Numerical integration of high order variational equations of ODE

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Abstract

Variational equations of Ordinary Differential Equations (ODEs) are differential equations that describe the sensitivity of a given ODE solution with respect to initial conditions or parameters. In essence, the variational equations quantify how small perturbations to initial conditions or parameters translate into changes in the solution of the ODE. Variational equations can be derived by taking derivatives of the original ODE, and solving them provides the Taylor expansion of the flow with respect to these variables.

However, computing high order derivatives of the vector field can be a daunting task, particularly for high dimensional systems. This is where Automatic differentiation, a powerful computational technique for computing derivatives of functions, comes into play. Automatic differentiation breaks down a function into a sequence of elementary operations and applies the chain rule of differentiation to each of these operations, allowing for efficient and accurate computation of the derivatives.

In this talk, we will delve deeper into the practical implications of Automatic differentiation in the context of ODE solvers. Usually, we call jet transport to applying Automatic differentiation to the steppers of ODE solvers. Hence, jet transport provides a way to compute the Taylor expansion of the ODE solution with respect to initial conditions or parameters. We will demonstrate that jet transport of a certain order on a wide class of steppers is algorithmically equivalent to applying the stepper to the whole set of variational equations. This equivalence implies that the error on the high order derivatives of the flow produced by the jet transport is of the same order as the stepper.

Furthermore, we will discuss the practical implications of jet transport, particularly in the choice of step size, and provide several applications of this technique.