Flow map parameterization methods for whiskered tori in time-dependent Hamiltonian systems

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Abstract

Studying invariant manifolds constitutes the centerpiece in understanding a dynamical system. It is a rather natural first approach—and often the only hope—to unveil the qualitative behavior of a time-evolving system. Besides the intrinsic interest of invariant manifolds, such structures have found their "real-world" analogues in celestial mechanics, astrodynamics and mission design, plasma physics, semi-classical quantum theory, magnetohydrodynamics, neuroscience, and the list goes on.

In this work, we consider non-autonomous Hamiltonian systems that depend on time quasi-periodically with an arbitrary number of frequencies. Under the parameterization method paradigm, we develop theory and efficient algorithms to compute non-resonant partially hyperbolic invariant tori and their invariant manifolds using flow maps. As opposed to the order by order methods, we compute invariant tori and their invariant manifolds simultaneously with a quasi-Newton method. To this end, it is essential to consider the geometrical properties of the system and of whiskered tori which results in the synergy between mathematical rigor and numerical computation. In geometrical jargon, we consider the extended phase space as a symplectic bundle and we leverage the geometrical properties the systems (i.e. exact symplecticity) and of the whiskered tori (i.e. isotropicity, Lagrangianity). These properties lead to a Newton step that is decomposed into substeps that require $\mathcal{O}(N)$ operations either in grid space or Fourier space, where the $\mathcal{O}(NlogN)$ cost comes from the FFT performed in order to switch representation spaces.

An important ingredient in the design of algorithms based on the parameterization method is the presence of "magic cancellations". These cancellations come as well from the geometrical properties and they allow the solution of the so-called small divisors equations. In order to prove some of these cancellations, we sketch definitions and results on fiberwise symplectic deformations and construct the corresponding so-called moment maps which can be seen as generating Hamiltonians of the deformations.

For the numerical experiments, we apply our algorithms in the Elliptic Restricted Three Body Problem and compute non-resonant 3-dimensional whiskered tori keeping an eye on applications to space mission design.

Keywords: Invariant tori · Whiskers · Parameterization method · Hamiltonian systems · ERTBP