CRM Workshop

Kähler packages and their combinatorial significance

S. Xambó

FME&IMTech

15/7/2024

To begin ...

These slides can be downloaded from

https://mat.web.upc.edu/people/sebastia.xambo/99/s-KP.pdf

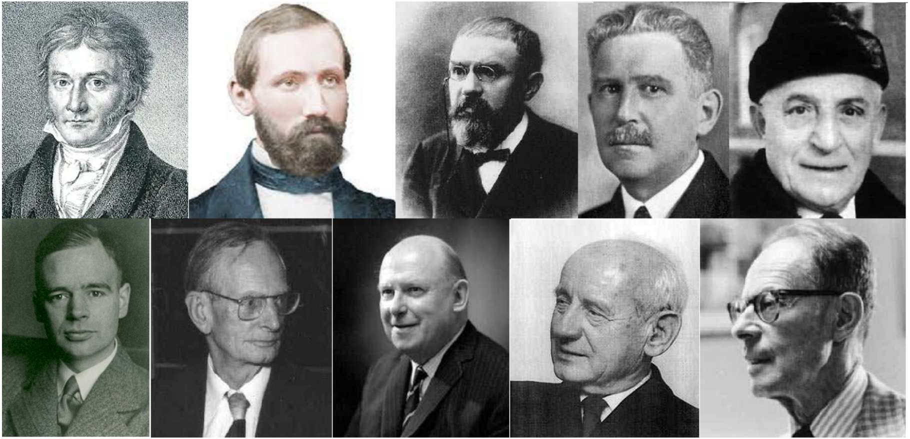
For a companion paper, a glossary, and a timeline of J. Huh's works, replace s-KP by p-KP, g-KP and t-KP, respectively.

References to p-KP will be given in the form §4.32, where 4 indicates the section and 32 the paragraph number (paper-wide correlative from 1 to 53), and §4 refers to the 4th section.

The Prelude of p-KP ends with a list of Acknowledgements, which includes my thanks to Anna de Mier's teachings on graphs and matroids.

Now let me add my thanks to the organizers of this workshop: Juanjo Rué and Marc Masdeu.

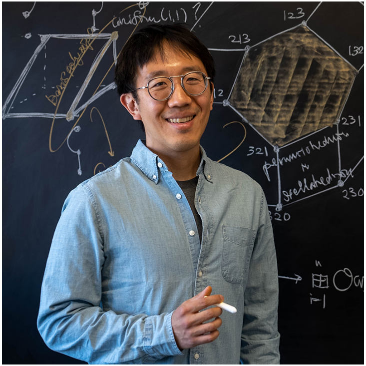
We are in good company



1st row: Carl F. Gauss (1777-1855), Bernhard Riemann (1826-1866), Henri Poincaré (1854-1912), Solomon Lefschetz (1884-1972), Oscar Zariski (1889-1986). *2nd row*: Bartel L. van der Waerden (1903-1996); Georges de Rham (1903-1990), William D. Hodge (1903-1975), Eric Kähler (1906-2000), André Weil (1906.1998).

Hassler Whitney (1907-1989), Willian Thomas Tutte (1917-2002), Richard P. Stanley (1944-).

1st row; Laurent Schwartz (1950), Kunihiko Kodaira (1954), Jean-Pierre Serre (1954), John Milnor (1962), Alexander Grothendieck (1966), Heisuke Hironaka (1970). *2nd row*: Pierre Deligne (1978), Grigori A. Margulis (1978), Curtis T. McMullen (1998), Andrei Okounkov (2006), Terence Tao (2006), June Huh (2022).



June Huh's FM mention: "For bringing the ideas of Hodge theory to combinatorics, the proof of the Dowling–Wilson conjecture for geometric lattices, the proof of the Heron–Rota–Welsh conjecture for matroids, the development of the theory of Lorentzian polynomials, and the proof of the strong Mason conjecture.

Index

- **The notion of Kähler package (KP)**
- **Examples of KPs, particularly combinatorial**
- **Introduction to the theory of Lorentzian polynomials**
- **Lorentzian polynomials derived from KPs**
- **Combinatorial applications**
- **Other recent contributions**

The notion of Kähler package

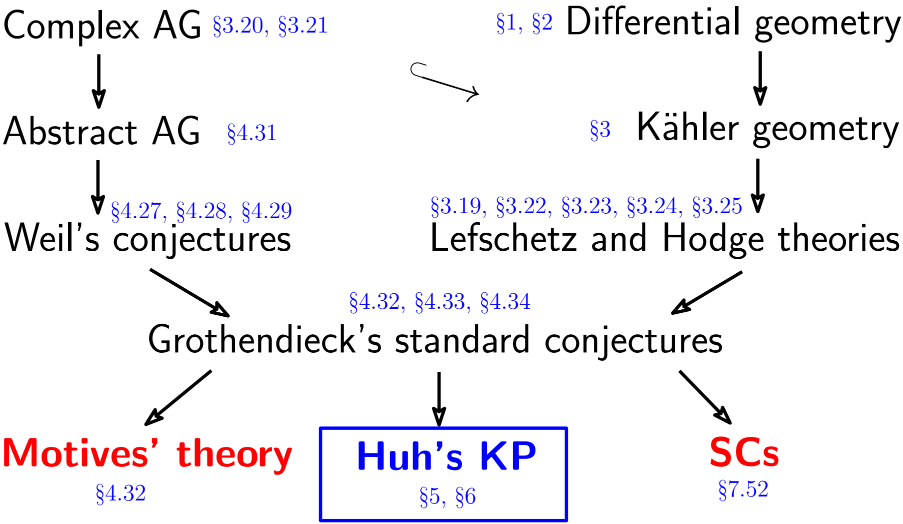
Historical streams

Weil cohomologies

Grothendieck's SCs

Ingredients and postulates for a KP

Examples


Complex AG §3.20, §3.21
§1, §2 Differential geometry
Abstract AG §4.31
§3 Kähler geometry
§4.27, §4.28, §4.29
Weil's conjectures
§3.19, §3.22, §3.23, §3.24, §3.25
Lefschetz and Hodge theories
§4.32, §4.33, §4.34
Grothendieck's standard conjectures
Motives' theory
§4.32
Huh's KP
§5, §6
SCs
§7.52

$\mathcal{X}$: Class of non-singular projective algebraic ***varieties*** X over an algebraically closed field $\boldsymbol{\kappa}$ (*ground field*).

$\mathcal{Z}^k(X)$: Group of codimension k cycles of X.

K: Field of characteristic 0 ($\mathbb{Q}$, $\mathbb{Q}_\ell$, $\mathbb{R}$, $\mathbb{C}$,...)

Weil cohomology (§4.33): A contravariant functor $X \mapsto H^*(X)$ from varieties to finite-dimensional graded K-algebras such that

- $H^k(X) = 0$ for $k < 0$ and $k > 2n$ ($n = n_X = \dim X$),
- endowed with $\int_X : H^{2n}(X) \simeq K$ (also called deg),
- and with a *class homomorphism* $\mathrm{cl} = \mathrm{cl}_X : \mathcal{Z}^k(X) \to H^{2k}(X)$,

so that the following properties are satisfied:

Poincaré duality: The map $H^k(X) \times H^{2n-k}(X) \to K$, $(\alpha, \alpha') \mapsto \int_X \alpha \cdot \alpha'$ is non-degenerate for all k.

In particular, $H^{2n-k}(X) \simeq H^k(X)^*$, or, defining $H_k(X) = H^k(X)^*$, $H^{2n-k}(X) \simeq H_k(X)$ (the duals are as K-vector spaces).

If $f : X \to X'$ be a morphism of varieties, the adjoint of $f^* : H^*(X') \to H^*(X)$ with respect to Poincaré duality is a K-linear map $f_* : H^*(X) \to H^{*+r}(X')$, where $r = n_{X'} - n_X$ (dubbed the *codimension* of f).

These two maps are related by the *projection formula*:

$f_*(\alpha \cdot f^*\alpha') = f_*\alpha \cdot \alpha'$ ($\alpha \in H^*(X)$, $\alpha' \in H^*(X')$).

Künneth formula: The natural map

- $H^*(X) \otimes H^*(X') \to H^*(X \times X')$, $\alpha \otimes \alpha' \mapsto \pi_X^*(\alpha) \cdot \pi_{X'}(\alpha')$,

is an isomorphism for any varieties X and X' (π_X and $\pi_{X'}$ are the projection maps of $X \times X'$ onto X and X', respectively).

Functoriality of the class map:

- $\mathrm{cl}_X(f^*Z') = f^*(\mathrm{cl}_{X'}Z')$ for any morphism of varieties $f : X \to X'$ and any $Z' \in \mathcal{Z}^k(X')$ (functoriality of cl);
- $\mathrm{cl}_{X\times X'}(Z \times Z') = \mathrm{cl}_X(Z) \otimes \mathrm{cl}_{X'}(Z')$ for any varieties X, X' and any cycles $Z \in \mathcal{Z}^*(X), Z' \in \mathcal{Z}^*(X')$);

- $\int_X \mathrm{cl}(Z) = \deg(Z)$ for any $Z \in \mathcal{Z}^n(X)$

(here $\deg(Z) = \sum n_i$ if $Z = \sum_i n_i P_i$, $P_i \in X$).

One instance of Weil cohomology is Grothendieck's ℓ-*adic cohomology*. In this theory K is the field $\mathbb{Q}_\ell$ of ℓ-adic rational numbers, with ℓ different from the characteristic of the ground field $\boldsymbol{\kappa}$. §4.33, §7.52.

The elements of $H^*(X)$ are called *cohomology classes* of X and the classes in the subring $\mathcal{A}^*(X) = \mathrm{cl}(\mathcal{Z}^*(X)) \subseteq H^{2*}(X)$ are said to be *algebraic*.

Two algebraic cycles are said to be *homologically equivalent* when they define the same algebraic class.

To note that $\int_X \alpha \cdot \alpha' \in \mathbb{Z}$ for any $\alpha, \alpha' \in \mathcal{A}^*(X)$,

with the convention that $\int_X \xi = \int_X \xi_n$ for any

$$\xi = \xi_0 + \xi_1 + \cdots + \xi_n \in \mathcal{A}^*(X).$$

We extract from Grothendieck's original paper [1] just the conjectural statements that play a role in the description of a KP, namely:

- *The hard Lefschetz property*,
- *The Hodge–Riemann relations*,

which themselves mimic for algebraic varieties the homonymous results for Kähler manifolds. §3.22, §3.24.

Let Y by a hyperplane section of X and $\xi = \mathrm{cl}(Y) \in H^2(X)$

Let $L : H^k(X) \in H^{k+2}(X)$ be the multiplication by ξ. Thus $L^j : H^k(X) \to H^{k+2j}(X)$ is multiplication by ξ^j.

Hard Lefschetz theorem

- $L^j : H^{n-j}(X) \to H^{n+j}(X)$ is an isomorphism for all $j \in [n]$.

  $\Leftrightarrow L^{n-k} : H^k(X) \to H^{2n-k}(X)$ is an isomorphism for all $k \in [n]$.

Corollary: $L^j : H^k(X) \to H^{k+2j}(X)$ is injective for $j \leqslant n-k$ and surjective for $j \geqslant n-k$.

Indeed, in the first case, composing $L^j : H^k(X) \to H^{k+2j}(X)$ with $L^{n-k-j} : H^{k+2j}(X) \to H^{2n-k}(X)$ is $L^{n-k} : H^k(X) \to H^{2n-k}(X)$, which is an isomorphism; so the first map must be injective.

In the second case, the map $L^{k-(n-j)} : H^{n-(k+2j-n)}(X) \to H^{2k}(X)$ followed by $L^j : H^k(X) \to H^{k+2j}(X) = H^{n+(k+2j-n)}$ produces the map $L^{j+(j-(n-k))} : H^{n-(k+2j-n)}(X) \to H^{n+(k+2j-n)}(X)$, which is an isomorphism. Therefore the second map must be surjective.

Hodge–Riemann relations

For $j \leqslant n/2$, let $\mathcal{A}^j_0(X) = \{\alpha \in \mathcal{A}^j \mid L^{n-2j+1}\alpha = 0\}$

(the *primitive part* of $\mathcal{A}^j(X)$; note that $\mathcal{A}^0_0(X) = \mathcal{A}^0(X)$).

Then the intersection pairing $\mathcal{A}^j_0(X) \times \mathcal{A}^j_0(X) \to \mathbb{Z}$ given by $(-1)^j \int_X L^{n-2j}(\alpha \cdot \beta)$ is positive definite.

This statement holds (*Hodge theory*) for complex algebraic varieties (3.24).

As presented by Huh, the scheme has three ingredients and three postulates (dubbed the *Kähler package* by Huh, for "Kähler first emphasized the importance of the respective objects in topology and geometry").

For the Kähler geometry background that inspires these definitions, see §3.

Ingredients

(1) A graded finite-dimensional real vector space $A = \bigoplus_{j=0}^{d} A^j$;
(2) a convex cone K of graded linear maps $L : A^\star \to A^{\star+1}$; and
(3) symmetric bilinear pairings $P_k : A^k \times A^{d-k} \to \mathbf{R}$, in the sense that $P_{n-k}(\alpha_{n-k}, \alpha_k) = P_k(\alpha_k, \alpha_{n-k})$ and with $P_k(L\alpha_{k-2}, \alpha'_{d-k}) = P_{k-2}(\alpha_{k-2}, L\alpha'_{d-k})$.

Postulates

For any $j \leqslant d/2$,

- *Poincaré Duality* (PD): $P_j : A^j \to (A^{d-j})^*$ is an isomorphism;
- *Hard Lefschetz Propery* (HL): For any $L \in K$, $L^{d-2j} : A^j \to A^{d-j}$ is an isomorphism;
- *Hodge-Riemann Relations* (HR): The pairing

$$A^j \times A^j \to \mathbf{R}, \quad (x, y) \mapsto (-1)^j P(x, L^{d-2j} y) = \langle x, y \rangle,$$

is positive definite on the kernel A_0^j of L^{d-2j+1}
(*primitive part* of A^j, to borrow the name from Lefschetz theory).

In the cases we will consider, A is a graded commutative algebra generated by A^1, with $A^0 = \mathbb{R}$, and the maps L have the form $\alpha \mapsto \alpha \cdot \xi$, for some $\xi \in A^1$.

Assuming that we have Poncaré duality, an element $\xi \in A^1$ defines the map $L : A^* \to A^{*+1}$, $\alpha \mapsto \alpha \cdot \xi$, and we say that the KP holds for ξ if L satisfies HL and HR.

$\mathrm{HR}_{\leqslant 1}$: It is clear that $A_0^0 = A^0$, for $A^{d+1} = 0$, and HR_0 just says that $\langle x, x \rangle > 0$ for any nonzero $x \in A^0$. We define $\deg : A^d \to \mathbb{R}$ by the formula $\deg(\alpha) = P(\mathbf{1}, \alpha)$. In particular, $\deg(L^d \mathbf{1}) = P(\mathbf{1}, L^d \mathbf{1}) > 0$ by HR_0. Now $A_0^1 = \{x \in A^1 \mid L^{d-1} x = 0\}$ and for any such x we have $\langle L\mathbf{1}, x \rangle = P(L\mathbf{1}, L^{d-2} x) = P(\mathbf{1}, L^{d-1} x) = 0$, which implies that $A^1 = \langle L\mathbf{1} \rangle \perp A_0^1$ with respect to the bilinear form $\langle x, y \rangle$. For any nonzero $x \in A_0^1$, we have $\langle x, x \rangle = P(x, L^{d-2} x) < 0$, by HR_1, while $\langle L\mathbf{1}, L\mathbf{1} \rangle = P(L\mathbf{1}, L^{d-2} L\mathbf{1}) = P(\mathbf{1}, L^d \mathbf{1}) > 0$. In other words, $\langle \, , \rangle$ has Lorentzian signature $(+ - \cdots -)$.

Remark. When Poincaré duality for X is known, the HR for X is stronger than HL, in the sense that for every q, $\mathrm{HR}_q \Rightarrow \mathrm{HL}_{\leqslant q}$, [2] In the examples known so far, $A = A(X)$ depends on the objects X of some species.

- X a smooth projective variety, $A(X)$ a cohomology ring (ℓ-adic, for instance). The package agrees essentially with GROTHENDIECK's standard conjectures. So this KP remains conjectural.

- X is a convex polytope and $A(X)$ its combinatorial cohomology [3] Karu (2004) / [[45]] .

- X a matroid and $A(X)$ can be its:
(i) Chow ring [4] Adiprasito, Huh, Katz (2018) / [[1]];
(ii) Conormal Chow ring [5] Ardilla, Denham, Huh (2022) / [[6]]; or
(iii) Intersection cohomology [6] Braden, Huh, Matherne, Proudfoot, Wang (2020) / [[12]].

- X is an element of a Coxeter group and $A(X)$ its Soergel bimodule [7] Elias, Williamson (2014) / [[26]]. Other references: [8], [9].

We are in good company: June Huh' collaborators

E. Katz
B. Wang
K. Adiprasito
T. Braden
J. P. Matherne

N. Proudfoot
C. Eur
P. Brändén
F. Ardila
G. Denham

K. Mészáros
A. St. Dizier
C. Crowley
M. Larson
C. Simpson

Eric Katz (Stanford PhD 2004: *A Formalism for Relative Gromov-Witten Invariants*, Y. Eliashberg & R. Vakil)

Botong Wang (Purdue PhD 2012: *Parabolic Higgs Bundles and Filtered Local Systems*, D. Arapura)

Karim Adiprasito (Freie UB PhD 2013: *Methods from Differential Geometry in Polytope Theory*, G. M. Ziegler & G. Kalai)

Tom Braden (MIT PhD 1995: *Characteristic Cycles of Toric Varieties: Perverse Sheaves on Rank Stratifications*, R. D. MacPherson)

Jacob P. Matherne (Louisiana SU PhD 2016: *Derived Geometric Satake Equivalence, Springer Correspondence, and Small Representations*, P. N. Achar $<$ D. A. Vogan, Jr.)

NICHOLAS PROUDFOOT (UC Berkeley PhD 2004: *Hyperkähler Analogues of Kähler Quotients*, A. I, Knutson)

CHRISTOPHER EUR (UC Berkeley PhD 2020: *The Geometry of Divisors on Matroids*, D. Eisenbud)

PETTER BRÄNDÉN (Göteborg PhD 2005: *On Unimodality and Real-Rootedness of Polynomials in Combinatorics*, E. Steingrímsson)

FEDERICO ARDILA (MIT PhD 2003: *Enumerative and Algebraic Aspects of Matroids and Hyperplane Arrangements*, R. P. Stanley)

GRAHAM DENHAM (U Michigan PhD 1999: *Local Systems on the Complexification of an Oriented Matroid*, P. J. Hanlon $<$ O. Tausky-Todd)

Karola Mészáros (MIT PhD 2010: *Root polytopes, triangulations, and subdivision algebras*, R. P. Stanley)

Avery St. Dizier (Cornell PhD 2020: *Combinatorics of Schubert Polynomials*, K. Mészáros)

Colin W. Crowley (U Wisconsin PhD 2023: ?, B. Wang)

Matt Larson (Stanford PhD 2024: *K-theoretic positivity for wonderful varieties and matroids*, J. Huh & R. Vakil)

Connor Simpson (U Wisconsin PhD 2024: ?, B. Wang)

Lorentzian polynomials

Provide a link between combinatorics and KP's

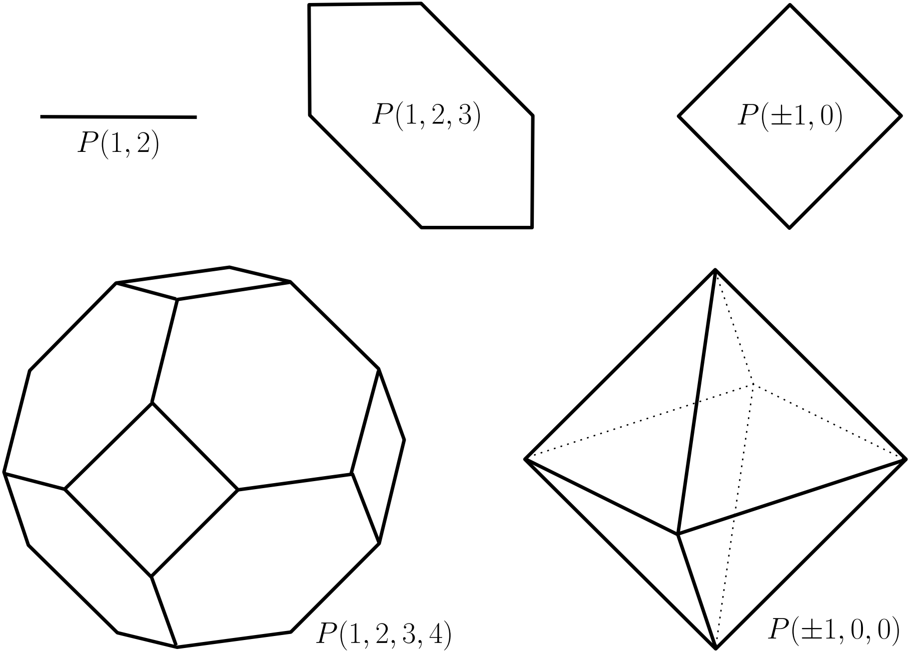
The following quotation from [10, page 4] reveals interesting aspects of its author research temper (emphasis not in the source):

> The known proofs of the Poincaré duality, the hard Lefschetz property, and the Hodge–Riemann relations for the objects listed above have certain structural similarities, but there is no known way of deducing one from the others. *Could there be a Hodge-theoretic framework general enough to explain this miraculous coincidence?*
>
> A related goal is to produce a flexible analytic theory that would reflect certain basic features of the unified theory: If one postulates the existence of the satisfactory cohomology $A(X)$, what can we say about X at an elementary and numerical level? *This is a worthwhile question because, depending on X, the construction and the study of $A(X)$ might be beyond the reach of our current understanding*. A step in this direction is taken in a joint work with Petter Brändén [namely [11]], where the difficult goal of finding $A(X)$ is replaced by an easier goal of producing a Lorentzian polynomial from X. *Such a Lorentzian polynomial can be used to settle and generate conjectures on various* X (Section 2) and, *sometimes, leads to a satisfactory theory of* $A(X)$ (Section 3).

A *generalized permutahedron* is a polytope in $\mathbb{R}^E$ (E a finite set) all of whose edges are in the direction $e_i - e_j$ for some $i, j \in E$ ($\{e_j\}_{j\in E}$ denotes the standard basis of $\mathbb{R}^E$), and it is said to be *integral* if its vertices belong to the lattice $\mathbb{Z}^E$.

Examples: the *standard permutahedron* $P(1, 2, \ldots, n)$, which is the convex hull of all the permutations of $(1, 2, \ldots, n)$, and the *hyperoctahedron* $P(\pm 1, 0, \ldots, 0)$, which is the convex hull of all the permutations of $(\pm 1, 0, \ldots, 0)$.

According to [12], *all generalized permutahedra in $\mathbb{R}^n$ are obtained from the standard permutahedron by moving its vertices so that all the edge directions are preserved*. See also [13].


$P(1,2)$
$P(1,2,3)$
$P(\pm 1, 0)$
$P(1,2,3,4)$
$P(\pm 1, 0, 0)$

We can identify the subsets of a finite set E with $\mathbb{B}^E$ (also called zero-one vectors). A subset J of the lattice $\mathbb{Z}^E$ is said to be *M-convex* if it is the set of lattice points of an integral generalized permutohedron. A finite set $J \subset \mathbb{N}^E$ is M-convex when it satisfies the *symmetric basis exchange property*: For any $\alpha, \beta \in J$ and an index i such that $\alpha_i > \beta_i$, there is an index j such that $\alpha_j < \beta_j$ and $\alpha - e_i + e_j, \beta + e_i - e_j \in J$.

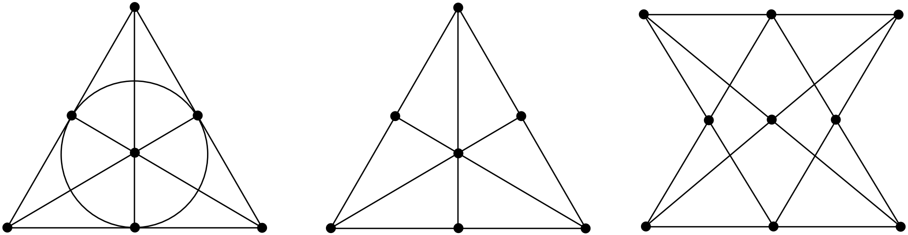
Examples: given a positive integer d, the set of vectors in $\alpha \in \mathbb{N}^E$ (or $\alpha \in \mathbb{B}^E$) such that $\sum \alpha_i = d$.

Importantly, *matroids on the set E can be defined as M-convex subsets of $\mathbb{B}^E$* (see [10]).

The set $J_G \subseteq \mathbb{B}^E$ of spanning trees of a graph $G = (V, E)$ is a matroid (in other words, it is the set of lattice points of an integral generalized permutahedron).

G  $\Rightarrow J_G = \{$  $\}$

Let $\mathcal{L}$ be a geometric lattice of rank d and $E = \mathcal{L}^1$ the set of points (or atoms) of $\mathcal{L}$ (elements of rank 1). Let $J_{\mathcal{L}} \subset \mathbb{B}^E$ be the set of bases of $\mathcal{L}$ (recall that a basis is a set of d points whose join has rank d). Then $J_{\mathcal{L}}$ is the set of lattice points of an integral generalized permutahedron.



Three graphical geometries which are not graphic. The first, the Fano plane, has 7 points, 7 lines and $\binom{7}{3} - 7 = 28$ bases and it is realizable over $\mathbb{F}$ if and only if char($\mathbb{F}$) is 2. The second, has 7 points, 6 lines, and 29 bases, and is realizable over $\mathbb{F}$ if and only the char($\mathbb{F}$) is not 2. The third has 9 points, 8 lines, and 84 bases, and it is not realizable over any field. See [14, Prop. 6.4.8] and [15].

Let H_n^d be the space of real homogeneous polynomials of degree d in n variables.

The set of *Lorentzian polynomials* $L_n^d \subseteq H_n^d$ is defined as follows.

The elements of L_n^2 are specified by two conditions:
(a_2) their coefficients are non-negative, and
(b_2) their signature has at most one positive sign.

For degrees $d > 2$ the set L_n^d is defined recursively by the following conditions:
(a_d) $\partial_j f \in L_n^{d-1}$ for all $j \in [n]$, and
(b_d) the set of (exponents of) monomials of f is the set of lattice points of an *integral generalized permutahedron*

One of the crucial results in [[17]] [11] Brändén, Huh (2020) is that L_n^d is the closure of $\mathring{L}_n^d$, a set defined by the conditions:
($\mathring{a}_2$) their coefficients are *positive* real numbers,
($\mathring{b}_2$) their signature has *exactly* one positive sign, and, for $d > 2$,
($\mathring{a}_d$) $\partial_j f \in \mathring{L}_n^{d-1}$ for all $j \in [n]$.

Theorem 2.28 of the same paper proves that the compact set $\mathbb{P}L_n^d \subset \mathbb{P}H_n^d$ is contractible, with contractible interior $\mathbb{P}\mathring{L}_n^d$, and conjectured that it is homeomorphic to a closed Euclidean ball (proved by Brändén [[16]] [16] Brändén (2021)).

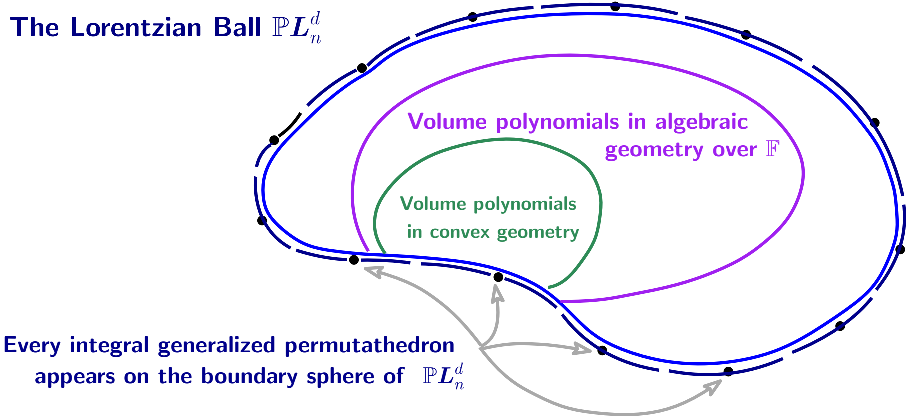
Example. If $C = C_1, \ldots, C_n$ are convex bodies in $\mathbb{R}^d$, $\mathrm{vol}_C : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}$, $w \mapsto \frac{1}{d!}\mathrm{vol}(w_1 C_1 + \cdot + w_n C_n)$ is a Lorentzian polynomial [10] Huh (2022), Example 6.

Example. Let $D = D_1, \ldots, D_n$ be nef Cartier divisors on d-dimensional irreducible projective variety X over an algebraically closed field. Consider the polynomial function

$$\mathrm{vol}_D : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}, \quad w \mapsto \frac{1}{d!}\deg(w_1 D_1 + \cdot + w_n D_n)^d.$$

If X admits a resolution of singularties Y and the Hodge-Riemann relations hold in degree $\leqslant 1$ for the ring of algebraic cycles $A(Y)$, then $\mathrm{vol}_D(w)$ is Lorentzian [10] Huh (2022), Example 7.

The Lorentzian Ball $\mathbb{P}\boldsymbol{L}_n^d$



This figure intends to be a reproduction of slide number 13 of Huh's lecture at the ICM-22. Note the statement on the boundary sphere.

The general strategy was summarized in slide number 14 of [17] Huh (2022), while pointing out [[40]] [18] Huh, Matherne, Meszaros, Stdizier (2022) and [[27]] [19] Eur, Huh (2020) for examples and conjectures for various X:

(1) Given X, search for interesting multivariate generating functions from it;

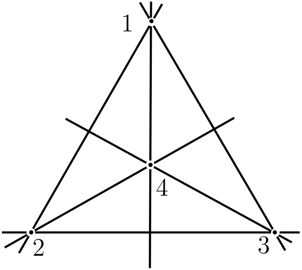
(2) Do we see any generalized permutohedra?

(3) Do we see any Lorentzian polynomials?

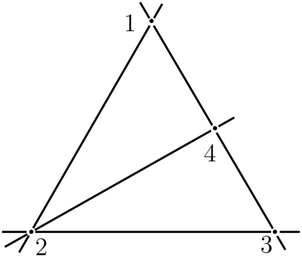
(4) Can we guess $A(X)$, $K(X)$, $P(X)$?

Let us end by describing how the Dowling–Wilson conjecture was solved, after [17] Huh22-lecture.

Given a geometric lattice $\mathcal{L}$ of rank d, consider the set $\mathcal{B}$ of its *bases*, that is, subsets of size d of $E = \mathcal{L}^1$ (the set of atoms) whose join has rank d. Then $\mathcal{B}$ is the set of *lattice points of an integral generalized permutohedron*, and the basis generating function $g = \sum_{\nu \in \mathcal{B}} w^\nu$ is a Lorentzian polynomial.



$g = w_1w_2w_3 + w_1w_2w_4 + w_1w_3w_4 + w_2w_3w_4$



$g = w_1w_2w_3 + w_1w_2w_4 + w_2w_3w_4$

Now define $\mathbf{H}(\mathcal{L}) = \{f : \mathcal{L} \to \mathbb{Q}\} = \bigoplus_{F\in\mathcal{L}} \mathbb{Q}\delta_F$ and make it a graded $\mathbb{Q}$-algebra (the *Möbius algebra* of $\mathcal{L}$) with the multiplication determined by

$$\delta_F \cdot \delta_{F'} = \begin{cases} \delta_{F\vee F'} & \text{if } r(F \vee F') = r(F) + r(F') \\ 0 & \text{otherwise.} \end{cases}$$

The *bases generating function* of $\mathcal{L}$ is $\frac{1}{d!}(\sum_{j\in E} w_j\delta_j)^d$. This suggests taking:

- $A(\mathcal{L}) = \mathbf{H}(\mathcal{L})$;
- $K(\mathcal{L})$, the set of multiplications by positive linear combinations of the δ_j; and
- $P(\mathcal{L})$, multiplication in $\mathbf{H}(\mathcal{L})$ composed with $\mathbf{H}^d(\mathcal{L}) \simeq \mathbb{Q}$.

But $\mathbf{H}(\mathcal{L})$ already fails to satisfy Poincaré duality, for $\dim \mathbf{H}^j(\mathcal{L}) = |\mathcal{L}^j|$ and in general $|\mathcal{L}^j| \neq |\mathcal{L}^{d-j}|$.

As shown in [[12]] [6] Braden, Huh, Matherne, Proudfoot, Wang (2020), the rescue from this failure came from the *intersection cohomology* of $\mathcal{L}$, $\mathbf{IH}(\mathcal{L})$, which is an indecomposable graded $\mathbf{H}(\mathcal{L})$-module endowed with a map $P : \mathbf{IH}(\mathcal{L}) \to \mathbf{IH}(\mathcal{L})^*$ that satisfies the following properties for every $j \leqslant d/2$ and every $L \in K(\mathcal{L})$:

Poincaré duality $P : \mathbf{IH}^j(\mathcal{L}) \to \mathbf{IH}^{d-j}(\mathcal{L})^*$ is an isomorphism;
Hard Lefschetz $L^{d-2j} : \mathbf{IH}^j(\mathcal{L}) \to \mathbf{IH}^{d-j}(\mathcal{L})$ is an isomorphism; and
Hodge-Riemann relations: The pairing $\mathbf{IH}^j(\mathcal{L}) \times \mathbf{IH}^j(\mathcal{L}) \to \mathbb{Q}$, $(x, y) \mapsto (-1)^j P(x, L^{d-2j} y)$ is positive definite on the kernel of L^{d-2j+1}. In addition, $\mathbf{IH}^0(\mathcal{L})$ generates a submodule isomorphic to $\mathbf{H}(\mathcal{L})$.

The construction relies on the resolution of singularities of algebraic varieties, and in particular on [20] Concini, Procesi (1995) 'wonderful models' (see [21] Concini, Procesi (2010), a wonderful book).

Since the composition of $\mathbf{H}^j(\mathcal{L}) \hookrightarrow \mathbf{IH}^j(\mathcal{L})$ with the Hard-Lefschetz isomorphism $\mathbf{IH}^j(\mathcal{L}) \simeq \mathbf{IH}^{d-j}(\mathcal{L})$ is injective, it follows that $L^{d-2j} : \mathbf{H}^j(\mathcal{L}) \to \mathbf{H}^{d-j}(\mathcal{L})$ composed with $\mathbf{H}^{d-j} \to \mathbf{IH}^{d-j}(\mathcal{L})$ is injective (see diagram below) and consequently $L^{d-2j} : \mathbf{H}^j(\mathcal{L}) \to \mathbf{H}^{d-j}(\mathcal{L})$ is injective, which proves that $|\mathcal{L}^j| \leqslant |\mathcal{L}^{d-j}|$.

$$\begin{array}{ccc} \mathbf{H}^j(\mathcal{L}) & \hookrightarrow & \mathbf{IH}^j(\mathcal{L}) \\ L^{n-2j} \downarrow & & \downarrow L^{n-2j} \\ \mathbf{H}^{d-j}(\mathcal{L}) & \to & \mathbf{IH}^{d-j}(\mathcal{L}) \end{array}$$

- Find applications of HR_q for $q > 1$.
- Lefschetz properties in algebraic contexts: [23], [24]:
- There are multiple feedbacks from the discrete to the continuous, for example in the form of combinatorial approaches to algorithmic learning, like *graph learning* in general and *manifold learning* in particular: [25] (and the references mentioned there) and the surveys [26] and [27].
- Please send corrections and suggestions to sebastia.xambo@upc.edu

Thank you!

References I

[1] A. Grothendieck, "Standard conjectures on algebraic cycles," 1969.
In [28], 193-199; in the pdf, 203-211.

[2] J. Huh, "Combinatorial applications of the hodge–riemann relations," in *Proceedings of the International Congress of Mathematicians: Rio de Janeiro 2018*, pp. 3093–3111, World Scientific, 2018.
arXiv pdf.

[3] K. Karu, "Hard Lefschetz theorem for nonrational polytopes," *Inventiones Mathematicae*, vol. 157, no. 2, pp. 419–447, 2004.
arXiv pdf (2002, v4).

References II

[4] K. Adiprasito, J. Huh, and E. Katz, "Hodge theory for combinatorial geometries," *Annals of Mathematics*, vol. 188, no. 2, pp. 381–452, 2018.

arXiv pdf, v2 2018.

[5] F. Ardila, G. Denham, and J. Huh, "Lagrangian geometry of matroids," *Journal of the American Mathematical Society*, 2022.

JAMS pdf.

[6] T. Braden, J. Huh, J. P. Matherne, N. Proudfoot, and B. Wang, "Singular Hodge theory for combinatorial geometries," 2020.

arXiv pdf.

References III

[7] B. Elias, S. Makisumi, U. Thiel, and G. Williamson, "The Hodge theory of Soergel bimodules," *Annals of Mathematics*, pp. 1089–1136, 2014.

[8] N. Libedinsky, "Gentle introduction to Soergel bimodules I: the basics," *Sao Paulo Journal of Mathematical Sciences*, vol. 13, pp. 499–538, 2019.

[9] B. Elias, S. Makisumi, U. Thiel, and G. Williamson, *Introduction to Soergel bimodules*, vol. 5 of *RSME Springer Series*.

Springer, 2020.

xxv+588p.

References IV

[10] J. Huh, "Combinatorics and Hodge theory," in *Proceedings of the ICM 2022*, IMU, 2022.

pdf.

[11] P. Brändén and J. Huh, "Lorentzian polynomials," *Annals of Mathematics*, vol. 192, no. 3, pp. 821–891, 2020.

[12] A. Postnikov, "Permutohedra, associahedra, and beyond," *International Mathematics Research Notices*, vol. 2009, no. 6, pp. 1026–1106, 2009.

[13] J. S. Doker, *Geometry of generalized permutohedra*.

PhD thesis, UC Berkeley, 2011.

pdf.

References V

[14] J. G. Oxley, *Matroid theory (2nd edition)*, vol. 21 of *Oxford Graduate Texts in Mathematics*.

Oxford University Press, 2011.

xiv+684p. The 1st edition was published in 1992.

[15] A. W. Ingleton, “Representation of matroids,” in *Combinatorial mathematics and its applications* (D. J. A. Welsh, ed.), pp. 149–167, Academic Press, 1971.

[16] P. Brändén, “Spaces of Lorentzian and real stable polynomials are Euclidean balls,” in *Forum of Mathematics, Sigma*, vol. 9, p. e73, Cambridge University Press, 2021.

Σ/CUP pdf.

References VI

[17] J. Huh, "Combinatorics and Hodge theory," 2022.

ICM Fields Lecture, 6 July 2022 Video (2:34–3:37).

[18] J. Huh, J. Matherne, K. Mészáros, and A. St Dizier, "Logarithmic concavity of Schur and related polynomials," *Transactions of the American Mathematical Society*, vol. 375, no. 06, pp. 4411–4427, 2022.

TAMS pdf.

[19] C. Eur and J. Huh, "Logarithmic concavity for morphisms of matroids," *Advances in Mathematics*, vol. 367, p. 107094, 2020.

arXiv pdf.

References VII

[20] C. De Concini and C. Procesi, “Wonderful models of subspace arrangements,” *Selecta Mathematica*, vol. 1, pp. 459–494, 1995.

[21] C. De Concini and C. Procesi, *Topics in hyperplane arrangements, polytopes and box-splines*.

Universitext, Springer Science & Business Media, 2010.

[22] M. Michalek, “Enumerative Geometry Meets Statistics, Combinatorics, and Topology,” *Notices of the AMS*, vol. 70, no. 4, 2023.

NAMS pdf.

References VIII

[23] J. Migliore and U. Nagel, "Survey article: a tour of the weak and strong Lefschetz properties," *J. Commut. Algebra*, vol. 5, no. 3, pp. 329–358, 2013.

[24] T. Harima, T. Maeno, H. Morita, Y. Numata, A. Wachi, J. Watanabe, T. Harima, T. Maeno, H. Morita, Y. Numata, *et al.*, *Lefschetz properties*.

No. 2080 in Lecture Notes in Mathematics, Springer, 2013.

[25] S.-T. Yau, "Manifold fitting, an invitation to machine learning –a mathematician's view," 2024.

Lecture at the 2024 b-M2L seminar html, organized by Roberto Rubio. slides pdf.

References IX

[26] Z. Wu, S. Pan, F. Chen, G. Long, C. Zhang, and S. Y. Philip, "A comprehensive survey on graph neural networks," *IEEE transactions on neural networks and learning systems*, vol. 32, no. 1, pp. 4–24, 2020.

[27] F. Xia, K. Sun, S. Yu, A. Aziz, L. Wan, S. Pan, and H. Liu, "Graph learning: A survey," *IEEE Transactions on Artificial Intelligence*, vol. 2, no. 2, pp. 109–127, 2021.

[28] M. S. Narasimhan, ed., *Algebraic Geometry*, Tata Institute Studies in Mathematics, Oxford University Press & Tata Institute of Fundamental Research, 1969.

WS Introduction

Since the seminal works of the MIT professors Gian-Carlo Rota in the 1960's, and later Richard Stanley, combinatorics has become an important branch of modern mathematics where interplay of different areas arise. In particular, the Rota-Hero-Welsh's Conjecture, proposed by Gian-Carlo Rota in the 1960s for graphs and later by Hero and Welsh for matroids, has long stood as one of the most tantalizing problems in combinatorics and algebraic geometry. It posited a deep connection between the combinatorial structure of matroids and the algebraic structure of projective varieties. Specifically, it conjectured that certain relations, now known as the Hodge Riemann relations, hold true for the intersection numbers of divisors on a projective variety defined by the matroid. Despite decades of effort from mathematicians around the world, the conjecture remained stubbornly unresolved, serving as a formidable challenge at the intersection of these two fields.

June Huh's initial breakthrough came with his groundbreaking proof of Rota's Conjecture [1], achieved through a novel synthesis of techniques from algebraic geometry, combinatorics, and homological algebra. His work unveiled the intricate connections between combinatorial objects such as matroids and algebraic varieties, providing a deep understanding of the underlying structures governing their intersection. Through meticulous analysis and innovative reasoning, Huh demonstrated that the Hodge Riemann relations indeed hold true, thereby resolving a decades-old mystery and opening up new avenues for exploration in combinatorial algebraic geometry.

Subsequently, Huh's collaboration with Karim Adiprasito and Eric Katz [2] further illuminated the rich mathematical landscape that emerged from their initial breakthrough. Together, they delved into the study of positivity for matroids and polytopes, uncovering deep connections between combinatorial structures and algebraic geometry. Their research yielded not only elegant proofs of fundamental results but also novel mathematical objects that bridge the gap between discrete and continuous mathematics. Through their collaborative efforts, they uncovered new perspectives on toric varieties, matroid theory, and tropical geometry, among other areas.

On another direction it has become nowadays more and more popular formal logic systems used to formalize mathematical proofs. In this context, LEAN is getting more and more popular due to its is versatile, allowing for a wide range of applications and uses. This tool is growing day by day and we believe it would be good to show to a wide mathematical audience the state of the art of this tool.

Anna de Mier: An introduction to matroid theory

Matroids originated in the 1930's as an abstraction of the notion of linear independence in vector spaces. Whereas matrices and graphs provide the first examples and were the motivation for the developing of the field, matroid theory has many connections to other areas, such as matchings, optimization, geometry or criptography. In this talk we will give an overview of matroids, focusing on those aspects that justify their alternative name, albeit not so common, of combinatorial (pre)geometries.

One special feature of matroids is that they have several equivalent definitions. We will go over the main ones (bases, indepedents sets, flats, rank, circuits. . .), and then we will survey the basic operations (minors and duality) and present some important classes of matroids. We will then move to matroid polynomials, with an emphasis on the characteristic polynomial and its properties.

No previous knowledge of matroid theory will be assumed.

Julian Pfeifle: Matroid polytopes, tropical geometry, and log-concavity

We continue the previous talk and examine some geometric incarnations of a matroid M, namely matroid polytopes, Bergman fans, and tropical linear spaces. More algebraically, the rational cohomology of the complement of a certain hyperplane arrangement associated to M is called the Orlik-Solomon algebra, whose Poincaré polynomial is related to the characteristic polynomial of M.

We hope that these examples will build some intuition for understanding the essence of Huh–Adiprasito–Katz's proof of the log-concavity of the coefficients of this characteristic polynomial, namely that the tropical variety or Bergman fan associated to a matroid has the structure of the cohomology ring of a smooth projective variety.

Souvik Goswami: Survey of Chow groups

The study of algebraic cycles is at the cornerstone of Algebraic Geometry, and shares common ground with Arithmetic Geometry, Algebraic K-theory, and Analytic geometry. The definition of the free group of algebraic cycles is easy to understand, and can be generalized to different categories. The non-trivial part is to define a suitable equivalence relation, so that the quotient group becomes equipped with a ring structure. Such an equivalence relation is provided by rational functions on subvarieties, and the resultant quotient is defined as the Chow group attached to an algebraic variety.

The purpose of this talk is to give an introduction and survey of this topic. More aligned towards the topic of this conference, I will introduce Chow group of a matroid.

Sebastià Xambó: Kähler packages and their combinatorial significance

The notion of Kähler Package will be presented and illustrated with several examples, particularly those introduced by June Huh and collaborators with which they could prove various long-standing combinatorial conjectures, but also including its first appearance in Grothendieck's standard conjectures —a scheme to approach the Weil conjectures about algebraic varieties defined over a finite field. Then the theory of Lorentzian polynomials will be considered, with emphasis on its role as a bridge to Kähler packages and on how it is used to settle or suggest conjectures. In the final part, a sample of recent contributions in closely related themes will be discussed.

María Inés de Frutos Fernández: Algebra in the Lean mathematical library

We will start by introducing the interactive theorem prover Lean and its mathematical library Mathlib. We will next give an overview of the algebra hierarchy in the Mathlib library (how groups, rings, modules, etc. are formalized, and the dependencies between these objects). Finally, we will discuss the formalizations of several topics that appear in Huh's work, including polynomial rings and graded algebras.

Yaël Dillies: Combinatorics in Mathlib and beyond

As the main Lean library of mathematics, Mathlib acquires new formalisations at an increasing pace. Curiously, one area of mathematics is lagging behind: combinatorics.

I will first present the combinatorics that Mathlib does have (basic graph theory, the regularity lemma, set families) before touring significant projects outside of Mathlib (discrete Fourier analysis, matroid theory, linear programming). Finally, I will offer some explanations on why Mathlib is so poor at incorporating combinatorics formalisations and how you can help.

Riccardo Brasca: Geometry in mathlib

We will give an overview of the status of geometry (both algebraic and analytic) in mathlib, the official mathematical library of Lean. We will explain what is already in mathlib and what is missing, focusing on what can be added rather easily. We will also explain what are the difficulties in formalizing geometric notions following mathlib's philosophy of being as general as possible and we talk about the solutions found by the mathlib community.