Higher-order superintegrable momentum-dependent Hamiltonians on curved spaces from the classical Zernike system

We consider the classical momentum- or velocity-dependent two-dimensional Hamiltonian given by

$$\mathcal{H}_N = p_1^2 + p_2^2 + \sum_{n=1}^{N} \gamma_n (q_1 p_1 + q_2 p_2)^n,$$

where q_i and p_i are generic canonical variables, γ_n are arbitrary coefficients, and $N \in \mathbb{N}$. For $N = 2$, being both γ_1, γ_2 different from zero, this reduces to the classical Zernike system. We prove that $\mathcal{H}_N$ always provides a superintegrable system (for any value of γ_n and N) by obtaining the corresponding constants of the motion explicitly, which turn out to be of higher-order in the momenta. Such generic results are not only applied to the Euclidean plane, but also to the sphere and the hyperbolic plane. In the latter curved spaces, $\mathcal{H}_N$ is expressed in geodesic polar coordinates showing that such a new superintegrable Hamiltonian can be regarded as a superposition of the isotropic $1:1$ curved (Higgs) oscillator with even-order anharmonic curved oscillators plus another superposition of higher-order momentum-dependent potentials. Furthermore, the symmetry algebra determined by the constants of the motion is also studied, giving rise to a $(2N-1)$th-order polynomial algebra. As a byproduct, the Hamiltonian $\mathcal{H}_N$ is interpreted as a family of superintegrable perturbations of the classical Zernike system. Finally, it is shown that $\mathcal{H}_N$ (and so the Zernike system as well) is endowed with a Poisson $\mathfrak{sl}(2,\mathbb{R})$-coalgebra symmetry which would allow for further possible generalizations that are also discussed.