Poisson Advanced School, Barcelona — July 18-22, 2022

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Singular Foliations : Problems sessions

Solutions

We intend to study vector fields that are tangent to a subvariety $W \subset M$. We shall deal with the smooth, complex, algebraic settings altogether.

Exercice 1

The smooth setting.

Let M be a smooth manifold and $W \subset M$ an embedded closed sub-manifold. We denote by $\mathfrak{X}_W(M)$ the space of all compactly supported vector fields on M that are tangent to the sub-manifold W.

1. Check that $\mathfrak{X}_W(M)$ is a singular foliation on M. Give a set of local generators.

A. By definition $\mathfrak{X}_W(M) = \{X \in \mathfrak{X}(M) \mid X|_m \in T_mW, \text{ for all } m \in W\}$. We need to check that $\mathfrak{X}_W(M)$

— is a $C^\infty(M)$-module, which is obvious.

— is stable under the Lie bracket. Let $X, Y \in \mathfrak{X}_W(M)$. The commutator $[X, Y]$ is the vector field on M whose value at $m \in M$ is given by

$$[X,Y]_{|m} = (\mathcal{L}_X Y)_{|m} = \frac{d}{dt}_{|t=0} T_{\phi^X_{-t}(m)}\phi^X_t(Y_{\phi^X_{-t}(m)}),$$

where ϕ^X_t is the local flow of X. Since the local flow ϕ^X_t preserves W, $T\phi^X_t$ maps TW to TW, and $T_{\phi^X_{-t}(m)}\phi^X_t(Y_{\phi^X_{-t}(m)})$ belongs to T_mW for all t for which it is defined. As a consequence, $[X, Y]|_m \in T_mW$.

— is locally finitely generated : it suffices to consider local coordinates. Every point $m \in M \backslash W$ admits an open neighborhood $\mathcal{U}$ on which the restriction of $\mathfrak{X}_W(M)$ is the space of all vector fields. In particular, for any choice of local coordinates $z_1, \ldots, z_d$ near $m \in \mathcal{U}$, $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_d}$ is a set of local generators.

Now, for every $m \in W$, it is a classical result of smooth differential geometry that there exists local coordinates $(\mathcal{U}, x_1, \ldots, x_a, y_1, \ldots, y_b)$ around m on which W is given by $y_1 = \cdots = y_b = 0$. Without any loss of generality, we can assume that $x_i, y_j \in]-1, 1[$ for all indices. We claim that $\mathfrak{X}_W(M)$ is generated by

$$\left\{ \frac{\partial}{\partial x_i}, y_j \frac{\partial}{\partial y_k} \middle| 1 \leq i \leq a \text{ and } 1 \leq j, k \leq b \right\}.$$

A vector field $X \in \mathfrak{X}(\mathcal{U})$ is tangent to W if and only if it reads

$$X = \sum_{i=1}^{a} h_i(x_1, \ldots, x_a, y_1, \ldots, y_b)\frac{\partial}{\partial x_i} + \sum_{j=1}^{b} f_j(x_1, \ldots, x_a, y_1, \ldots, y_b)\frac{\partial}{\partial y_j}$$

with $f_j(x_1, \ldots, x_a, 0, \ldots, 0) = 0$. The proof then relies on the so-called Hadamard Lemma, which states that for every smooth function $f_j(x_1, \ldots, x_a, y_1, \ldots, y_b)$ such that $f_j(x_1, \ldots, x_a, 0, \ldots, 0) = 0$, there exists smooth functions $g^j_1(x_1, \ldots, x_a, y_1, \ldots, y_b), \ldots, g^j_b(x_1, \ldots, x_a, y_1, \ldots, y_b)$ on $\mathcal{U}$ such that

$$f_j(x_1, \ldots, x_a, y_1, \ldots, y_b) = \sum_{k=1}^{b} y_j g^k_j(x_1, \ldots, x_a, y_1, \ldots, y_b).$$

This implies that

$$X = \sum_{i=1}^{a} h_i(x_1, \ldots, x_a, y_1, \ldots, y_b) \frac{\partial}{\partial x_i} + \sum_{j,k=1}^{b} g_j^k(x_1, \ldots, x_a, y_1, \ldots, y_b)\, y_j \frac{\partial}{\partial y_k}$$

This completes the proof.

2. Describe the singular distribution $m \mapsto T_m(\mathfrak{X}_W(M))$.

A. For $m \notin W$, $T_m(\mathfrak{X}_W(M)) = T_m M$. Take any local coordinates $z_1, \ldots, z_d$ on an coordinate neighborhood that does not intersect W. For every $v = \sum_{i=1}^{d} v_i \frac{\partial}{\partial z_i}_{|m} \in T_m W$ the local vector field

$$X = \phi_m(z_1, \ldots, z_d) \sum_{i=1}^{d} v_i \frac{\partial}{\partial z_i}$$

with $\phi_m(z_1, \ldots, z_d)$ a compactly supported "bump" function equal to 1 at m extends by 0 to a compactly supported vector field on M tangent to W (because it vanishes at every point on W) and satisfies $X|_m = v$.

For $m \in W$, we claim that $T_m(\mathfrak{X}_W(M)) = T_m W$. Take local coordinates $(x_1, \ldots, x_a, y_1, \ldots, y_b)$ as in the previous question. Let $v = \sum_{i=1}^{a} v_i \frac{\partial}{\partial x_i}|_m \in T_m W$. Consider ψ_m a compactly supported bump function which is 1 at m, and let

$$X = \psi_m \sum_{i=1}^{a} v_i \frac{\partial}{\partial x_i}$$

that we extend by 0 on the whole manifold M. We have $X|_m = v$, and X is a compactly supported vector field tangent to W. This completes the argument.

3. Describe
   (a) the leaves of $\mathfrak{X}_W(M)$?

   A. The connected components of submanifold W and the connected components of $M \setminus W$ are the leaves of $\mathfrak{X}_W(M)$, because these are submanifolds whose tangent space coincide at every point with the tangent space of $\mathfrak{X}_W(M)$ computed in the previous question. Moreover, if $\mathrm{codim}(W) \geq 2$ and M, W are connected, then $\mathcal{F}$ has only 2 leaves : W and $M \setminus W$. If M, W are connected and $\mathrm{codim}(W) = 1$, one can have one or two connected component in $M \setminus W$.

   (b) The set of all regular points.

   A. The set of regular points is $M \setminus W$.

   (c) The transverse singular foliation to each one of these leaves.

   A. For a leaf L which is a connected component of $M \setminus W$, the transverse foliation is a point. For a leaf L which is a connected component of W, the transverse foliation is represented by vector fields on $\mathbb{R}^b$ vanishing at 0, with b the codimension of W in M. This can be seen as follows. Let $(x_1, \ldots, x_a, y_1, \ldots, y_b)$ be local coordinates on which W is given by $y_1 = \cdots = y_d = 0$ as in the first question. We saw then that vector fields in $\mathfrak{X}_W(M)$ are, after restriction to that open set, vector fields of the form :

$$X = \sum_{i=1}^{a} h_i(x_1, \ldots, x_a, y_1, \ldots, y_b) \frac{\partial}{\partial x_i} + \sum_{j,k=1}^{b} g_j^k(x_1, \ldots, x_a, y_1, \ldots, y_b)\, y_j \frac{\partial}{\partial y_k}.$$

Consider the transverse submanifold $x_1 = \cdots = x_a = 0$. By restricting to the transversal vector fields of the previous form which are tangent to that transversal, one obtains vector fields of the form

$$\sum_{j,k=1}^{b} g_j^k(0,\ldots,0,y_1,\ldots,y_b)\, y_j \frac{\partial}{\partial y_k}$$

This is exactly vector fields on a neighborhood of 0 in $\mathbb{R}^b$ that vanish at 0.

4. Compute

   (a) the isotropy Lie algebras of $\mathcal{F} = \mathfrak{X}_W(M)$ at every point.

   A. Again, we will distinguish two types of points.

   — If $m \in W$, $\mathcal{F}(m)$ is spanned by the vector fields $(x_i - x_i(m))\frac{\partial}{\partial x_l}, y_i \frac{\partial}{\partial x_l}$, and $y_j \frac{\partial}{\partial y_k}$ for $1 \leq i, l \leq a$ and $1 \leq j, k \leq b$. Also, $\mathcal{I}_m \mathcal{F}$ is spanned by the $(x_i - x_i(m))\frac{\partial}{\partial x_l}$, $y_j \frac{\partial}{\partial x_l}$, $(x_i - x_i(m)) y_j \frac{\partial}{\partial y_j}$, $y_j y_l \frac{\partial}{\partial y_k}$'s. Hence,

   $$\mathfrak{g}_m = \frac{\mathcal{F}(m)}{\mathcal{I}_m \mathcal{F}} \simeq \mathfrak{gl}(\mathbb{R}^b).$$

   We can use a fancier argument. The isotropy Lie algebra at m is the same as the isotropy Lie algebra at the origin for the transverse singular foliation at m. This transverse singular foliation is the space of all vector fields on $\mathbb{R}^d$ vanishing at 0. Therefore

   $$\mathfrak{g}_m \simeq \frac{vetors\,fields\,of\,\mathbb{R}^b\,vanishing\,at\,0}{vector\,fields\,vanishing\,quadratically\,at\,0}$$

   is the quotient of the space of vector fields on $\mathbb{R}^d$ vanishing at 0 by those that vanish at order at least two. As a Lie algebra, this is easily seen to be isomorphic to $\mathfrak{gl}(\mathbb{R}^b)$.

   — If $m \in M \setminus W$, then m is a regular point :

   $$\mathfrak{g}_m \simeq \{0\}.$$

   (b) The rank of $\mathfrak{X}_W(M)$ at every point .
   A. The rank of $\mathfrak{X}_W(M)$ is the dimension of M at every point in $M \setminus W$, and is

   $$\dim(W) + (\mathrm{codim}(W))^2$$

   at every point in W.

5. Is $\mathfrak{X}_W(M)$ the image through the anchor map of a Lie algebroid ?
   A. Let us study different cases, of varying difficulties.

   (a) In a neighborhood of every point, the answer is "yes". This can be proved as follows : $\mathfrak{X}_W(M)$ is, locally, the direct product of the singular foliation of all vector fields on $\mathbb{R}^a$ (which is obviouly a singular foliation coming from a Lie algebroid) with the singular foliation of all vector fields on $\mathbb{R}^b$ that vanish at zero (which is the image through the anchor map of the transformation Lie algebroid for the action of $\mathrm{gl}(\mathbb{R}^b)$ on $\mathbb{R}^b$). Hence it is the image through the anchor map of a Lie algebroid.

   (b) On the whole manifold M, the question is open : it does not seem obvious.

   (c) There is a neighborhood of W on M where the answer is yes. This follows from the tubular neighborhood theorem, which gives a diffeomorphism between a neighborhood of W in M and a neighborhood of the zero section in the normal bundle N_W. Under this diffeomorphism, $\mathfrak{X}_W$ becomes the space of all vector fields on N_W tangent to the zero section. It is easy to check that :

i. Fiberwise linear vector field on N_W are the sections of a transitive Lie algebroid (called $CDO(N_W)$ in Kirill Mackenzie) over W,

ii. that this Lie algebroid acts on N_W,

iii. and that the anchor of the transformation Lie algebroid admits for image all vector fields tangent to the zero section.

(d) If the codimension of W is one, then $\mathfrak{X}_W(M)$ is Debord and the answer is yes.

6. Describe the holonomy groupoid of $\mathfrak{X}_W(M)$.

A. For the sake of simplicity, we assume that W and $M \setminus W$ are connected. Let us describe the Androulidakis-Skandalis holonomy groupoid for each one of the leaf L of $\mathfrak{X}_W(M)$. Recall that the holonomy groupoid is a topological groupoid, whose orbits are the leaves of the singular foliation, and whose restriction to each leaf is a smooth Lie groupoid. It suffices therefore to describe that restriction for the two leaves $M \setminus W$ and W.

(a) For $L = M \setminus W$, the restriction of the holonomy groupoid to L is the pair groupoid $(M \setminus W) \times (M \setminus W) \rightrightarrows (M \setminus W)$.

(b) For $L = W$, the restriction of the holonomy groupoid is the Lie groupoid

$$GL(N_W) \rightrightarrows W$$

where $N^W = TM_{|_W}/TW$ is the normal bundle (which is a vector bundle over W) and $GL(N^W)$ is the Lie groupoid of all invertible linear endomorphisms from a fiber of N^W to an other fiber of N^W. More precisely,

$$GL(N^W) := \cup_{m,n \in W} GL(N^W_{|_m}, N^W_{|_n}).$$

This identification is beyond the scope of these exercises. However, it is easy to check that the Lie algebroid of these Lie groupoid are, indeed, the holonomy Lie algebroid of the leaf.

Exercice 2

The complex setting.

Readjust Exercice 1 to the complex case.

Exercice 3

The algebraic setting.

Now, we will allow $W \subset M$ to be a singular subvariety. This mainly makes sense while working within the context of complex algebraic geometry (but also in the complex setting, as we shall explain). More precisely, we will make the simplifying assumption that $M = \mathbb{C}^d$. And we define $W \subset \mathbb{C}^d$ to be an an affine variety, i.e. W is the zero locus of some polynomial functions $\varphi_1, \ldots, \varphi_r \in \mathbb{C}[x_1, \ldots, x_d]$. We denote by $\mathcal{I}_W$ the ideal of vanishing functions on W. Without any loss of generality, we can assume that $\mathcal{I}_W$ is the ideal generated by $\varphi_1, \ldots, \varphi_r$, so we will make this assumption. Let $\mathcal{F} = \mathfrak{X}_W(M)$ be the space of all vector fields X on $M = \mathbb{C}^d$, with polynomial coefficients, that satisfy :

$$X[\mathcal{I}_W] \subset \mathcal{I}_W.$$

1. Explain and justify why it makes sense to define $\mathfrak{X}_W(M)$ as above.

(For instance, consider $X \in \mathfrak{X}_W(M)$ as a complex vector field and show that its flow - when it exists - preserves W.)

A. Let X be a vector field such that $X[\mathcal{I}_W] \subset \mathcal{I}_W$. This implies that X is tangent to the regular part of W, which is complex submanifold of $\mathbb{C}^d$.

This also implies that the flow of X, if it exists, is a biholomorphism that maps W to itself. Let us show this point : there exists a matrix $A = (A_i^j)_{i,j=1}^k$, with coefficients in $\mathbb{C}[x_1, \ldots, x_d]$ such that :

$$L_X \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_r \end{pmatrix} = \begin{pmatrix} & A & \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_r \end{pmatrix}$$

with the understanding that "L_X" means that one applies the derivation X to any one of the functions in the column vector. In turn, for every time t for which the flow of X is defined :

$$(\phi_X^t)^* \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_r \end{pmatrix} = e^{tL_X} \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_r \end{pmatrix} = A(t) \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_r \end{pmatrix},$$

for some invertible matrix $A(t)$, well defined for t near 0. Translated geometrically, it means that the flow of X at time t maps $\mathcal{I}_W$ to itself.

We remind the reader that it is not true in general, for X a smooth vector field and $\mathcal{I}$ an ideal of smooth functions that $X[\mathcal{I}] \subset \mathcal{I}$ implies $(\phi_t^X)^*[\mathcal{I}] \subset \mathcal{I}$. Fr instance, take $M = \mathbb{R}$, $\mathcal{I}$ to be the ideal of functions vanishing on $\mathbb{R}_-$ and $X = \frac{\partial}{\partial x}$. It is therefore crucial, in the previous argument, to make use of the fact that $\mathcal{I}_W$ is finitely generated.

2. Show that $\mathcal{F}$ is a singular foliation over the algebra of polynomials in d variables.

A. Let us check all axioms.

— It it clearly a $\mathbb{C}[x_1, \ldots, x_d]$-module.

— Stability under Lie bracket : for all $X, Y \in \mathfrak{X}_W(M)$, $[X,Y][\varphi_i] = X[Y[\varphi_i]] - Y[X[\varphi_i]]$, since $Y[\varphi_i], X[\varphi_i] \in \mathcal{I}_W$, we have $X[Y[\varphi_i]], Y[X[\varphi_i]] \in \mathcal{I}_W$. Hence $[X,Y] \in \mathfrak{X}_W(M)$.

— Finitely generated : By construction, $\mathcal{F}$ is a sub-$\mathbb{C}[x_1, \ldots, x_d]$-module of $\mathfrak{X}(M)$ (with $M = \mathbb{C}^d$). Now, as a module $\mathbb{C}[x_1, \ldots, x_d]$-module, $\mathfrak{X}(M)$ is isomorphic to

$$\mathfrak{X}(M) \simeq \underbrace{\mathbb{C}[x_1, \ldots, x_d] \oplus \cdots \oplus \mathbb{C}[x_1, \ldots, x_d]}_{d\text{-times}}, \tag{1}$$

because any vector fields on $M = \mathbb{C}^d$ decomposes in a unique manner as a sum

$$P_1(x_1, \ldots, x_d)\frac{\partial}{\partial x_1} + \cdots + P_d(x_1, \ldots, x_d)\frac{\partial}{\partial x_d}. \tag{2}$$

Since $\mathbb{C}[x_1, \ldots, x_d]$ is Noetherian, every sub-$\mathbb{C}[x_1, \ldots, x_d]$-module of (1) is finitely generated. (Notice that the generators are global).

3. What is

(a) $T_m\mathcal{F}$ for all $m \in M \setminus W$?

A. let $(v_1, \ldots, v_d) \in \mathbb{C}^d$. For $a \notin W$, we can assume that $\varphi_1 \in \mathcal{I}_W$ satisfies $\varphi_1(a) \neq 0$. The vector field

$$X = \frac{\varphi_1}{\varphi_1(a)} \sum_{i=1}^d v_i \frac{\partial}{\partial x_i}$$

belongs to $\mathfrak{X}_W(\mathbb{C}^d)$ and satisfies by construction $X(a) = (v_1, \ldots, v_d)$.

(b) $T_m\mathcal{F}$ for m in the subset $W_{reg} \subset W$ of regular points of W.

A. Do we still have $T_a\mathcal{F} = T_aW$ as in the smooth case? Let us find this out. The local ring at a is by definition the localisation $\mathcal{O}_a$ of $\mathbb{C}[x_1 \dots, x_d]$ with respect to the multiplicative set of all polynomials that do not vanish at a. It is a classical property that $a \in W$ is a regular point if and only if there exists "local coordinates" $y_1, \dots, y_d \in \mathcal{O}_a$. such that W is of the form

$$y_1 = \dots = y_k = 0,$$

i.e. the localization of $\mathcal{I}_W$ is generated by these variables. Hence the tangent space at m is the vector space, $span\{\frac{\partial}{\partial y_i}_{|m}, i \geq k+1\}$. Therefore, for $v \in T_aW$ the local vector field

$$X = \sum_{i=1}^{\dim W} v_i \frac{\partial}{\partial y_{k+i}}$$

maps $\mathcal{O}_a$ to $\mathcal{O}_a$, in particular it maps $\mathcal{O}$ to $\mathcal{O}_a$ and we have $X[\mathcal{I}_W] \subset (\mathcal{I}_W)_{\mathfrak{m}_a}$. Therefore, for every $i \in \{1, \dots, d\}$ there exists a polynomial function g_i that does not vanish at a such that $g_iY[x_i] \in \mathbb{C}[x_1, \dots, x_d]$. Hence, the vector field $\hat{X} = \frac{g_1 \cdots g_r}{g_1(a) \cdots g_r(a)} X$ is tangent to W satisfies $\hat{X}(a) = v$ and $\hat{X}[\mathcal{I}_W] \subset \mathcal{I}_W$.

(c) $T_m\mathcal{F}$ for m an isolated singularity of W, i.e. a point m where $d_m\varphi_i = 0$ for $i = 1, \dots, r$ and which is isolated among such points.

A. It is zero.

(d) (Requires the notion of strata of an affine variety). Show that the tangent space of $\mathcal{F}$ at $m \in W$ is included into the tangent space of the strata of W at this point.

A. It suffices to prove that if $X[\mathcal{I}_W] \subset \mathcal{I}_W$ then $X[\mathcal{I}_{W_{sing}}] \subset \mathcal{I}_{W_{sing}}$ where $\mathcal{I}_{W_{sing}}$ is the ideal of functions on the singular part of W. Since $\mathcal{I}_{W_{sing}}$ is obtained by considering the minors of order $k \geq d - \dim W$ of k elements chosen into the generators $\varphi_1, \dots, \varphi_r$. That is, W_{sing} is given the ideal

$$\left\langle \varphi_1, \cdots \varphi_r, P[\varphi_{i_1}, \cdots, \varphi_{i_k}],\ P \in \mathfrak{X}^k(\mathbb{C}^d) \text{ for integers } \begin{array}{c} 1 \leq i_1 < \cdots < i_k \leq r \\ \\ d - \dim W \leq k \leq r \end{array} \right\rangle \quad (3)$$

Let us explain why the vector fields that tangent to W are also tangent to its singular locus. We recall that a k-multivector field $P \in \mathfrak{X}^k(\mathbb{C}^d)$ is of the form $\eta_1 \wedge \cdots \wedge \eta_k$ for some vector fields $\eta_1, \dots, \eta_k \in \mathfrak{X}(\mathbb{C}^d)$ and is defined as follows

$$(\eta_1 \wedge \cdots \wedge \eta_k)[\varphi_{i_1}, \cdots, \varphi_{i_k}] := \begin{vmatrix} \eta_1[\varphi_{i_1}] & \cdots & \eta_1[\varphi_{i_k}] \\ \vdots & & \vdots \\ \eta_k[\varphi_{i_1}] & \cdots & \eta_k[\varphi_{i_k}] \end{vmatrix}.$$

For a vector field $X \in \mathfrak{X}_W(\mathbb{C}^d)$ one has,

$$X\left[P[\varphi_{i_1}, \cdots, \varphi_{i_k}]\right] = (\mathcal{L}_X P)[\varphi_{i_1}, \cdots, \varphi_{i_k}] + \sum_{j=1}^{k} P[\varphi_{i_1}, \dots, X[\varphi_{i_j}], \dots, \varphi_{i_k}] \quad (4)$$

Notice that $(\mathcal{L}_X P)[\varphi_{i_1}, \cdots, \varphi_{i_k}] \in \mathcal{I}_{sing}$ since $(\mathcal{L}_X P) \in \mathfrak{X}^k(\mathbb{C}^d)$. On the other hand, for every j there exists polynomial functions $f_1, \dots, f_r$ such that $X[\varphi_{i_j}] = \sum_{l=1}^{r} f_l \varphi_l$. Since P

is a multi-derivation one has,

$$P[\varphi_{i_1},\dots,X[\varphi_{i_j}],\dots,\varphi_{i_k}]=\sum_{l=1}^{r}\varphi_l P[\varphi_{i_1},\dots,f_l,\dots,\varphi_{i_k}]+ \\ \sum_{i=1}^{r} f_l P[\varphi_{i_1},\dots,\varphi_l,\dots,\varphi_{i_k}]$$

It is now clear that the rhs of the equation above is in the ideal $\mathcal{I}_{sing}$.

Show that in the coming example, the latter inclusion is strict :

$$W=\{(x,y,z)\in\mathbb{C}^3\,|\,xy(x+y)(x+yz)=0\}$$

for any point in the straight line $x=y=0$

A. The straight line $x=y=0$ is a strata of the the previous affine variety W. Any vector field tangent to W is tangent to this straight line. Let us show that it has to vanish at every point of this straight line. If not, its flow at time t would map a point $(0,0,z_0)$ to a point $(0,0,z_1)$ with $z_1\neq z_0$. Its differential then induce a linear automorphism of the normal bundle of that straight line that has to preserve the straight lines $x=0, y=0, x+y=0$. Since a linear endomorphism of $\mathbb{C}^2$ preserving three straight lines has to be a multiple of the identity map, this differential cannot map the straight line $x+z_0y$ to the straight line $x+z_1y$. This concludes the proof.

From now on, we assume that $r=1$ and $\varphi:=\varphi_1$ is a homogeneous polynomial[1] and admits an isolated singularity at zero.

A.

(a) Show that the complex singular foliation generated by $\mathcal{F}$ admits three leaves in this case : $M\setminus W$, $W\setminus\{0\}$ and $\{0\}$.

A.

We invite the reader to start with the weight homogeneous polynomial $\varphi(x,y,z)=xy-z^n$ with $n\geq 2$ (the weights of x,y,z being $n,n,2$ respectively) in order to understand the logic of the construction.

A.

(b) Let $\overrightarrow{E}$ be the *Euler vector field* :

$$\overrightarrow{E}:=\sum_{i=1}^{d} n_i x_i\frac{\partial}{\partial x_i}.$$

Show that the Euler vector field is in $\mathfrak{X}_W(M)$.

A.

(c) Show that any vector field of the form $P^{\#}(d\varphi)$ is in $\mathfrak{X}_W(M)$, with P a bivector field on $\mathbb{C}^d$, and $P^{\#}$ the corresponding $1-1$ tensor from T^*M to TM.

A.

1. The variables may have non-negative weights $n_1,\dots,n_d$

(d) Give a set of generators of $\mathcal{F}$.

A.

(e) Compute the isotropy Lie algebras of $\mathcal{F}$ at the origin

i. when $\varphi(x_1, \dots, x_d) = \sum_{i=1}^d x_i^2$,

A.

ii. when $\varphi(x_1, \dots, x_d) = \sum_{i=1}^d x_i^3$

A.

(f) We consider the singular foliation

$$\mathcal{F}_\varphi := \{X \in \mathfrak{X}(V) \mid X[\varphi] = 0\}.$$

i. Give a set of generators of $\mathcal{F}_\varphi$.

A.

ii. Give an almost algebroid structure $(A, [\cdot\,, \cdot]_A, \rho)$ for $\mathcal{F}_\varphi$.

A.

Look for the notion of a "Koszul resolution", and show that k-vector fields with $k \geq 2$, equipped with the contraction by $d\varphi$ form a geometric resolution of $\mathcal{F}_\varphi$.

A.

iii. The almost Lie algebroid is the beginning of a Lie ∞-algebroid structure on a geometric resolution : compute a 3-ary bracket.

A.

(g) Apply the previous question to $\varphi(x_1, \dots, x_d) = \sum_{i=1}^d x_i^3$.

A.