# Singular Foliations

## Problem session

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July 21, 2022

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## Outline

**Topic: Vector fields tangent to a subset**

1. The smooth setting

2. The algebraic setting

## Smooth setting

- $M$ is a smooth manifold and $W \subset M$ an embedded closed sub-manifold.
- $\mathfrak{X}_W(M)$ is the space of all compactly supported vector fields on $M$ that are tangent to the sub-manifold $W$.

## Algebraic setting

- $M = \mathbb{C}^d$ and $W \subset M$ an affine variety i.e. $W$ is the zero locus of some polynomial functions $\varphi_1, \ldots, \varphi_r \in \mathbb{C}[x_1, \ldots, x_d]$.
- We denote by $\mathfrak{X}_W(M)$ the space of all vector fields on $M$ that are tangent to $W$.

Exercice 1: **Vector fields tangent to a sub-manifold $W$**

## Vector fields tangent to a sub-manifold $W$

1. Check that $\mathfrak{X}_W(M)$ is a singular foliation on $M$. Give a set of local generators.
   - By definition

$$\mathfrak{X}_W(M) = \{X \in \mathfrak{X}(M) \mid X|_m \in T_mW, \text{ for all } m \in W\}$$

   - For all $X, Y \in \mathfrak{X}(M)$,

$$[X,Y]_{|_m} = (\mathcal{L}_X Y)_{|_m} = \frac{d}{dt}_{|t=0} T_{\phi^X_{-t}(m)}\phi^X_t(Y_{\phi^X_{-t}(m)}),$$

   where $\phi^X_t$ is the flow of $X$.

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## Vector fields tangent to a sub-manifold $W$

- For every $m \in W$, there exists local coordinates $(\mathcal{U}, x_1, \dots, x_a, y_1, \dots, y_b)$ around $m$ on which $W$ is given by

$$y_1 = \cdots = y_b = 0.$$

## Vector fields tangent to a sub-manifold $W$

- The so-called Hadamard Lemma, states that every smooth function

$$y_1, \dots, y_b \mapsto f_j(x_1, \dots, x_a, y_1, \dots, y_b)$$

such that $f_j(x_1, \dots, x_a, 0, \dots, 0) = 0$, there exists smooth functions $g_1^j, \dots, g_b^j$ on $\mathcal{U}$ such that

$$f_j(x_1, \dots, x_a, y_1, \dots, y_b) = \sum_{k=1}^{b} y_j \, g_j^k(x_1, \dots, x_a, y_1, \dots, y_b).$$

## Vector fields tangent to a sub-manifold $W$

2 Describe the singular distribution $m \mapsto T_m(\mathfrak{X}_W(M))$.

- For $m \notin W$, $T_m(\mathfrak{X}_W(M)) = T_mM$:
- For $m \in W$, $T_m(\mathfrak{X}_W(M)) = T_mW$:

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## Vector fields tangent to a sub-manifold $W$

3.a the leaves of $\mathfrak{X}_W(M)$

- The connected components of submanifold $W$ and the connected components of $M \setminus W$ are the leaves of $\mathfrak{X}_W(M)$
- Moreover,
  1. if $\mathrm{codim}(W) \geq 2$ and $M, W$ are connected, then $\mathfrak{X}_W(M)$ has only 2 leaves: $W$ and $M \setminus W$.
  2. If $M, W$ are connected and $\mathrm{codim}(W) = 1$, one can have one or two connected component in $M \setminus W$.

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## Vector fields tangent to a sub-manifold $W$

3.b The set of all regular points.

- $M \setminus W$.

3.c The transverse singular foliation to each one of these leaves.

- For a leaf $L \subset M \setminus W$, the transverse foliation is a point.
- For a leaf $L \subset W$, it is represented by vector fields on $\mathbb{R}^b$ vanishing at 0, with $b$ the codimension of $W$ in $M$: How to see this?

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## Vector fields tangent to a sub-manifold $W$

1. Let $(x_1, \ldots, x_a, y_1, \ldots, y_b)$ be local coordinates on which $W$ is given by $y_1 = \cdots = y_d = 0$
2. Consider the transverse submanifold $x_1 = \cdots = x_a = 0$.
3. Vector fields in $\mathfrak{X}_W(M)$ are, after restriction to that open set, vector fields of the form:

$$X = \sum_{i=1}^{a} h_i \frac{\partial}{\partial x_i} + \sum_{j,k=1}^{b} g_j^k \, y_j \frac{\partial}{\partial y_k}.$$

## Vector fields tangent to a sub-manifold $W$

- By restricting to the transversal, one obtains vector fields of the form

$$\sum_{j,k=1}^{b} g_j^k(0,\ldots,0,y_1,\ldots,y_b)\, y_j \frac{\partial}{\partial y_k}$$

  This is exactly vector fields on a neighborhood of 0 in $\mathbb{R}^b$ that vanish at 0.

## Vector fields tangent to a sub-manifold $W$

4.a The isotropy Lie algebras of $\mathcal{F} = \mathfrak{X}_W(M)$ at every point.

## Vector fields tangent to a sub-manifold $W$

4.b The rank of $\mathfrak{X}_W(M)$ at every point .

- $rk_m(\mathcal{F}) = \dim(L_m) + \dim(g_m)$

## Vector fields tangent to a sub-manifold $W$

5. Is $\mathfrak{X}_W(M)$ the image through the anchor map of a Lie algebroid?
   - There is a neighborhood of $W$ on $M$ where the answer is yes.

## Vector fields tangent to a sub-manifold $W$

- This follows from the tubular neighborhood theorem, which gives a diffeomorphism between a neighborhood of $W$ in $M$ and a neighborhood of the zero section in the normal bundle $N_W$. Under this diffeomorphism, $\mathfrak{X}_W$ becomes the space of all vector fields on $N_W$ tangent to the zero section. It is easy to check that:
  1. Fiberwise linear vector field on $N_W$ are the sections of a transitive Lie algebroid (called $CDO(N_W)$ in Kirill Mackenzie) over $W$,
  2. that this Lie algebroid acts on $N_W$,
  3. and that the anchor of the transformation Lie algebroid admits for image all vector fields tangent to the zero section.

## Vector fields tangent to a sub-manifold $W$

6. Describe the holonomy groupoid of $\mathfrak{X}_W(M)$.
   - Assume that $W$ and $M \setminus W$ are connected.
     1. For $L = M \setminus W$, the restriction of the holonomy groupoid to $L$ is the pair groupoid $(M \setminus W) \times (M \setminus W) \rightrightarrows (M \setminus W)$.
     2. For $L = W$, the restriction of the holonomy groupoid is the general linear groupoid of the normal bundle $N_W = TM_{|W}/TW$.

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## The algebraic setting

Exercice 3: **Vector fields tangent to an affine variety** $W$

## Vector fields tangent to an affine variety $W$

1. Explain and justify why it makes sense to define $\mathfrak{X}_W(M)$ as above.
2. Show that $\mathcal{F}$ is a singular foliation over the algebra of polynomials in $d$ variables.
   - Any vector fields on $M = \mathbb{C}^d$ decomposes in a unique manner as a sum

$$P_1(x_1, \ldots, x_d)\frac{\partial}{\partial x_1} + \cdots + P_d(x_1, \ldots, x_d)\frac{\partial}{\partial x_d}.$$

3. What is
   - $T_m\mathcal{F}$ for all $m \in M \setminus W$?
   - Do we still have $T_a\mathcal{F} = T_aW$ as in the smooth case?

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## Vector fields tangent to an affine variety $W$

- It is a classical property that $a \in W_{reg}$ there exists "local coordinates" $y_1, \ldots, y_d \in \mathcal{O}_a$. such that $W$ is of the form
$$y_1 = \cdots = y_k = 0,$$
i.e. the localization of $\mathcal{I}_W$ is generated by these variables. For $v \in T_aW$ the local vector field
$$X = \sum_{i=1}^{\dim W} v_i \frac{\partial}{\partial y_{k+i}}$$
maps $\mathcal{O}_a$ to $\mathcal{O}_a$.

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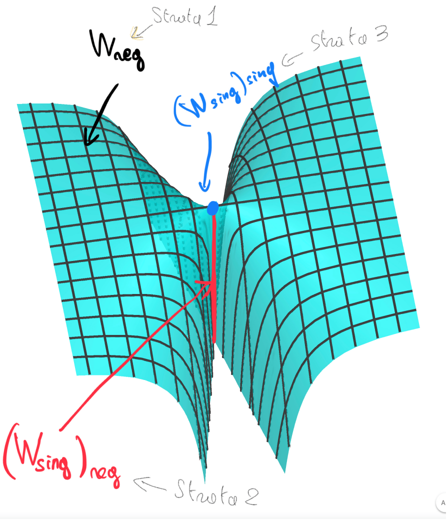
## Vector fields tangent to an affine variety $W$

- $T_m\mathcal{F}$ for $m$ an isolated singularity of $W$, i.e. a point $m$ where $d_m\varphi_i = 0$ for $i = 1, \ldots, r$ and which is isolated among such points.

4. Show that the tangent space of $\mathcal{F}$ at $m \in W$ is included into the tangent space of the strata of $W$ at this point.

# Frame Title

Whitney's umbrella $x^2 + zy^2 = 0$



## Vector fields tangent to an affine variety $W$

$\mathcal{I}_{sing}$ is generated by the minors of order $k = d - \dim W$ of $k$ elements chosen into the generators $\varphi_1, \ldots, \varphi_r$. That is, $W_{sing}$ is given the ideal

$$\left\langle \mathcal{I}_W, P[\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}],\ P \in \mathfrak{X}^k(\mathbb{C}^d),\ \begin{array}{c} 1 \leq i_1 < \cdots < i_k \leq r \\ \\ d - \dim W \leq k \leq r \end{array} \right\rangle$$

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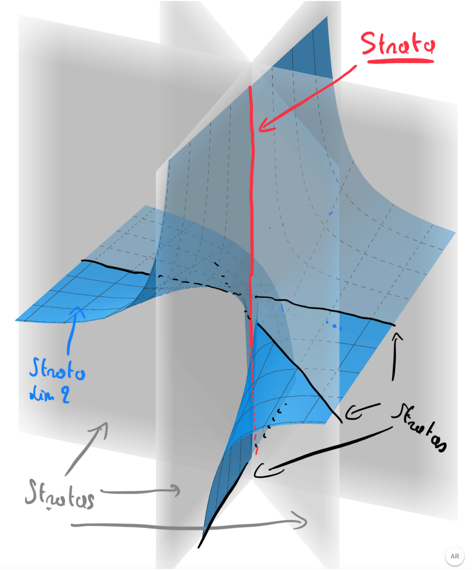

## Vector fields tangent to an affine variety $W$

From now on, we assume that $r = 1$ and $\varphi := \varphi_1$ is a homogeneous polynomial and admits an isolated singularity at zero.

5. Let $\overrightarrow{E}$ be the *Euler vector field*:

$$\overrightarrow{E} := \sum_{i=1}^{d} n_i x_i \frac{\partial}{\partial x_i}.$$

   Show that the Euler vector field is in $\mathfrak{X}_W(M)$.

## Vector fields tangent to an affine variety $W$

Consider the singular foliation

$$\mathcal{F}_\varphi := \{X \in \mathfrak{X}(V) \mid X[\varphi] = 0\}.$$

6. Give a set of generators of $\mathcal{F}_\varphi$.
7. Give an almost algebroid structure $(A, [\cdot\,,\cdot]_A, \rho)$ for $\mathcal{F}_\varphi$.
   - $$[\bar{E}, \bar{H}_{ij}]_2 := (|\varphi| - 2)\bar{H}_{ij}$$
     $$[\bar{H}_{ij}, \bar{H}_{kl}]_2 := \frac{\partial^2\varphi}{\partial x_i \partial x_l}\bar{H}_{jk} - \frac{\partial^2\varphi}{\partial x_i \partial x_k}\bar{H}_{jl} + \frac{\partial^2\varphi}{\partial x_j \partial x_k}\bar{H}_{il} - \frac{\partial^2\varphi}{\partial x_j \partial x_l}\bar{H}_{ik}.$$

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