Almost global existence for some nonlinear Hamiltonian PDEs on Toric Manifolds

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I will present a result of almost global existence for small and smooth solutions of some abstract nonlinear Hamiltonian partial differential equations on toric manifolds. We apply it to some relevant equations, namely the nonlinear Schrodinger equation with a convolution potential and the beam equation. Moreover, the abstract result is used to ensure effective stability of plane waves in the NLS equation.

In the literature, results of this kind are well known for equations defined for one dimensional space variable. The main novelty of our work consists in the application to higher dimensional problems. In particular, it extends to some more general manifolds previous results holdings on flat torus [1].

On a toric manifold the geodesic flow is completely integrable in the classical sense, and it admits global action variables. In particular, the Laplace-Beltrami operator Δ_g can be written as a quadratic function of some first order pseudodifferential operators. We call it the "quantum actions", since they appear as the Weyl quantization of the global action variables. Some relevant examples of toric manifolds are flat tori, Zoll manifolds, Lie Groups and their homogeneous spaces, rotation invariant surfaces.

A result of almost global existence ensures that a solution corresponding to initial data with H^s norm of order $\epsilon \ll 1$, has H^s norm of order ϵ for any $t < \epsilon^{-r}$. This is true for any $r > 0$. In our work, this is the dynamical consequence of the existence of a Birkhoff normal form at any order, provided that some non-trivial non-resonances conditions are fulfilled by the frequencies of the linear dynamics, namely the eigenvalues of the Laplacian. Two ingredients are essential in order to consider higher dimensional systems:

- a key breakthrough is the existence of a partition of the set of the frequencies that generalizes a seminal result of Bourgain. It provides a cluster structure which connects the distance between eigenvalues of the Laplacian and the distance between the corresponding Fourier indexes. This is proven thanks

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to a quantum version of the classical Nekhoreshev construction, contained in [2] .

- the second essential ingredient is a non-trivial decay property for the product of the eigenfunctions of the Laplacian. This property replaces the usual localization of the eigenfunctions with respect to the exponential, that is the typical property exploited in one dimensional system, and that has no equivalent in higher dimension. A similar property was already fruitfully introduced for the simpler case of Zoll manifolds in [3].

This is joint work with Dario Bambusi, Roberto Feola and Beatrice Langella. At the moment of sending, the work is not available as a preprint. It should be within days.

References

[1] D. Bambusi, R. Feola, R. Montalto. *Almost global existence for some Hamiltonian PDEs with small Cauchy data on general tori*, preprint: arXiv: 2208.00413. (2022)

[2] D. Bambusi, B. Langella *Growth of Sobolev norms in quasi integrable quantum systems*, preprint: arXiv: 2202.04505. (2022)

[3] D. Bambusi, J.M. Delort, B. Grébert, and J. Szeftel. *Almost global existence for Hamiltonian semilinear Klein-Gordon equations with small Cauchy data on Zoll manifolds.* Comm. Pure Appl. Math., 60(11):1665–1690, (2007)