Projects for the “Mathematical Research 2024 Summer Program”

ANALYSIS, PDE, & DYNAMICAL SYSTEMS

Xavier Cabré ***The existence theorem of minimal surfaces (or soap films)***
The student will be able to follow all details in a proof of the existence of solution to the Plateau's problem. This consists of proving the existence of a minimal surface (also called a "soap film") spanning a given boundary (that is, spanning a closed "wire" in space). Even that this is a central and advanced theorem of Analysis, we will be able to understand a rather elementary (but deep and beautiful) proof presented in the Lecture Notes on minimal surfaces of M. Cozzi and A. Figalli. This will bring us to consider similar ideas for the simplest existence proof for the Dirichlet problem associated to Poisson's equation $-\Delta u = f(x)$ in a domain of Euclidean space. All together will serve as an instructive introduction to important tools from the Calculus of Variations, Functional Analysis, and Geometric Measure Theory.

Andrew Clarke*:* ***Nonexistence of invariant curves in convex planar billiards***
Billiards are a family of dynamical systems that were invented by Birkhoff a century ago. They concern the motion of a particle inside a bounded region, where the motion in the interior is along straight lines, and the collisions with the boundary are determined by the optical reflection law. They have been studied extensively (especially in the last 50 years), used to model many different physical phenomena (in celestial mechanics, thermodynamics, the Lorentz gas etc.), and are the subject of many important open problems and conjectures in dynamical systems theory (for example the Birkhoff conjecture, the Ivrii conjecture, “Can one hear the shape of a drum?”). In his famous paper from 1973, Lazutkin proved the existence of a family of invariant curves called caustics for all sufficiently smooth and strictly convex billiards. An important question is: what happens when we loosen the conditions of Lazutkin’s theorem? It has been proved by Mather that if the boundary has a flat point then there are no caustics. Likewise, Hubacher established the same conclusion if there is a point where the boundary is only C^1. The goal of this project is to gain an understanding of the billiard dynamics and caustics, and to read and comprehend the papers of Mather and Hubacher.

Gyula Csato: ***The Riesz Rearrangement Inequality***
The goal is to learn some basic concepts on rearrangements, which is a fundamental tool for finding optimal shapes and for solving many questions in analysis, geometry, and partial differential equations. The focus will be on the Riesz rearrangement inequality and the aim is to understand at least one proof in 1 and one proof in 2 dimensions. If there is time and interest, we will also think about possible formulations on the sphere, which is still an open problem. This project requires only some basic knowledge in differential and integral calculus of one and several variables.

Kostya Drach ***Reverse isoperimetric inequalities in the hyperbolic space***
The isoperimetric problem is probably one of the oldest active areas of research in mathematics. The classical isoperimetric problem in the Euclidean plane asks to find a closed curve of given length enclosing the maximal possible area. The answer, known already to the Ancient Greeks, is a circle; many generalizations of this problem have proved to be of fundamental importance in various areas of mathematics. Instead of maximal area, the reverse isoperimetric problem asks to find the minimal possible area. This is a highly non-trivial problem that is well-posed, for example, under some additional assumptions on the curvature of the curve. The aim of the project is to discuss possible assumptions. The theory of reverse isoperimetric problems under curvature constraints has many open questions. An immediate goal of the project would be to attack one of them in the setting of curves in the hyperbolic plane, and possibly to go further. The project requires only a little knowledge of the 2-dimensional hyperbolic geometry.

Kostya Drach **_Length spectral rigidity for Blaschke products_**
A finite-degree Blaschke product is a holomorphic self-map of the open unit disk that frequently appears in one-dimensional complex dynamics as a local model of the dynamics in the Fatou components. Restricted to the unit disk, such Blaschke products behave like expanding maps of the circle – objects studied in one dimensional real dynamics. In this project, we will draw a third parallel – between expanding maps of the circle (that are restrictions of Blaschke products) and closed hyperbolic surfaces of constant negative curvature -1. In both of the setups, one can give a meaning to the term "the length of a closed geodesic line". The main question, still open in many contexts, is whether the set of lengths of closed geodesics define the object (the surface, or the expanding map) up to some change of coordinates. The goal of the project would be to show that the answer is "yes" for (most of the) expanding maps coming from Blaschke products. Along the way, all relevant concepts, both geometric and dynamical, will be discussed.

Gissell Estrada **_Overdamped limit of the Langevine equation and its mean-field limit_**
In the last decades, the behaviour of large particle systems and their mean-field limits have been intensively studied. Of great interest are the simultaneous attractive and repulsive interactions in flocks of fish, swarms of birds or herds of sheep. We start by studying the behaviour of particles acting according to the Langevin equation. We will discuss how they behave under different frictions and, in doing so, we will first direct our focus to the one-particle case. Our emphasis will be on the overdamped case, particularly highlighting its significance. Subsequently, we will investigate the behaviour of friction on the basis of several particles by looking at the mean-field limit. In particular, we will look at how the particles interact with each other and whether and how this affects the potential field on which the Langevine equation is based. This project involves knowledge of ordinary and partial differential equations. It may also involve numerical simulations to compare the dynamics for different damping factors.

Anna Jové **_Exponential dynamics are chaotic_**
We will study the discrete complex dynamical systems given by the iterates of the exponential map f(z)=e^z. As it was conjectured by Fatou in 1926 and proved by Misiurewicz 55 years later, such a dynamical system is chaotic, in the sense that the eventual behavior of a point is impossible to predict numerically. More precisely, the Julia set of the exponential is the whole complex. To prove this, we will follow a recent approach, introduced by Shen and Rempe-Gillen, based on properties of the hyperbolic metric in some slit domain. Can this proof be adapted for other transcendental maps?

Alberto Maione: **_Gamma-convergence and applications_**
The Gamma-convergence was introduced in 1975 by Ennio De Giorgi and Tullio Franzoni and occupies a prominent place in the world of variational convergences by its application in applied analysis. Its importance is underlined by the fact that almost all other notions of convergence can be expressed in its language. In this project, we will first devote ourselves to the theoretical study of Gamma-convergence, through the famous monographies by Dal Maso and Braides and some notes written by the lecturer. In particular, we will study a proof of the so-called "Fundamental theorem of Gamma-convergence", by De Giorgi and Franzoni. Subsequently, we will apply this theoretical study to the understanding of recent applications of Gamma-convergence, ranging from applications in the theory of elliptic partial differential equations, to homogenisation problems in the mathematical theory of composite materials and to the study of phase transitions. These lectures are recommended for students in mathematics who wish to specialise in the field of analysis.

Martí Prats: **_Composition operator using quasiconformal mappings_**
When pre-composing with a bi-Lipschitz mapping, the integrability properties of a function are preserved. Also the integrability properties of its derivatives. If the mapping is smooth enough, then

the same will happen with higher order derivatives. When the composition operator is less than bi-Lipschitz, the situation is more delicate, and one expects a loss of integrability of the mapping and its derivatives. We will study a couple of articles regarding this situation for Lipschitz and quasiconformal mappings and, time permitting, we will try to provide new proofs of known results (and perhaps some unknown ones).

Rafael Ramírez Ros*:* ***Billiards***
A particle moves following straight lines inside a planar smooth convex domain and when the particle hits the boundary, it reflects in such a way that the angle of incidence just before the collision is equal to the angle of reflection just after the collision. G. D. Birkhoff introduced this problem almost 100 years ago. He claimed that in this problem the formal side, usually so formidable in dynamics, almost completely disappears, and only the interesting qualitative questions need to be considered. B. H. Neumann and J. K. Moser introduced outer billiards as a toy model for Celestial Mechanics in the late 1950s. There are many more types of billiards: symplectic billiards, projective billiards, wire billiards, coin billiards, high-dimensional billiards. Nowadays, billiards have become a popular subject in Dynamical Systems. More than 100 preprints with the word billiard in the abstract have been uploaded to arXiv just in 2023. The goal of this project is to learn some basic properties of some billiards. The choice of properties and/or billiards will be based on the student's interests: numerical computations, analysis, geometry, integrable dynamics, nearly integrable dynamics, chaotic dynamics.

Xavier Ros Oton ***Elementary proof of the isoperimetric inequality***
Among all sets in the plane with fixed area, which one has the smallest perimeter? The answer to such a classical question, and its corresponding generalization to higher dimensions, is the so-called isoperimetric inequality. We propose to discuss an elementary proof of such inequality in 2D (requiring almost no prerequisites), to then study modern proofs in higher dimensions and their connections to other areas of Analysis.

Xavier Ros Oton ***Eigenvalues of the Laplacian***
The Dirichlet problem for the Laplacian and the corresponding eigenvalue problem arise in a variety of contexts, both within mathematics (Analysis, Geometry, and Probability) or in other sciences (especially Physics). In this project we propose to study the proof of existence of eigenvalues, their basic properties, and their connections to other areas of Mathematics.

Olli Saari: ***Fourier restriction theory***
Fourier restriction theory is a part of harmonic analysis that studies norm inequalities for surface carried measures. The goal of the project is to study the basic principles and tools in restriction theory and get acquainted with a well-known open problem called the local smoothing conjecture and in particular its two dimensional case that was recently resolved.

Odí Soler: ***The High indices theorem***
Abel's Theorem is a classical result relating the asymptotics of a power series and the sum of its coefficients. In particular, for power series with radius of convergence 1 it holds that if the series of the coefficients converges to S, then the power series has nontangential limit at point z = 1 equal to S. Even though one cannot have a full converse to this result, there are several partial converses, called tauberian theorems. The goal of this project is to study the High Indices Theorem, which is the converse to Abel's Theorem for lacunary power series. Depending on time and interest, variations with different sets of sufficiency conditions will be considered.

Clara Torres Latorre ***Brunn-Minkowski and related geometric inequalities***

The Brunn-Minkowski is a classical inequality for sets in Euclidean spaces which plays a central role in convex analysis and related topics. We propose to study the proof of such inequality, its relation to other geometric and functional inequalities, as well as some generalizations and open problems in this context.

ALGEBRA, GEOMETRY, AND NUMBER THEORY

Francesc Bars: ***On automorphism group of the curves given by introducing torsion of a Drinfeld module***

Finite separable field extensions of the field of functions of the polynomial ring in one variable over a finite field could be considered as curves defined over a finite field. Such curves have a lot of interesting properties. The project consists on understanding the relation between fields of transcendence degree 1 and algebraic curves, learning rank-d Drinfeld modules, begin the study of the field extensions obtained by adjoining the torsion of such Drinfeld modules (an analog of adjoining roots of unity, or torsion of elliptic curves for finite extensions of the rationals), and study the automorphism group for such field extensions, as a curve over a finite field.

Robert Cardona ***Open book decompositions of three-manifolds***

Three-dimensional manifolds admit a topological structure named "open book decomposition". This is a beautiful way of decomposing any three-manifold into the union of simpler pieces, such as surfaces and solid tori. These decompositions are relevant to different areas of geometry and topology. After understanding open book decompositions, and depending on the student's interests, their applications to different topics can be discussed: continuous dynamical systems, geometric topology or contact geometry. Time permitting, open problems and alternative proofs can be introduced. The only requirement for this project is the understanding of the notion of manifold.

Robert Cardona ***Topological methods in hydrodynamics***

The study of hydrodynamics is a very rich subject at the crossroads of different areas of mathematics. Geometrical and topological methods proved very useful in the last decades in studying the qualitative properties of certain fluid equations. In this project, we will introduce the Euler equations, which model the evolution of an ideal fluid, and study them from a geometrical point of view. Depending on the student's interests, relations to symplectic geometry, dynamical systems, knot theory or geometric analysis can be pursued. Time permitting, open problems can be introduced. Basic knowledge in differential geometry is required.

Carlos d'Andrea: ***Positive polynomials as a sum squares***

Let $p(t) \in R[t]$, a univariate polynomial, and assume that $p(t) \geq 0$ for all $t \in R$. It is known and easy to prove that then there exist $p_1(t), p_2(t) \in R[t]$ such that $p(t) = p_1^2(t) + p_2^2(t)$.
The situation over $Q[t]$ is quite interesting. For instance, the polynomial $q(t) = t^2 + 3 \in Q[t]$ is always positive but cannot be written as a sum of two squares. However, it can be written as a sum of four squares: $q(t) = t^2 + 1^2 + 1^2 + 1^2$. It has been proven more than a 100 years ago by Landau that every polynomial in $Q[t]$ which is non-negative is a sum of 8 squares of rational polynomials. This result was improved by Pourchet, lowering the bound to the optimal value of 5.
The main goal of this project is reading and understanding the proof of Pourchet, with the idea of trying to characterize which are the polynomials in $Q[t]$ that can be described as the sum of two/three/four/five squares. Over the integers, we have that an integer m is a sum of two squares if and only if in its factorization there are no primes congruent to 3 modulo 4. It would be interesting to find a characterization like this for polynomials in $Q[t]$ and/or $Z[t]$.

Another direction of research is to look at what is called the "local situation": the description of the polynomials $f(t) \in Q[t]$ which are positive (or non-negative) over the real roots of another polynomial $q(t) \in Q[t]$ (the "global situation" is recovered when one sets $q(t) = 0$).

Rosa M. Miró Roig **_Lefschetz properties in algebraic geometry, commutative algebra and combinatorics_**

The study of the Lefschetz properties in algebra and geometry was motivated by the hard Lefschetz theorem, a breakthrough in algebraic topology and geometry: For any smooth complex projective variety, the cup product with powers of the hyperplane class yields an isomorphism between the corresponding cohomology classes.

Lefschetz properties of graded artinian algebras are algebraic generalizations of the Hard Lefschetz property of the cohomology ring of a smooth projective complex variety. The investigation of the Lefschetz properties of graded artinian algebras was started in the mid 1980's and nowadays is a very active area of research.

Let $A = R/I$ be a graded artinian algebra, where $R = k[x_1, . . . , x_r]$ and k is a field. We say that A has the Weak Lefschetz Property (WLP, for short) if, for a general linear form L, the multiplication $(\times L) : A_i \to A_{i+1}$ has maximal rank, for all i. We say that A has the Strong Lefschetz Property (SLP, for short) if the same holds for any power L^d.

Though many artinian algebras are expected to have the WLP, establishing this property is often rather difficult and there has been a lot of effort to prove Lefschetz properties for a wide range of classes of Artinian algebras. It is then of interest to pass to important classes of algebras, and ask which among these have WLP or SLP. The most important of these is that of complete intersections and Gorenstein and I propose to start analyzing these two cases.

Joan Porti: **_Hyperbolic geometry_**

After developing the geometric tools of hyperbolic plane and hyperbolic space, the student shall study isometric group actions and the interactions with low dimensional topology and geometric group theory. The main goal is to study how negative curvature of hyperbolic space is reflected in topological properties of the manifolds involved, and/or in algebraic properties of the groups acting by isometries.

Julian Pfeifle: **_Convex hulls in the Hopf fibration_**

The Hopf fibration $S^1 \to S^3 \to S^2$, a non-trivial fiber bundle which presents the 3-sphere as a 2-sphere's worth of circles, is a gemstone at the intersection of differential and discrete geometry. In analogy to the charming paper [1], we propose to study the relationship between Hopf circles, their convex hull, and the convex hulls of their duals. While [1] analyzes the combinatorial types of convex hulls of two circles in R^3, the convex hull of two Hopf circles is always the direct sum of the circles because the planes containing them intersect only in one point --- can you prove this? But what about the convex hull of three circles? And the intersection of three cycloplanes, the convex bodies $D^1 \times R^2$ polar dual to a Hopf circle? Is it true that this intersection is always a circle, and if so, which one is it, in terms of the projection to S^2?

[1] E. Nash, A.F. Pir, F. Sottile, and L. Ying, The Convex Hull of Two Circles in R^3, https://doi.org/10.1007/978-1-4939-7486-3_14

Simone Virili: **_Topological extensions of algebraic dualities_**

Given a field **k**, it is well-known that a **k**-vector space V is naturally isomorphic to its *double dual space* V**:=Hom(Hom(V,**k**),**k**) if, and only if, the dimension of V is finite. The isomorphism is constructed sending each vector v in V to the "evaluation at v". Whenever dim(V) is infinite, this map is still injective, but it fails to be surjective.

On the other hand, there is a standard trick to extend the duality of finite dimensional vector spaces to the infinite dimensional case: for any **k**-vector space V, the *dual space* V*:=Hom(V,**k**) can be naturally endowed with a Hausdorff topology (which is discrete precisely when V is finite dimensional). One can then consider the space CHom(V*,**k**) of continuous and **k**-linear maps from V* (endowed with the mentioned topology) to **k** (endowed with the discrete topology). The following facts then hold true: CHom(V*,**k**)≤V**, and CHom(V*,**k**)=V** if and only if dim(V) is finite; and the canonical morphism from V to V** induces an isomorphism from V to CHom(V*,**k**), independently on the dimension of V.

MATHEMATICAL MODELING AND NUMERICAL SIMULATIONS

Jezabel Curbelo & Álvaro Meseguer: ***From chaotic dynamical systems to fluid turbulence***

In this project, the student will study theoretical and numerical methodologies currently used by applied mathematicians and theoretical physicists to understand transition to turbulence from a deterministic point of view. The project will encompass reading fundamental bibliography on dynamical systems, hydrodynamic stability theory, numerical computation of Navier-Stokes equations (or other nonlinear partial differential equations arising in pattern formation, such as Swift-Hohenberg or Kuramoto-Sivashinsky, for example). In a second stage, the project will also involve hands on computer experimentation to explore complex dynamics (bifurcations, instabilities, etc.), to be later understood from the point of view of dynamical systems theory.

Jezabel Curbelo & Arantxa Alonso: ***Geophysical Fluid Dynamics***

A full understanding of any transitional fluid flow requires analyzing and computing the underlying coherent structures that govern the dynamics. These structures appear in a vast variety of fluid flow problems, and their identification and accurate computation is fundamental because they play a crucial role in the global transport of mass, heat and momentum. Many fluid flows self-organize into regular or coherent patterns of varying spatio-temporal complexity but, in general, geophysical fluid flows are notoriously complex and difficult to predict. The diversity of analysis tools and computational techniques required to study such fluid involves numerical analysis, dynamical systems, mathematical modeling and computational methods. The proposal is focused on the role of coherent structures in geophysical fluid dynamics to understand the natural environment. Depending on the student's interest, it will be focused on an equation-based study of convection motion or a data-based study of atmospheric or oceanic dynamics.

Gissell Estrada: ***Mathematical modelling of opinion consensus and polarisation dynamics***

The use of mathematical models to study opinion formation has been rapidly growing and in particular, the use of kinetic equations. Many social aspects such as compromise, self-thinking, the effects of leaders, or external actions to control the opinion and the conviction of individuals have been included in these models. In this project we start by looking at the classical Hegselman-Krause bounded confidence model that describes the interaction of individuals only when their opinions are already sufficiently closed. Our aim is to study when opinion formation within an interacting group leads to consensus, polarisation or fragmentation. We are going to study the different scenarios by analytical methods as well as by numerical simulations. This project involves basic knowledge on dynamical systems.

Matteo Giacomini & Antonio Huerta. ***Towards artificial micro-swimmers: modelling and simulation of locomotion at small scale***

Personalised medicine treatments increasingly depend upon the development of innovative engineering solutions. In this context, understanding the swimming patterns of biological organisms at the micro- and nano-scale is essential to design artificial micro-robots. Biological micro-swimmers achieve self-propulsion by means of large geometric deformations of their bodies. To predict the *behaviour of these organisms and appropriately design artificial micro-swimmers, Mathematical modelling and numerical simulations are required. This project aims to study the effect of employing increasingly complex PDE models (Stokes, Oseen, and Navier-Stokes equations) to describe the incompressible flow past micro-swimmers at low Reynolds number and to assess them by means of numerical simulations.*

Roser Homs: ***Can algebra tell us whether we have enough data?***
Gaussian graphical models are statistical models in which a graph encodes conditional independence relations among the random variables represented by the nodes of the graph. They are used in a variety of applications, specially in biological studies, where data tends to be scarce because of the high cost of obtaining it or data protection issues. In this project we will learn how to use algebraic tools (such as elimination ideals) to determine what is the minimal number of observations that we need to ensure the existence of the maximum likelihood estimator for a given model. Our goal will be to perform a study with computer algebra software for certain families of models with additional symmetries.

Gemma Huguet: ***Mean field models for cortical neuronal networks***
Understanding the brain dynamics involves the study of the dynamics of large-scale networks of neurons. One can describe the collective dynamics of the neuronal network in terms of macroscopic measures, such as the mean firing rate or the mean voltage, using the so-called mean-field theories. In this project we will review the basic elements to model the activity of a network of excitatory neurons in the cortex and we will study the mathematical formulation of a mean-field model, which presents an exact description of the macroscopic activity of a large network of all-to-all coupled neurons. The project involves basic tools from dynamical systems and complex analysis. As a second stage, the project may involve numerical simulations of the system to check the accuracy of the mean field model to describe the macroscopic variables of the full network.

Marc Jorba & Daniel Pérez-Palau: ***Lunar Voyages Unveiled: Navigating with Dynamical Systems***
The exploration of the space captivates humanity's curiosity and thirst for discovery. Simplified models are a useful framework to address the complex missions and also to sketch their design. Among these models, the Circular Restricted Three Body Problem (CR3BP) stands out, offering insight into spacecraft motion under the gravitational influence of two massive bodies. This project endeavors to explore the CR3BP from both theoretical and practical perspectives, aiming to unravel various solutions for lunar travel. Through the utilization of advanced ODE solvers, we will use invariant manifolds (the skeleton of the dynamics) to navigate the pathways to the Moon.

Jose Muñoz: ***Numerical and analytical structure of Optimal Control Problems***
Optimal Control problems can be posed as the minimisation of a functional subjected to an initial value problem. Depending on the form of the functional, the solution has some geometrical properties such as a symplectic structure, or preservation of control Hamiltonian and generalised momenta. In this project students will explore the structure of the set of optimality conditions for some practical situations, and apply time-integration schemes that allow preserving these quantities also on the numerical results.

Patricia Sánchez: ***Bifurcations of equilibrium points in non-centered barred galaxies***
Dynamics of galaxies can be studied using a dynamical system where the different components of the galaxy are represented by potentials in the dynamical system. This dynamical system is similar to that of the Circular Restricted Three Body Problem. Barred galaxies, which are characterized by a central bar composed of stars, exhibit a notable dynamical behavior wherein the equilibrium points contribute to organizing the dynamics of the system. In some galaxies, such as the Large Magellanic Cloud, the

bar is displaced from its central position, which results in a bifurcation of the equilibrium points. In classical models, where the bar is centered, the model has five equilibrium points with two of them exhibiting a saddle x centre x centre behavior (with realistic parameters of the galaxy) around which the arms of the galaxy emanate. The present study aims to investigate the bifurcations of the equilibrium points depending on the position of the bar and other parameters of the system. This would allow us to explain galaxies with different numbers of arms.

DISCRETE MATHEMATICS

Patrick Morris: ***Monochromatic products in random sets of integers***

It is a well-known consequence of Schur's theorem that if n is large, then any red/blue-colouring of $[n]=\{1,2,...,n\}$ contains a monchromatic product, that is, three integers a,b and c that are all the same colour and such that ab=c. For some probability p=p(n), let $[n]_p$ denote the random set obtained by keeping each element of [n] with probability p independently. For what values of p does $[n]_p$ have the property (with high probability) that any red/blue-colouring results in a monchromatic product? When p=1, we certainly have this property as discussed above whilst when $p< (n\log n)^{-1/3}$, it turns out that with high probablity the set $[n]_p$ has no products at all (and so no monochromatic products in a colouring). What happens for values of p in between? This is a new line of questioning introduced recently by Mattos, Mergoni and Parczyk. However such questions have been well studied for sums a+b=c and so this project will try to adapt methods developed for sums to the product setting.