Invariant sets and periodic orbits of some piecewise linear maps

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Piecewise affine maps appear in many fields like in the study of mechanical systems with friction, power electronics, relay control systems or economics. Despite their apparent simplicity, they exhibit great dynamic richness and a variety of phenomena that are characteristic of these systems, see [1, 3, 4, 7, 8].

In this talk we will describe some of the dynamical features of the family of piecewise linear rotations

$$F_\alpha(x,y) = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} x - \operatorname{sign}(y) \\ y \end{pmatrix}. \tag{1}$$

In [4], we proved that for the special cases $\alpha \in \mathcal{A} := \{\pi/3, \pi/2, 2\pi/3, 4\pi/3, 3\pi/2, 5\pi/3\}$, the maps (1) are *pointwise periodic*, i.e. bijective in $\mathbb{R}^2$ and such that each point is periodic, but they are not *globally periodic*. Despite this, the global dynamics is not simple. Also, for each of the maps with $\alpha \in \mathcal{A}$, we found a first integral. These first integrals exhibit unusual characteristics in the context of discrete dynamical systems: for instance, their *energy levels* are discrete, thus *quantized*. Furthermore, the level sets are bounded sets whose interior is like a necklace formed by a finite number of open tiles of a certain *regular or uniform tessellation* of the plane. For these cases, we will describe the action of the maps on each of the invariant sets defined by the quantized integrals in geometrical terms. More precisely, consider a map in (1) with $\alpha \in \mathcal{A}$ with first integral V, then:

(a) We prove that F induces a dynamics between the tiles of the necklace forming the level set $\{V = c\}$. Gometrically, F acts as a rotation of orther k among the tiles of the necklace.

(b) We prove that each tile is invariant by F^k, which is a rotation of order p around the center of the tile. As a consequence, on each tile there is a k-periodic point (the center) and the rest of the points are kp-periodic.

In all the cases, the values of k and p depend explicitly on the energy level c. This part of the talk can be found in [4]

The general properties of the maps F with $\alpha \in [0, 2\pi) \setminus \mathcal{A}$, being a rational multiple of π, are still not completely known. For instance, in [7] it is proved that for such cases there exists a sequence of open invariant nested necklaces that tend to infinity, whose beads are polygons, and where the dynamics of F is given by a product of two rotations. Remarkably, although the adherence of the union of all these invariant necklaces does not fill the full plane, it allows to prove that all orbits of F are bounded.

For these cases with $\alpha \notin \mathcal{A}$, we will present some sparse results. Consider the critical set $\mathcal{F} = \bigcup_{i\in\mathbb{N}} LC_{-i}$ formed by all the preimages of the critical line LC_0 where the discontinuty is located (here we use the notation in [2]). Our simulations indicate that, in these cases, the critical set seems to fractalize. We also consider the set $\mathcal{U} = \mathbb{R}^2 \setminus \overline{\mathcal{F}}$. We prove that any connected component of $\mathcal{U}$ is open, bounded and periodic. Moreover, any element of $\mathcal{U}$ is periodic. Furthermore, in the literature it has been claimed (but not proved) that in some cases there exists non-periodic orbits in $\mathcal{F}$. We present some evidences of this last fact, as well as of the fractalization of the critical set, and we prove that if $\overline{\mathcal{F}} \setminus \mathcal{F} \neq \emptyset$, then the elements of $\overline{\mathcal{F}} \setminus \mathcal{F}$ are aperiodic. This last part of the talk belongs to [5] .

Finally, if time permits, we will show some recent results about periodic orbits, chaotic behaviors and on some typologies of invariant sets that appear in other families of continuous piecewise affine maps. These results belong to [6].

References

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