Hyperbolic manifolds fibering over S^1

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Abstract

Fibrations over the circle are a well-studied topic in geometric topology, since they provide a useful tool to describe the topology of a manifold M. Unfortunately, fibering does not behave well with a hyperbolic structure on M, which is what we are most interested in. For one thing, an even-dimensional manifold with negative sectional curvature cannot fiber over the circle due to an Euler characteristic constraint. Moreover, if $F \hookrightarrow M$ is a fiber of a fibering hyperbolic manifold, then the induced metric on F cannot be hyperbolic, and in dimension ≥ 5 it cannot admit *any* negatively curved metric structure at all.

Despite this "incompatibility", Agol and Wise showed that every closed hyperbolic 3-manifold virtually fibers over the circle. However, what happens in greater odd dimension is still mysterious, since not a single example of fibering hyperbolic manifold was known until recently.

In this talk, we will show the existence of a hyperbolic 5-manifold which fibers over the circle. To this end, we will introduce a combinatorial game based on Bestvina-Brady theory and presented in 2017 by Jankiewicz, Norin, and Wise.

To play this game, we will need a right-angled hyperbolic polytope along with an additional combinatorial structure on its facets (which is the choice of a *colouring* and a *state*). We will employ the 5-dimensional polytope P^5, belonging to a family of polytopes introduced by Potyagailo and Vinberg in 2005. These polytopes possess a remarkable number of symmetries, which play a crucial role in checking that the conditions required to obtain a fibration are actually satisfied.

This first example of a fibering hyperbolic 5-manifold implies that there exists a hyperbolic group G containing a subgroup H of finite type which is not hyperbolic, therefore solving a well known open problem in geometric group theory.

This is joint work with Giovanni Italiano and Bruno Martelli.

References

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