Weakly geometric graphs and their properties

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Recently, network geometry [1] has become a hot research field in network science. Latent geometric spaces underlying real complex networks provide the simplest explanation to many of their observed topological properties, including degree distribution, small-worldness, clustering, community structure, etc. The key topological property within the geometric framework is clustering—the tendency of the network to form cycles of length three—due to the triangle inequality in the latent geometry.

Interestingly, in geometric models clustering undergoes a transition when the coupling between the underlying geometry and the topology is tuned [2]. As the coupling is weakened we move from a regime of finite clustering to one where the clustering vanishes in the thermodynamic limit. Because real networks tend to be highly clustered, the weak coupling regime has mostly been ignored in the literature. Here, we argue that this is not entirely justified.

First, we study through analytic and numerical analyses the decay of clustering coefficient as a function of the system size N [3]. The coupling strength can be modeled as an inverse temperature β, and we prove that in the regime of weak coupling the clustering decays as a power-law. For a substantial range of β, the exponent of this power-law is temperature dependent: $\sigma(\beta) = 2 - 2/\beta$. This implies an extremely slow decay of the clustering coefficient, with the exponent vanishing as $\beta \to 1$, the transition point between the weak and strong coupling regimes. The slow decay implies strong geometric effects, which is why we refer to this region as the "quasi-geometric" regime.

Second, by extending the graph embedding tool mercator [4] to the weak coupling regime, we show that, in the quasi-geometric regime, geometric information can be recovered from the connectivity alone, underlying its importance in shaping the topology of the network [5]. In contrast, when the coupling is weaker, network topologies are no longer distinguishable from those generated with the configuration model. In Fig. 1 we show mercator's ability to recover the geometric coordinates θ for various values of β from the connectivity alone. The quasi-geometric regime can clearly be distinguished. Additionally, we show that many real networks can best be described as living in the weak coupling regime.

Finally, we extend the geometric renormalization group for networks [6] to the weak coupling regime. We show that, in the quasi-geometric regime, geometric information is essential for creating self similar down-scaled network replicas. This information is not necessary when the coupling is weaker, as there any procedure that preserves the degree distribution is enough to obtain self-similarity.

These three results illustrate a need to extend the scope of research on geometric random graphs to all coupling regimes.

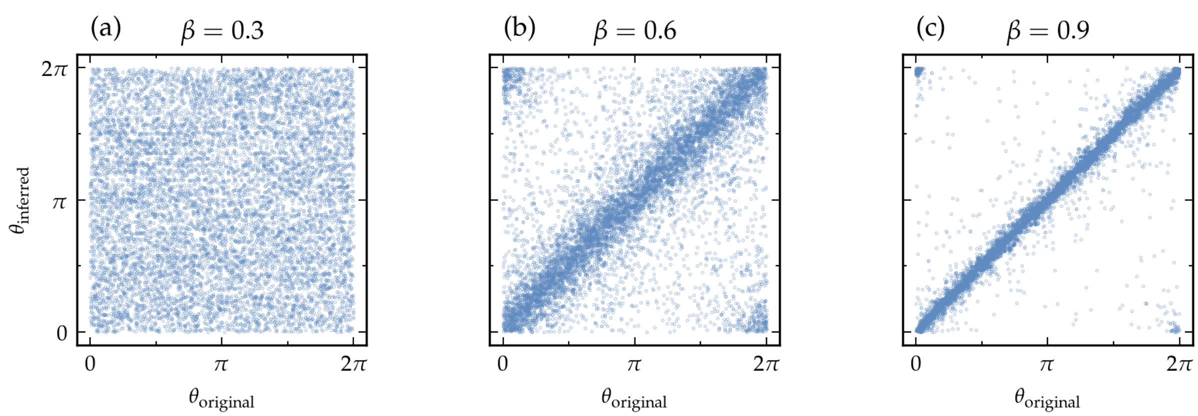

Figure 1: The geometric coordinate as inferred by mercator as a function of the original coordinate used to generate the network with $N = 10000$ nodes and a power-law degree distribution. Various β's in the weak-coupling regime are shown.

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