Introduction to Singular Foliations

(And mainly to its Geometry)

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Schedule :

1. Tuesday : What are singular foliations ?
2. Wednesday : What structures do they hide ?
3. Thursday : exercises, symmetries of a subset.
4. **Friday : More structures they hide + open questions**.

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Where are we ?

Definition

A bi-submersion over $(M, \mathcal{F})$ is a triple (M, s, t) where :

1. X is a manifold,
2. $s, t \colon X \to M$ are submersions, respectively called source and target,

such that

1. The pull-back singular foliations $s^{-1}\mathcal{F}$ and $t^{-1}\mathcal{F}$ are equal,
2. and $s^{-1}\mathcal{F} = t^{-1}\mathcal{F}$ coincides with the sheaf of sections of the form $X + Y$ with $X \in \Gamma(\ker(Ts))$ and $Y \in \Gamma(\ker(Tt))$.

Bi-submersions over $(M, \mathcal{F})$ shall be denoted by $M \xleftarrow{s} X \xrightarrow{t} M$.

Example : Lie groupoids, anchored bundles.

Is it like Lie groupoids ?

YES	NO
$\exists \; s, t, \epsilon$	There is no product nor inverse !
$\exists$ inverse X^{-1}	Not an inner rule.
$\exists$ product $X \circ Y$	Still not inner rule
Inner on fundamental atlases	But it is not a real inverse !
$\exists$ Morita Equivalence	Not like Morita at all !
Bisections induce symmetries	✓
A quotient is a groupoid	But not Lie !
Debord : on each leaf "Yes".	Not on transversal !

This is how **AS** holonomy Lie groupoid is defined.

Compactness

Question

What does properness (or compactness) of the AS holonomy Lie groupoid implies ? For instance, does it imply that, for any point, the transverse singular foliation is given by a linear action of a compact Lie group by isometries of a finite dimensional Euclidian space ?

Even simpler :

Question

Does $\mathfrak{g}_m(\mathcal{F})$ compact Lie algebra implies it is a linear action of a compact Lie group ?

Origin : Cerveau, Conn, Zung, Crainic, Fernandes, Marcut $(\dots)$'s linearization results...

Question (A.-Zambon)

Is any singular foliation the image through the anchor map of a Lie algebroid ?

Even more basic.

Question

It the singular foliation of vector fields on $\mathbb{R}^2$ vanishing quadratically at 0 the image $\cdots$?

A personal take :

1. Probably not,
2. ...but it is not the good question.

Proposition

Most singular foliation admit a geometric resolution of finite length .

+ Unique up to homotopy equivalence. The next theorem has a long story :

Theorem (CLG, Lavau, Strobl)

There is a Lie ∞-algebroid on any one of these geometric resolutions, and any two are homotopy equivalent.

Call them universal Lie ∞-algebroid of $\mathcal{F}$

The PhD thesis of Lavau : one more hidden structure (☠)

Proposition

A locally real analytic singular foliation admit a geometric resolution of finite length on any relatively compact open subset.

+ Unique up to homotopy equivalence. The next theorem has a long story :

Theorem (CLG, Lavau, Strobl)

There is a Lie ∞-algebroid on any one of these geometric resolutions, and any two are homotopy equivalent.

Call them universal Lie ∞-algebroid of $\mathcal{F}$

Various uses of this structure

Question (Lavau,CLG)

Is there a Maurer-Cartan equation in a DGLA that would classify deformations of a singular foliation ?

Question (Louis)

Can a Lie algebra action on an affine variety be extended to the ambient space ?

Question

(Open) What is the Molino-Atiyah class of a singular foliation and what is its geometrical interpretation ?

More open questions

Question

How to classify neighborhood of leaves ?

The regular case is too easy : only $\pi_1(L)$. Partial results by CLG and Ryvkin. Involves $\pi_n(L)$...

Question

What are the good cohomologies ?