Linear Flow on the Infinite Torus

Lorenzo Baroni
Università degli Studi Roma Tre
lorenzo.baroni@uniroma3.it

Abstract

We discuss the characterization of linear flows on the infinite torus. In analogy with finite-dimensional Hamiltonian systems, many Hamiltonian PDEs and systems on infinite lattices admit invariant tori of possibly infinite dimension, and the dynamics on them is linearizable. However, a satisfactory characterization is still lacking, unlike in the finite-dimensional case, which is thoroughly understood: the dynamics is reducible to a linear flow with frequencies that are linearly independent over the integers, which is equivalent to being topologically transitive, to minimality, and to unique ergodicity. The existence of such tori is fundamental in KAM theory. In the general context of topological dynamics, we prove that this characterization still holds when the dimension of the torus is infinite if and only if the integer (finite) linear combinations of the frequencies form a free abelian group, which means that there exists a basis. Then, we investigate the topological properties of a special class of linear flows on the torus and conclude that in the infinite-dimensional setting there exist orbits whose closure is locally homeomorphic to the product of an interval and a Cantor set. Intuitively, such orbits are well approximated by periodic orbits whose period diverges as the approximation becomes more accurate. Our work recovers and puts in a more general framework a recent result by Sakbaev and Volovich concerning the existence of a new type of trajectories for the infinite harmonic oscillator that is absent in the finite-dimensional case. We also discuss the relation with the (intractable) classification problem for countable abelian groups without torsion, i.e., without elements $a \neq 0$ that admit a non-zero integer n such that $na = 0$. It turns out that our investigation yields a characterization and a concrete explicit construction of all linear flows on the infinite torus whose (torsion-free) abelian group generated by the frequencies admits a decomposition into additive subgroups of the rationals, that is the largest class of countable abelian groups without torsion that has been completely classified to date.