Lie's distinguished functionals as Noether quantities of the Euler–Poincaré variational principle

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Abstract

Lie's distinguished functionals, also referred to as Casimirs, commute with every functional in the Lie-Poisson bracket, thereby serving as integrals of motion. They are considered "off-shell" integrals, providing a description of the geometric structure inherent to a generally nonlinear dynamical model. The Hamiltonian, on the other hand, determines the specific type of motion that the model supports.

The conservation of Lie's distinguished functionals of the hydrodynamic Lie–Poisson bracket, in particular, has long been associated with particle relabeling symmetry, which is inherently concealed within the Eulerian framework. The enstrophy moments of quasi-geostrophic (QG) flow are significant examples of such functionals, which, for instance, are useful in deriving stability theorems using Arnold's method. In this work, we demonstrate the explicit correspondence of Lie's distinguished functionals with Noether's quantities arising from the Euler–Poincaré variational principle when variations are not restricted to vanish at the endpoints of integration.

The models that support this connection, so far, are embodied in the generalized thermal QG (or TQG) model [BVL24], which will be introduced systematically. Notable instances include the adiabatic QG equation, the TQG itself [BV21b], and its novel variant, the stratified TQG [BV21a]. The renewed interest in thermal ocean models stems from their capability to simulate circulatory patterns akin to submesoscale motions within a two-dimensional context (Figure 1). These motions influence the transport of passive tracers, such as chlorophyll and light oil, and active tracers, like *Sargassum* seaweed.

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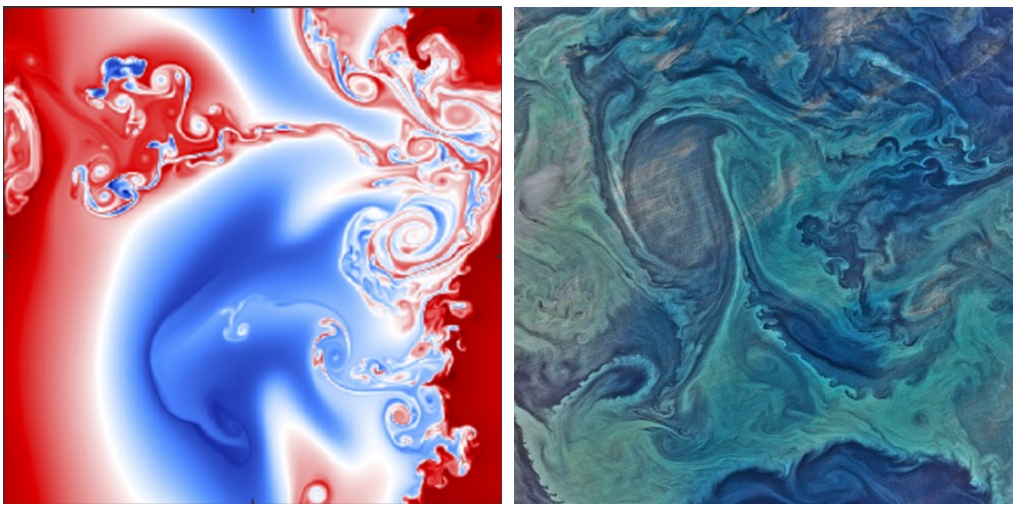

Figure 1: (left panel) Kelvin–Helmholtz-like vortices manifesting in a direct numerical simulation of the TQG within a doubly periodic domain with a side length corresponding to the baroclinic deformation radius. (right panel) Ocean color image acquired by VIIRS on 1 January 2015 west of the Drake Passage in the Southern Ocean, revealing vortices with diameters ranging from a couple of km to a couple of hundred km. Image credit: NASA Ocean Color Web.

References

[BV21a] F. J. Beron-Vera. Extended shallow-water theories with thermodynamics and geometry. *Phys. Fluids*, 33:106605, 2021.

[BV21b] F. J. Beron-Vera. Nonlinear saturation of thermal instabilities. *Phys. Fluids*, 33:036608, 2021.

[BVL24] F. J. Beron-Vera and Erwin Luesink. Dual Euler–Poincaré/Lie–Poisson formulation of subinertial stratified thermal ocean flow with identification of Casimirs as Noether quantities. Preprint, 2024.