Free boundary regularity for the inhomogeneous Stefan Problem

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The Stefan problem describes the phenomenon of freezing and melting of a material with a solid-liquid interphase, such as ice and water. It can be formulated as a parabolic free boundary problem where the interface between the solid and the liquid regions evolves over time and is an unknown of the problem. After introducing the problem and some related literature, the poster focuses on presenting some recent techniques to obtain free boundary regularity results starting from an initial suitable flatness assumption for the one-phase Stefan problem with the presence of a right hand side extending the result of the homogeneous case due to De Silva, Forcillo and Savin [3].

Let $\Omega \subset \mathbb{R}^n$, $T > 0$ and given a heat source $f \in C(\Omega \times [0,T]) \cap L^\infty(\Omega \times [0,T])$, the two-phase Stefan problem is modelled by the following free boundary problem

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \Omega_T^+(u) := (\Omega \times [0,T]) \cap \{u > 0\}, \\ \partial_t u - \Delta u = f & \text{in } \Omega_T^-(u) := (\Omega \times [0,T]) \cap \{u \leq 0\}^0, \\ \dfrac{\partial_t u^+}{|\nabla u^+|} = \left|\nabla u^+\right| - \left|\nabla u^-\right| = -\dfrac{\partial_t u^-}{|\nabla u^-|} & \text{on } F(u) := (\Omega \times [0,T]) \cap \partial\{u > 0\}. \end{cases} \tag{1}$$

A solution u of (1) represents the temperature of the system. $F(u)$ is the free boundary set and moves according to the energy balance condition given by

$$V_\nu = \frac{\partial_t u^+}{|\nabla u^+|} = -\frac{\partial_t u^-}{|\nabla u^-|} = \left|\nabla u^+\right| - \left|\nabla u^-\right|$$

Here $\nu = \frac{\nabla u^+}{|\nabla u^+|}$ and $V_\nu(\cdot, \tau)$ is the speed of the surface $F_\tau(u) = F(u) \cap \{t = \tau\}$ in the direction ν.

If the temperature vanishes identically in one of the phases (the zero phase now has to be understood as the unmelted region of ice) one often speaks about a *one-phase* Stefan problem.

This will be based on a recent collaboration with F. Ferrari, N. Forcillo and D. Jesus.

References

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