Conformal and Projective Transformations on Integrable Mechanical Billiards

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Abstract

This presentation focuses on the study of mathematical billiards in the presence of non-constant potentials and their integrability.

For free motion in 2-dimensional plane $\mathbb{R}^2$, there are two types of integrable billiard systems: circular and elliptic free billiards. The integrability of such systems has been shown by Birkhoff. Additionally, a conjecture attributed to Birkhoff and Poritsky states that any closed convex reflection wall of an integrable billiard system is either a circle or an ellipse. In the presence of specific potential (such as Kepler potential and Hooke potential) defined in the plane, there are various known integrable billiard systems. These integrable examples have been found independently in different contexts. In our study, we illustrate how some of these integrable billiard systems are related to each other by conformal transformations. As an application, we obtain infinitely many billiard systems defined in central force problems which are integrable on a particular energy level. We then explain that the classical Hooke-Kepler correspondence extends to the correspondence between integrable Hooke and Kepler billiards. As a result, we show that any focused conic sections give rise to integrable Kepler billiards which give new examples of integrable Kepler billiards. The conformal transformation technique is applied to Stark-type problems and Euler's two-center problem and provides new examples of integrable mechanical billiards.

As well as mechanical billiard systems in the plane, we also consider systems on curved surfaces. We use the projective dynamical approach to integrable mechanical billiards to establish the integrability of natural mechanical billiards with the Lagrange problem, which is the superposition of two Kepler problems and a Hooke problem, with the Hooke center at the middle of the Kepler centers, as the underlying mechanical systems, and with any combinations of confocal conic sections with foci at the Kepler centers as the reflection wall, in the plane, on the sphere, and in the hyperbolic plane. This covers many previously known integrable mechanical billiards, especially the integrable Hooke, Kepler and two-center billiards in the plane as subcases. The approach based on conformal correspondence has been also applied to integrable Kepler billiards in the hyperbolic plane to illustrate their equivalence with the corresponding integrable Hooke billiards on the hemisphere and in the hyperbolic plane as well.