. We therefore modify W around points $(x,t) \in \Omega$ where $\mathrm{d}\varphi(x,t) \wedge \mathrm{d}\psi(x,t) = 0$ but $(\mathrm{d}\varphi(x,t), \mathrm{d}\psi(x,t)) \neq 0$ making W look essentially constant there.

Lemma 7.5. *Suppose $V \in X_0$, and let $\varepsilon > 0$. Then there exists $\tilde{V} = (\tilde{u}, \tilde{S}, \mathrm{d}\tilde{\varphi} \wedge \mathrm{d}\tilde{\psi}) \in X_0$ such that $\|V - \tilde{V}\|_{L^\infty} < \varepsilon$ and*

$$|\{(x,t) \in \Omega\colon \mathrm{d}\tilde{\varphi}(x,t) \wedge \mathrm{d}\tilde{\psi}(x,t) \neq 0\} \cup \mathrm{int}(\{(x,t) \in \Omega\colon \mathrm{d}\tilde{\varphi}(x,t) = \mathrm{d}\tilde{\psi}(x,t) = 0\})| > (1-\varepsilon)\,|\Omega|\,.$$

Proof. Assume, without loss of generality, that $0 < \varepsilon < \min_\Omega \mathrm{dist}(V, \partial\mathcal{U}_{r,s})$. Then the inequality $\|V - \tilde{V}\|_{L^\infty} < \varepsilon$ ensures that $\tilde{V}$ takes values in $\mathcal{U}_{r,s}$.

Since W is absolutely continuous, we cover Ω by all the cubes $Q_i \subset \Omega$ with centers (x_i, t_i) and the following properties:

- If $\mathrm{d}\varphi(x_i,t_i) \wedge \mathrm{d}\psi(x_i,t_i) \neq 0$, then $\mathrm{d}\varphi \wedge \mathrm{d}\psi \neq 0$ in Q_i.
- If $\mathrm{d}\varphi(x_i,t_i) \wedge \mathrm{d}\psi(x_i,t_i) = 0$, then we have $\sup_{(x,t)\in Q_i} |W(x,t) - W(x_i,t_i)| < \varepsilon^2/[C(\|W\|_{L^\infty} + 1)]$.

Such cubes exist for every $(x_i, t_i) \in \Omega$, and therefore they form a Vitali cover of Ω. By the Vitali Covering Theorem, we may choose a finite, pairwise disjoint subcollection $\{Q_1, \ldots, Q_N\}$ with $|\Omega \backslash \cup_{i=1}^N Q_i| < \varepsilon/2$.

We intend to modify V in each Q_i where $\mathrm{d}\varphi(x_i,t_i) \wedge \mathrm{d}\psi(x_i,t_i) = 0$. Fix such Q_i, and let $R \subset Q_i$ be a subcube with center (x_i, t_i) and $|R| = (1 - \varepsilon/2)\,|Q_i|$. Choose $\delta > 0$ such that $(1-\delta)^4 = 1 - \varepsilon/2$; now $l(r) = (1-\delta)l(Q_i)$. Choose a smooth cutoff function χ_R with $\chi_R|_R = 1$ and $|\nabla\chi_R| \leq C/[\delta l(Q_i)]$. Define $g \in C^\infty(Q; Q)$ by

$$g(x,t) := (x,t) + \chi_R(x,t)[(x_i,t_i) - (x,t)]$$

so that $g(x,t) = (x_i, t_i)$ is constant in R and $g = \mathrm{id}$ near ∂Q_i. Set $\tilde{\varphi} := \varphi \circ g$ and $\tilde{\psi} := \psi \circ g$ so that

$$\nabla_{x,t}\,\tilde{\varphi} = D^T_{x,t}\, g \nabla_{x,t}\, \varphi \circ g, \qquad \nabla_{x,t}\,\tilde{\psi} = D^T_{x,t}\, g \nabla_{x,t}\, \psi \circ g. \tag{7.2}$$

Thus $|\{(x,t) \in Q_i : \mathrm{d}\tilde{\varphi}(x,t) = \mathrm{d}\tilde{\psi}(x,t) = 0\}| \geq (1 - \varepsilon/2)\,|Q_i|$.

The claim will be proved once we show that $\|\mathrm{d}\tilde{\varphi} \wedge \mathrm{d}\tilde{\psi}\| < \varepsilon/2$ in Q_i; then $\|V - \tilde{V}\|_{L^\infty} < \varepsilon$. To this end, we fix $(x,t) \in Q_i$ and estimate

$$|Dg(x,t)| = |(1 - \chi_R(x,t))I + [(x_i, t_i) - (x,t)] \otimes \nabla \chi_R(x,t)| \leq \frac{C}{\delta}$$

and

$$\begin{aligned}
|\mathrm{d}\varphi(g(x,t)) \wedge \mathrm{d}\psi(g(x,t))| &\leq |\mathrm{d}\varphi(g(x,t)) \wedge (\mathrm{d}\psi(g(x,t)) - \mathrm{d}\psi(g(x_i,t_i))| \\
&\quad + |(\mathrm{d}\varphi(x,t)) - \mathrm{d}\varphi(x_i,t_i)) \wedge \mathrm{d}\psi(x_i,t_i)| \\
&\leq C' \,\|W\|_{L^\infty}\, |W(x,t) - W(x_i,t_i)| < \varepsilon^2/C.
\end{aligned}$$

Now $\delta = 1-(1-\varepsilon/2)^{1/4}$ implies that $\varepsilon/2 = 1-(1-\varepsilon/2) = \delta(1+(1-\varepsilon/2)^{1/4} + (1-\varepsilon/2)^{2/4} + (1-\varepsilon/2)^{3/4}) < 4\delta$. Thus, whenever $|v_1| = |v_2| = 1$, we have

$$\begin{aligned}
&|[\mathrm{d}\tilde{\varphi}(x,t) \wedge \mathrm{d}\tilde{\psi}(x,t)](v_1, v_2)| \\
&\quad = |[\mathrm{d}\varphi(g(x,t)) \wedge \mathrm{d}\psi(g(x,t))](D^T g(x,t)v_1, D^T g(x,t)v_2)| < \frac{\varepsilon}{2}.
\end{aligned}$$

□

By Lemma 7.5 and a standard diagonal argument, it suffices to prove Proposition 7.2 for every $\varepsilon > 0$ and every mapping $V \in X_0$ such that

$$\begin{aligned}
\tilde{\Omega} &:= \{(x,t) \in \Omega\colon \mathrm{d}\varphi(x,t) \wedge \mathrm{d}\psi(x,t) \neq 0\} \\
&\cup \mathrm{int}(\{(x,t) \in \Omega\colon \mathrm{d}\varphi(x,t) = \mathrm{d}\psi(x,t) = 0\})
\end{aligned}$$

satisfies

$$|\tilde{\Omega}| > (1 - \varepsilon)\,|\Omega|\,, \tag{7.3}$$

$$\begin{aligned}
&\int_\Omega \left(\frac{r^2 + s^2}{2} - |u(x,t)|^2 - |B(x,t)|^2\right) \mathrm{d}x\,\mathrm{d}t \\
&\quad \leq 2 \int_{\tilde{\Omega}} \left(\frac{r^2 + s^2}{2} - |u(x,t)|^2 - |B(x,t)|^2\right) \mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{7.4}$$

7.4. Proof of Proposition 7.2

Assuming that (7.3)–(7.4) hold, we wish to construct the mappings $V_\ell = p(W_\ell)$ of Proposition 7.2 by suitably modifying a discretisation argument from [21]. Given $V = p(W) \in X_0$ we cover $\tilde{\Omega}$, up to a small set, by cubes $Q_i \subset \tilde{\Omega}$ with center (x_i, t_i) such that W varies very little in Q_i. We then approximate W by $W(x_i, t_i)$ in each Q_i. Now $V(x_i, t_i) = \bar{\nu}$ for some $\nu = \sum_{\mathbf{j}\in\{1,2\}^N} \mu_{\mathbf{j}} \delta_{V_{\mathbf{j}}} \in \mathcal{L}_g(\mathscr{O}_\tau)$ with τ close to 1, and we set $W_\ell := [W - W(x_i, t_i)] + [W(x_i, t_i) + (\bar{u}_\ell, \bar{S}_\ell, \mathrm{d}\bar{\varphi}_\ell, \mathrm{d}\bar{\psi}_\ell)]$, where $(\bar{u}_\ell, \bar{S}_\ell, \mathrm{d}\bar{\varphi}_\ell, \mathrm{d}\bar{\psi}_\ell)$ is given by Proposition 7.4.

On one hand, the discretisation needs to be fine enough that $V_\ell = p(W_\ell)$ does not take values outside $\mathcal{U}_{r,s}$, and on the other hand, the cubes need to cover a substantial proportion of $\tilde{\Omega}$. Both properties are ensured by the following application of the Vitali Covering Theorem.

Lemma 7.6. *Suppose* $\varepsilon > 0$ *and* $V = p(W) \in X_0$ *satisfies* (7.3)–(7.4). *Let* $\gamma > 0$. *Then there exist pairwise disjoint cubes* $Q_i \subset \tilde{\Omega}$ *with centers* (x_i, t_i) *and parameters* $\delta_i > 0$ *with the following properties:*

(i) *For every* $i \in \mathbb{N}$, *there exists* $\tau_i \in (0, 1)$ *and* $\nu = \sum_{\mathbf{j}\in\{1,2\}^N} \mu_{\mathbf{j}} \delta_{V_{\mathbf{j}}} \in \mathcal{L}_g(\mathscr{O}_{\tau_i})$ *with barycentre* $\bar{\nu} = V(x_i, t_i)$, *where* ν *given by Proposition 7.3.*
(ii) $B_{\mathscr{M}}([V_{\mathbf{j}',1}, V_{\mathbf{j}',2}]), \delta_i) \subset \mathcal{U}_{r,s}$ *for all* $i \in \mathbb{N}$ *and* $\mathbf{j}' \in \{1, 2\}^k$, $1 \le k \le N - 1$.
(iii) $\sup_{(x,t)\in Q_i} |W(x, t) - W(x_i, t_i)| < \gamma\delta_i$.
(iv) $|\tilde{\Omega}\setminus \cup_{i=1}^{\infty} Q_i| = 0$.

Proof. Let $(x, t) \in \tilde{\Omega}$. Since $V(x, t) \in \mathcal{U}_{r,s}$, by Theorem 6.7 there exist $\tau \in (0, 1)$ and $\nu = \sum_{\mathbf{j}\in\{1,2\}^N} \mu_{\mathbf{j}} \delta_{\mathbf{j}} \in \mathcal{L}_g(\mathscr{O}_\tau)$ with barycentre $\bar{\nu} = V(x, t)$. Furthermore, there exists $\delta > 0$ such that $B_{\mathscr{M}}([V_{\mathbf{j}',1}, V_{\mathbf{j}',2}]), \delta) \subset \mathcal{U}_{r,s}$ whenever $\mathbf{j}' \in \{1, 2\}^k$, $1 \le k \le N - 1$. Since W is continuous, there exists a cube $Q \subset \tilde{\Omega}$ with center (x, t) such that $\sup_{(x',t')\in Q} \left|W(x', t') - W(x, t)\right| < \gamma\delta_i$.

The collection of cubes chosen above forms a Vitali cover of $\tilde{\Omega}$, and therefore, by the Vitali Covering Theorem, there exists a countable, pairwise disjoint subcollection $\{Q_i\}_{i\in\mathbb{N}}$ with $|\tilde{\Omega}\setminus \cup_{i=1}^{\infty} Q_i| = 0$. □

Proof of Proposition 7.2. Let $\varepsilon > 0$ and $V \in X_0$, and suppose V satisfies (7.3)–(7.4). Let $0 < \gamma = \gamma_V \ll [\min_{(x,t)\in\tilde{\Omega}}((r^2 + s^2)/2 - |u(x, t)|^2 - |B(x, t)|^2)]^{1/2}$ (to be determined later) and choose cubes $Q_i \subset \tilde{\Omega}$ via Lemma 7.6. At each Q_i define

$$W_\ell := W + (\bar{u}_\ell, \bar{S}_\ell, \mathrm{d}\bar{\varphi}_\ell, d\bar{\psi}_\ell) \in C^\infty(Q_i; \mathbb{R}^{17}),$$

where $(\bar{u}_\ell, \bar{S}_\ell, \mathrm{d}\bar{\varphi}_\ell, d\bar{\psi}_\ell) \in C_c^\infty(Q_i; \mathbb{R}^{17})$ is given by Proposition 7.4.

We now intend to show that

$$V_\ell = (u + \bar{u}_\ell, S + \bar{S}_\ell, (\mathrm{d}\varphi + \mathrm{d}\bar{\varphi}_\ell) \wedge (\mathrm{d}\psi + \mathrm{d}\bar{\psi}_\ell)) =: (u_\ell, S_\ell, \mathrm{d}\varphi_\ell \wedge \mathrm{d}\psi_\ell)$$

takes values in $\mathcal{U}_{r,s}$ for every $\ell \in \mathbb{N}$; then $V_\ell \in X_0$ by construction.

Fix a cube Q_i and write $V(x_i, t_i) = p(W(x_i, t_i)) \in \mathcal{U}_{r,s}$ as a barycentre of $\nu = \sum_{\mathbf{j}\in\{1,2\}^N} \mu_{\mathbf{j}} \delta_{V_{\mathbf{j}}} \in \mathcal{L}_g(\mathscr{O}_\tau)$. By Lemma 7.6, whenever $\mathbf{j}' \in \{1, 2\}^k$ with $1 \le k \le N - 1$, we have $B_{\mathscr{M}}([V_{\mathbf{j}',1}, V_{\mathbf{j}',2}], \delta_i) \subset \mathcal{U}_{r,s}$.

Let $(x,t) \in Q_i$. By Lemma 7.6, $|W(x,t) - W(x_i,t_i)| < \gamma\delta_i$. By Proposition 7.4, there exists $\tilde{W}$ such that $p(\tilde{W}) \in [V_{\mathbf{j}',1}, V_{\mathbf{j}',2}]$ for some $\mathbf{j}' \in \{1,2\}^k$ and some $k \leq N_1$, and $|W(x_i,t_i) + (\bar{u}_\ell, \bar{S}_\ell, \mathrm{d}\bar{\varphi}_\ell, d\bar{\psi}_\ell)(x,t) - \tilde{W}| < \gamma\delta_i$. Thus $|W_\ell(x,t) - \tilde{W}| < 2\gamma\delta_i$. Hence, whenever $\gamma > 0$ is small enough (independently of i), we conclude that $|V_\ell(x,t) - p(\tilde{W})| < \delta_i$ and $V_\ell(x,t) \in \mathcal{U}_{r,s}$.

Whenever $\gamma^2 < \min_{(x,t)\in\tilde{\Omega}}[(r^2+s^2)/2 - |u(x,t)|^2 - |B(x,t)|^2]/3$, condition (iii) of Lemma 7.6 and the property $|\tilde{\Omega}\setminus \cup_{i\in\mathbb{N}} Q_i| = 0$ yield a finite subcollection of cubes such that

$$\begin{aligned}
&\int_{\tilde{\Omega}} \left(\frac{r^2+s^2}{2} - |u(x,t)|^2 - |B(x,t)|^2\right) \mathrm{d}x\, \mathrm{d}t \\
&\quad \leq 2\sum_{i=1}^{M} \left(\frac{r^2+s^2}{2} - |u(x_i,t_i)|^2 - |B(x_i,t_i)|^2\right) |Q_i|\,.
\end{aligned}$$

Let $1 \leq i \leq M$ and write $V(x_i,t_i) = \bar{\nu}$, where $\nu = \sum_{\mathbf{j}\in\{1,2\}^N} \mu_{\mathbf{j}}\delta_{V_{\mathbf{j}}}$ is given by Proposition 7.3. In particular, $V_\ell(x,t) = V_{\mathbf{j}}$ in each $A_{\mathbf{j}} \subset Q_i$. Now Proposition 7.3 and Lemma 7.6 give

$$\begin{aligned}
&\left(\frac{r^2+s^2}{2} - |u(x_i,t_i)|^2 - |B(x_i,t_i)|^2\right) |Q_i| \\
&\leq 3 \sum_{\mathbf{j}\in\{1,2\}^N} \left(\left|u_{\mathbf{j}}\right|^2 + \left|B_{\mathbf{j}}\right|^2 - |u(x_i,t_i)|^2 - |B(x_i,t_i)|^2\right) \left|A_{\mathbf{j}}\right| \\
&\leq 4 \sum_{\mathbf{j}\in\{1,2\}^N} \int_{A_{\mathbf{j}}} (|u_\ell(x,t)|^2 + |B_\ell(x,t)|^2 - |u(x,t)|^2 - |B(x,t)|^2)\, \mathrm{d}x\, \mathrm{d}t.
\end{aligned}$$

Indeed, since $V_\ell = V_{\mathbf{j}}$ in each $A_{\mathbf{j}}$, the last estimate is equivalent to the inequality

$$\begin{aligned}
&4 \sum_{\mathbf{j}\in\{1,2\}^N} \int_{A_{\mathbf{j}}} (|u(x,t)|^2 + |B(x,t)|^2 - |u(x_i,t_i)|^2 - |B(x_i,t_i)|^2)\, \mathrm{d}x\, \mathrm{d}t \\
&\quad \leq \sum_{\mathbf{j}\in\{1,2\}^N} \left|A_{\mathbf{j}}\right| [\left|u_{\mathbf{j}}\right|^2 + \left|B_{\mathbf{j}}\right|^2 - |u(x_i,t_i)|^2 - |B(x_i,t_i)|^2],
\end{aligned}$$

which in turn is ensured if τ is large enough and $\gamma^2 \ll \min_{(x,t)\in\tilde{\Omega}}[(r^2+s^2)/2 - |u(x,t)|^2 - |B(x,t)|^2]$ is small enough. This proves the claim. □

7.5. Completion of the Proof of Theorem 1.1

Proof of Theorem 1.1. The functional $V \mapsto \int_{\mathbb{R}^4} |V(x,t)|^2\, \mathrm{d}x\, \mathrm{d}t$ is a Baire-1 map in X, and thus its poins of continuity are residual in X (see [21, Lemma 4.5]). Let now $V \in X$ be a point of continuity and choose a sequence of mappings $\tilde{V}_\ell \in X_0$ with $d(\tilde{V}_\ell, V) \to 0$. By Proposition 7.2 and a standard diagonal argument, we find $V_\ell \in X_0$ with $d(V_\ell, V) \to 0$ and

$$\liminf_{\ell\to\infty} \int_{\Omega} (|u_\ell(x,t)|^2 + |B_\ell(x,t)|^2 - |u(x,t)|^2 - |B(x,t)|^2)\, \mathrm{d}x\, \mathrm{d}t \tag{7.5}$$

$$\geq C \int_\Omega \left(\frac{r^2 + s^2}{2} - |u(x,t)|^2 - |B(x,t)|^2 \right) \mathrm{d}x\, \mathrm{d}t. \tag{7.6}$$

Since $V \mapsto \|V\|_{L^2}^2$ is continuous at V and $d(V_\ell, V) \to 0$, it follows that $\|V_\ell\|_{L^2}^2 \to \|V\|_{L^2}^2$. Thus $\|V_\ell - V\|_{L^2} \to 0$ which, combined with (7.5)–(7.6), gives $|u(x,t)|^2 + |B(x,t)|^2 = r^2 + s^2$ a.e. $(x,t) \in \Omega$. Thus

$$V(x,t) \in \overline{\mathcal{U}_{r,s}} \cap \{(u, S, B, E) \colon |u + B| = r,\ |u - B| = s\} \subset K_{r,s}$$

a.e. $(x,t) \in \Omega$. On the other hand, by the definition of X we have $V(x,t) = 0$ a.e. $(x,t) \in (\mathbb{R}^3 \times \mathbb{R}) \setminus \Omega$. Now V has all the sought properties. □

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Appendix A. The Ill-Definedness of Magnetic Helicity and Mean-Square Magnetic Potential in the Whole Space

We state two simple results which indicate that mean-square magnetic potential and magnetic helicity are *not* well-defined quantities for L^2-integrable solutions of ideal MHD in $\mathbb{R}^2$ or $\mathbb{R}^3$. We denote $L^2_\sigma(\mathbb{R}^n; \mathbb{R}^n) := \{v \in L^2(\mathbb{R}^n; \mathbb{R}^n) \colon \nabla \cdot v = 0\}$ when $n \in \{2, 3\}$.

Proposition A.1. *There exists $v \in L^2_\sigma(\mathbb{R}^2; \mathbb{R}^2)$ with the following property: if $\Psi \in \mathcal{D}'(\mathbb{R}^2)$ satisfies $\nabla^\perp \Psi = v$, then $\Psi \notin L^2(\mathbb{R}^2)$.*

Proposition A.1 is proved by choosing $\Theta \in \dot{W}^{1,2}(\mathbb{R}^2)$ such that $\Theta + C \notin W^{1,2}(\mathbb{R}^2)$ for every $C \in \mathbb{R}$ and setting $v := \nabla^\perp \Theta$. The 3D result requires somewhat more work. Here we choose a smooth v in order to make $\Psi \cdot v$ well-defined for all $\Psi \in \mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)$.

Proposition A.2. *There exists $v \in L^2_\sigma(\mathbb{R}^3, \mathbb{R}^3) \cap C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ with the following property: whenever $\Psi \in \mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)$ satisfies $\nabla \times \Psi = v$, we have $\Psi \cdot v \notin L^1(\mathbb{R}^3)$.*

Proof. Fix $\psi_0 \in C_c^\infty(B(0,1), \mathbb{R}^3)$ such that ψ_0 and $\varphi_0 := \nabla \times \psi_0$ satisfy $\int_{B(0,1)} \psi_0(x) \cdot \varphi_0(x)\, \mathrm{d}x \neq 0$. (Choose, for example, $\psi_0(x) = \chi(x)(1, x_3, 0)$, where $\chi \in C_c^\infty(B(0,1)$ with $\chi(0) > 0$.) Set $\psi_0(x) = \varphi_0(x) = 0$ outside $B(0,1)$.

Fix points $x_j \in \mathbb{R}^3$ and radii $R_j > 0$ such that the balls $B(x_j, R_j)$, $j \in \mathbb{N}$, are mutually disjoint. For every $j \in \mathbb{N}$ denote

$$\psi_j(x) := \frac{\psi_0\left(\frac{x-x_j}{R_j}\right)}{R_j^{1/2}}, \qquad \varphi_j(x) := \frac{\varphi_0\left(\frac{x-x_j}{R_j}\right)}{R_j^{3/2}},$$

so that $\operatorname{supp}(\psi_j) \subset B(x_j, R_j)$, $\nabla \times \psi_j = \varphi_j$ and $\|\varphi_j\|_{L^2} = \|\varphi_0\|_{L^2}$. Define $v \in L^2_\sigma(\mathbb{R}^3, \mathbb{R}^3)$ by

$$v(x) := \sum_{j=1}^{\infty} \frac{1}{j^2}\varphi_j(x).$$

Suppose now $\Psi \in \mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)$ satisfies $\nabla \times \Psi = v$ and $\Psi \cdot v \in L^1(\mathbb{R}^3)$. Given $j \in \mathbb{N}$, note that in $B(x_j, R_j)$ we have $\Psi = \psi_j + \nabla g_j$, where $g_j \in \mathcal{D}'(B(x_j, t_j))$. Thus, by using the fact that $\nabla g_j \cdot \varphi_j = 0$ for every $j \in \mathbb{N}$ we get

$$\begin{aligned}\int_{\mathbb{R}^3} |\Psi(x)\cdot v(x)|\,\mathrm{d}x &\geq \sum_{j=1}^{\infty} \frac{1}{j^2}\left|\int_{B(x_j,R_j)} \psi_j(x)\cdot\varphi_j(x)\,\mathrm{d}x\right| \\ &\geq \sum_{j=1}^{\infty} \frac{1}{j^2} R_j \left|\int_{B(0,1)} \psi_0(x)\cdot\varphi_0(x)\,\mathrm{d}x\right|,\end{aligned}$$

and the lower bound series diverges as soon as the radii satisfy $R_j \geq j$. □

References

1. ALUIE, H.: Hydrodynamic and magnetohydrodynamic turbulence: invariants, cascades, and locality. Doctoral Dissertation, Johns Hopkins University, 2009
2. ASTALA, K., FARACO, D., SZÉKELYHIDI Jr., L.: Convex integration and the Lp theory of elliptic equations. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **7**(1), 1–50, 2008
3. ARNOLD, V.I.: The asymptotic Hopf invariant and its applications. *Sel. Math. Soviet.* **5**(4), 327–345, 1986
4. ARNOLD, V.I., KHESIN, B.A.: *Topological Methods in Hydrodynamics. Applied Mathematical Sciences*, vol. 125. Springer, New York 1998
5. BEEKIE, R., BUCKMASTER, T., VICOL, V.: Weak solutions of ideal MHD which do not conserve magnetic helicity. arXiv:1907.10436
6. BISKAMP, D.: *Nonlinear Magnetohydrodynamics*. Cambridge University Press, Cambridge 1991
7. BISKAMP, D., WELTER, H.: Dynamics of decaying two-dimensional magnetohydrodynamic turbulence. *Phys. Fluids B Plasma Phys.* **1**(10), 1964–1979, 1989
8. BRONZI, A.C., LOPES FILHO, M.C., NUSSENZVEIG LOPES, H.J.: Wild solutions for 2D incompressible ideal flow with passive tracer. *Commun. Math. Sci.* **13**, 1333–1343, 2015
9. BUCKMASTER, T., DE LELLIS, C., SZÉKELYHIDI Jr., L., VICOL, V.: Onsager's conjecture for admissible weak solutions. *Commun. Pure Appl. Math.* **72**(2), 229–274, 2019
10. BUCKMASTER, T., SCHKOLLER, S., VICOL, V.: Nonuniqueness of weak solutions to the SQG equation. *Commun. Pure Appl. Math.* **72**(9), 1809–1874, 2019
11. BUCKMASTER, T., VICOL, V.: Nonuniqueness of weak solutions to the Navier–Stokes equation. *Ann. Math. (2)* **189**(1), 101–144, 2019
12. CAFLISCH, R.E., KLAPPER, I., STEELE, G.: Remarks on singularities, dimension and energy dissipation for ideal hydrodynamics and MHD. *Commun. Math. Phys.* **184**(2), 443–455, 1997
13. CHIODAROLI, E., DE LELLIS, C., KREML, O.: Global ill-posedness of the isentropic system of gas dynamics. *Commun. Pure Appl. Math.* **68**(7), 1157–1190, 2015

14. Chiodaroli, E., Michálek, M.: Existence and non-uniqueness of global weak solutions to inviscid primitive and Boussinesq equations. *Commun. Math. Phys*. **353**(3), 1201–1216, 2017
15. Choffrut, A., Székelyhidi Jr., L.: Weak solutions to the stationary incompressible Euler equations. *SIAM J. Math. Anal*. **46**(6), 4060–4074, 2014
16. Coifman, R., Lions, P.-L., Meyer, Y., Semmes, S.: Compensated compactness and Hardy spaces. *J. Math. Pures Appl. (9)* **72**(3), 247–286, 1993
17. Constantin, P.E.W., Titi, E.S.: Onsager's conjecture on the energy conservation for solutions of Euler's equation. *Commun. Math. Phys*. **165**(1), 207–209, 1994
18. Córdoba, D., Faraco, D., Gancedo, F.: Lack of uniqueness for weak solutions of the incompressible porous media equation. *Arch. Ration. Mech. Anal*. **200**(3), 725–746, 2011
19. Csató, G., Dacorogna, B., Kneuss, O.: The Pullback Equation for Differential Forms. Progress in Nonlinear Differential Equations and their Applications, 83. Birkhäuser/Springer, New York 2012
20. Dallas, V., Alexakis, A.: The signature of initial conditions on magnetohydrodynamic turbulence. *Astrophys. J. Lett*. **788**(2), 4, 2014
21. De Lellis, C., Székelyhidi Jr., L.: The Euler equations as a differential inclusion. *Ann. Math. (2)* **170**(3), 1417–1436, 2009
22. De Lellis, C., Székelyhidi Jr., L.: On admissibility criteria for weak solutions of the Euler equations. *Arch. Ration. Mech. Anal*. **195**(1), 225–260, 2010
23. De Lellis, C., Székelyhidi Jr., L.: The h-principle and the equations of fluid dynamics. *Bull. Am. Math. Soc. (N.S.)* **49**(3), 347–375, 2012
24. De Lellis, C., Székelyhidi Jr., L.: Dissipative continuous Euler flows. *Invent. Math*. **193**(2), 377–407, 2013
25. De Lellis, C., Székelyhidi Jr., L.: High dimensionality and h-principle in PDE. *Bull. Am. Math. Soc*. **54**(2), 247–282, 2017
26. Deschamps, G.A.: Electromagnetics and differential forms. *Proc. IEEE* **69**(6), 676–696, 1981
27. Eyink, G.L.: Dissipative anomalies in singular Euler flows. *Physica D* **237**(14–17), 1956–1968, 2008
28. Eyink, G.L.: Dissipation in turbulent solutions of 2D Euler equations. *Nonlinearity* **14**(4), 787–802, 2001
29. Faraco, D., Lindberg, S.: Proof of Taylor's conjecture on magnetic helicity conservation. *Commun. Math. Phys*., 2019. https://doi.org/10.1007/s00220-019-03422-7
30. Freire, A., Müller, S., Struwe, M.: Weak convergence of wave maps from $(1+2)$-dimensional Minkowski space to Riemannian manifolds. *Invent. Math*. **130**, 589–617, 1997
31. Gerbeau, J.-F., Le Bris, C., Lelièvre, T.: *Mathematical methods for the magnetohydrodynamics of liquid metals*. Oxford University Press, Oxford, Numerical Mathematics and Scientific Computation 2006
32. Gromov, M.: Partial differential relations. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 9. Springer, Berlin 1986
33. Isett, P.: A proof of Onsager's conjecture. *Ann. Math. (2)* **188**(3), 871–963, 2018
34. Isett, P., Vicol, V.: Hölder continuous solutions of active scalar equations. *Ann. PDE* **1**(1), 77, 2015
35. Kang, E., Lee, J.: Remarks on the magnetic helicity and energy conservation for ideal magneto-hydrodynamics. *Nonlinearity* **20**(11), 2681–2689, 2007
36. Khesin, B., Peralta-Salas, D., Yang, C.: A basis of Casimirs in 3D magnetohydrodynamics. arXiv: 1901.04404
37. Kim, S., Yan, B.: Convex integration and infinitely many weak solutions to the Perona–Malik equation in all dimensions. *SIAM J. Math. Anal*. **47**(4), 2770–2794, 2015
38. Kirchheim, B.: Rigidity and geometry of microstructures. Habilitation Thesis, Universität Leipzig 2003

39. Linkmann, M.F., Bereza, A., McComb, W., McKay, M.E.: Nonuniversality and Finite Dissipation in Decaying Magnetohydrodynamic Turbulence. *Phys. Rev. Lett*. **114**(23), 5, 2015
40. Lopes Filho M.C., Mazzucato, A.L., Nussenzveig Lopes, H.J.: Weak solutions, renormalized solutions and enstrophy defects in 2D turbulence. *Arch. Ration. Mech. Anal*. **179**(3), 353–387, 2005
41. Mininni, P.D., Pouquet, A.: Finite dissipation and intermittency in magnetohydrodynamics. *Phys. Rev. E* **80**(2), 4, 2009
42. Moffatt, H.K.: Degree of knottedness of tangled vortex lines. *J. Fluid Mech*. **35**(1), 1187–129, 1969
43. Müller, S., Šverák, V.: Convex integration with constraints and applications to phase transitions and partial differential equations. *J. Eur. Math. Soc*. **1**, 393–422, 1999
44. Nash, J.: C^1 isometric imbeddings. *Ann. Math*. **2**(60), 383–396, 1954
45. Onsager, L.: Statistical hydrodynamics. Nuovo Cimento (9) **6**. Supplemento, 2 (Convegno Internazionale di Meccanica Statistica), 279–287, 1949
46. Ortolani, S., Schnack, D.D.: *Magnetohydrodynamics of Plasma Relaxation*. World Scientific, Singapore 1993
47. Pedregal, P.: Laminates and microstructure. *Eur. J. Appl. Math*. **4**(2), 121–149, 1993
48. Roubícek, T.: Nonlinear partial differential equations with applications. Second edition. International Series of Numerical Mathematics, 153. Birkhäuser/Springer Basel AG, Basel 2013
49. Scheffer, V.: An inviscid flow with compact support in space-time. *J. Geom. Anal*. **3**(4), 343–401, 1993
50. Sermange, M., Temam, R.: Some mathematical questions related to the MHD equations. *Commun. Pure Appl. Math*. **36**(5), 635–664, 1983
51. Shnirelman, A.: On the nonuniqueness of weak solution of the Euler equation. *Commun. Pure Appl. Math*. **50**(12), 1261–1286, 1997
52. Shvydkoy, R.: Convex integration for a class of active scalar equations. *J. Am. Math. Soc*. **24**(4), 1159–1174, 2011
53. Székelyhidi Jr., L.: Relaxation of the incompressible porous media equation. *Ann. Sci. Éc. Norm. Sup'er. (4)* **45**(3), 491–509, 2012
54. Székelyhidi Jr., L.: From Isometric Embeddings to Turbulence, In: HCDTE Lecture Notes. Part II. Nonlinear Hyperbolic PDEs, Dispersive and Transport Equations, 4, American Institute of Mathematical Sciences, 1–66, 2014
55. Tartar, L.: The compensated compactness method applied to systems of conservation laws. In: Systems of Nonlinear Partial Differential Equations. Dordrecht, 263–285, 1977
56. Tartar, L.: Compensation effects in partial differential equations. *Accad. Naz. Sci. XL Rend. Mem. Mat. Appl. (5)* **29**, 395–453, 2005
57. Taylor, J.B.: Relaxation of toroidal plasma and generation of reverse magnetic fields. *Phys. Rev. Lett*. **33**, 1139–1141, 1974
58. Wang, Y., Zuo, B.: Energy and cross helicity conservation for the three-dimensional ideal MHD equations in bounded domain. arxiv:1811.00411
59. Woltjer, L.: A theorem on force-free magnetic fields. *Proc. Natl. Aca. Sci. USA* **44**, 6, 1958
60. Yu, X.: A note on the energy conservation of the ideal MHD equations. *Nonlinearity* **22**(4), 913–922, 2009

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