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Geometry of singular foliations: a draft of an introduction

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Contents

Introduction

Warning !

These notes are not finished. Please consider them as a draft. In particular:

1. Many citations are missing.
2. Many proofs are missing.
3. Notations are not yet 100 per cent consistent.
4. Among parts which are not written: Claire Debord's proof of longitudinal smoothness, Stepan-Sussmann theorem, proofs of existence in the construction of Androulidakis-Skandalis holonomy groupoid, and some open questions.

Flying under radar: Singular foliations.

Singular foliations are so common in mathematics that they often go unnoticed.

Regular foliations have been long studied; The Frobenius theorem is taught quite early in the differential geometry curriculum. Holonomy (or ”first return”) is a very classical notion. In contrast, singular foliations have never been studied with such an intensity. Still, there is a long story behind foliations that have leaves which are not all of the same dimension:

1. As pointed by Sylvain Lavau [Lav18], the 1960’s saw an intense debate about finding a correct definition of a singular foliation. The discussion led to some major discoveries by H. Hermann (1962), T. Nagano (1966), P. Stefan (1970), H. Sussmann (1973)[1]. See [Her62]-[Nag66]-[Ste74]-[Ste80]-[Sus73a]-[Sus73b].

2. Then the subject seems to have been slightly forgotten, or at least put aside. There were, still, important contributions to the linearization problem by Dominique Cerveau [Cer79] (where ”singular foliations” appear under the name of ”involutive distributions”) in 1977, and Pierre Dazord [Daz85], who defined a holonomy map for a singular leaf in 1984. There were other contributions coming from complex geometry. Also, Poisson geometers knew that symplectic leaves of a Poisson manifold, or Lie algebroid leaves, were a sort of ”singular foliation” [Lic77]-[Wei83], but, to our knowledge, rarely saw it as such.

3. Then, starting in the 2000’s, a ”singular foliations renewal” arose from non-commutative geometry, with pioneering and fundamental works by Iakovos Androulidakis, Claire Debord, and Georges Skandalis in particular. It is unfair to summarize their contributions in one sentence, but since we have to do so, let us claim that the main feat is the construction, by Androulidakis and Skandalis [AS09], of a holonomy groupoid of a singular foliation, that extends holonomy groupoids of regular foliations [MM03], and a smooth groupoid previously constructed by Claire Debord for projective singular foliations [Deb01]. A theorem of crucial importance was also made by Claire Debord: although Androulidakis-Skandalis holonomy groupoid is not smooth, it is longitudinally smooth [Deb13a].

   This holonomy groupoid, or more precisely its natural C^*-algebra, is used by this school to define elliptic pseudodifferential operators, their analytic indexes, to investigate its Baum-Connes conjecture [AS19], Boutet de Monvel calculus [DS21] - in one word, to do analysis of singular foliations.

The purpose of the present introduction is not to do analysis of singular foliations, although it is certainly the most active topic at the moment. We are not competent in non-commutative geometry, in index theorem and so on. Our purpose is to introduce to the geometry of a singular foliation.

Let us go back to the initial debate - in a very anachronistic manner: Should singular foliations be seen:

(0) as level sets (called ”leaves”) of non-independent functions?

1. as a partition of a manifold into submanifolds?

2. as the data, at each point, of sub-spaces of the tangent space, (satisfying an involutivity condition)?

3. or as an involutive $\mathcal{C}^\infty(M)$-module of vector fields morally thought to be tangent to the leaves?

Definition (0) (i.e. ”level set of non-independent functions”) is opposite to what we intend to study here: leaves would not be manifolds, and even if we work within the context of algebraic geometry (so that these level sets would be affine varieties) there is still a problem: exceptional leaves would be of bigger dimensions than the ”regular” ones. Such partitions into affine varieties seem to have very few interesting geometric properties. We do not claim that it is not interesting by itself, but this is clearly opposite to what we are looking for.

Now, the three remaining points of view (1) partition into submanifolds, 2) distribution, 3) defined through its tangent vector fields) have to be made more precise to yield a reasonable definition of a

[1] We refer to Sylvain Lavau’s excllent article [Lav18] for the historical aspects. [Lav18] can also be read as an introduction to the subject.

singular foliation. As we shall see in the first chapter, all of them allow counter-examples to properties that we wish to be true. This does not mean that they have to be rejected, but they have to be made precise.

We may dare to say that after that debate took place in the late 1960's, only two definitions survived to the XXI-st century:

($\star$) "Consensus definition" A singular foliation is a sub-sheaf of the sheaf of vector fields which is stable under Lie bracket, and under multiplication by a smooth function and locally finitely generated as a module over smooth functions. [2].

($\star\star$) "Variation" A singular foliation is a partition of a submanifold into leaves, such that through any vector tangent to a leaf there is at least a vector field tangent to all leaves [Sus73b, DLPR12].

We will work with the first of these definitions, for the following reasons:

1. Definition ($\star$) implies definition ($\star\star$): Singular foliations in the sense of ($\star$) do admit leaves which are honest submanifolds and partition the manifold[3] and the henceforth obtained partition satisfies (ii),

2. the tangent spaces of these leaves form a (singular) involutive distribution,

3. it is -according to us- general enough to contain most interesting examples,

4. but it is restrictive enough to be able to prove strong results,

   Singular foliations as in ($\star\star$) may not admit a AS-holonomy groupoid (at least, this is not known).

5. ($\star$) is used by a now well-established community of non-commutative geometers (Androulidakis, Debord, Mohsen, Skandalis, Yuncken, Skandalis, Zambon - to cite a few) and some theoretical physicists (e.g. Kotov, Strobl), while ($\star\star$) is less commonly used [**?**, Miy21].

In conclusion: we will present the theory of singular foliations using Definition ($\star$). We have no time to detail here, but it would in fact not be a very different theory using Definition ($\star\star$).

Are singular foliations worth studying?

Since we are given the opportunity to present our work to (mostly young) Poisson geometers, we have to argue: is there a point in studying singular foliations?

First, whoever studies Poisson geometry will encounter a highly non-trivial singular foliation: the symplectic leaves of a Poisson structure. But we claim more: whoever understands Poisson geometry has understood objects which are more or less analogous to those used in the geometry of singular foliations. Half of the way is behind you.

Below, we listed classical notion of Poisson geometry on the left, and its equivalent object in the SF-theory on the left[4]: if you know what the left hand column is about, understanding the right hand column should not be overly difficult.

[2]For those unfamiliar with or hostile to sheaves, this definition can be equivalently stated as: a locally finitely generated involutive sub-$\mathcal{C}^\infty(M)$-module of the module of compactly supported vector fields.

[3]and this is the least we can require to dare calling an object "singular foliation": leaves have to make sense!

[4]We use the abbreviations SF= Singular Foliations, AS =Androulidakis-Skandalis

Notion in Poisson geometry	The equivalent notion in Singular Foliation theory
Poisson manifold (M, π)	Singular foliation $\mathcal{F}$ on M
Hamiltonian flow are Poisson diffeo.	Flows of vector fields in $\mathcal{F}$ are symmetries of $\mathcal{F}$
Weinstein's splitting theorem	Singular Foliations' splitting theorem
Decomposition into symplectic leaves	Decomposition into leaves
Transverse Poisson structure (of a leaf)	Transverse singular foliation (of a leaf)
Poisson-Dirac reduction	Induced SF on a transverse submanifold
Lie algebroid structure on T^*M	(easy) almost Lie-algebroid structures covering $\mathcal{F}$ or (harder) universal Lie ∞-algebroids of $\mathcal{F}$
Isotropy Lie algebra $\ker \pi_m^\#$ at $m \in M$	(easy) isotropy Lie algebra of $\mathcal{F}$ at m (harder) isotropy Lie ∞-algebra of $\mathcal{F}$ at m.
Poisson cohomology	Longitudinal cohomology (easy) Cohomology of the universal Lie ∞-algebroid (harder)
Symplectic realization	Bi-submersions
Morita equivalences	Equivalences of Bi-submersions (arguably)
Symplectic Groupoid	AS holonomy groupoid

But please do not be mistaken and think that singular foliations are simply a generalization of Poisson geometry, like quasi-Poisson or Jacobi structures. The previous two-columns presentation is misleading: some of the objects on the right hand side are much more involved than those in the left hand side - although some are also easier. We therefore repeated the previous picture, but added comparison signs $>>>, >>, >, =, <, <<, <<<$ to indicate our (subjective) opinion about the difference in difficulty:

Notion in Poisson geometry		The equivalent notion in Singular foliation theory
Poisson manifold (M, π)	$=$	Singular foliation $\mathcal{F}$ on M
Hamiltonian flow are Poisson diffeo. (This is almost trivial)	$<<$	Vector fields tangent to $\mathcal{F}$ are symmetries of $\mathcal{F}$ (This is really hard, at least in the smooth case, Many existing proofs have gaps...)
Weinstein's splitting theorem	$=$	Singular Foliations' splitting theorem
Decomposition in symplectic leaves	$>$	Decomposition into leaves
Transverse Poisson structure (of a leaf)	$=$	Transverse singular foliation (of a leaf)
Poisson-Dirac reduction	$>$	Induced SF on a transverse submanifold
Lie algebroid structure on T^*M	$>$ $<<$	(easy) almost Lie-algebroid structures generating $\mathcal{F}$ or (harder) the universal Lie ∞-algebroid of $\mathcal{F}$
Isotropy Lie algebra $\ker \pi_m^\#$ at $m \in M$	$=$ $<<$	(easy) isotropy Lie algebra of $\mathcal{F}$ at m (harder) isotropy Lie ∞-algebra of $\mathcal{F}$ at m.
Poisson cohomology	$>$ $<<$	Longitudinal cohomology (easy) Cohomology of the universal Lie ∞-algebroid (harder)
Symplectic realization	$=$	Bi-submersions
Morita equivalences	$=$	Equivalences of Bi-submersions
Symplectic Groupoid (This is often a smooth groupoid at worst, a stacky groupoid)	$<<<$	AS holonomy groupoid (This is almost never a smooth groupoid, not even a stacky groupoid)

In particular, the AS holonomy groupoid is not like any Lie groupoid Poisson geometry has so far produced. Its non-smoothness is at the origin of the subtle analysis developed by non-commutative geometers. Although the AS holonomy groupoid is certainly the most studied aspect of singular foliation at the present time [AS09]-[Deb13a]-[AS11]-[AS19]-[AZ13a], we will in fact say very little about it: these notes are about geometry more than about analysis, while AS holonomy groupoid's complex beauty is better understood while trying to understand the meaning of notions in analysis in presence of a singular foliation.

To which area of mathematics do singular foliations belong to?

As we will see, singular foliations shall be defined as a sub-algebra $\mathcal{F}$ of vector fields, stable under the Lie bracket and under multiplication by a function, and the leaf through a point $m \in M$ shall be the set of points reachable from m following the flows of vector fields in $\mathcal{F}$. Those vector fields in $\mathcal{F}$ are, heuristically, vector fields "tangent to all leaves". But there is an additional assumption in the consensus definition: "locally finitely generated". Also the consensus definition works with compactly supported vector field. Before dealing with those, we have to address a more fundamental question: in which area of mathematics are we?

The present manuscript is mainly written having in mind the universe of smooth differential geometry. But singular foliations do make sense in real analytic differential geometry, in complex geometry, and in algebraic geometry as well. And we will try to deal with all three aspects altogether. For that purpose, we will use the language of sheaves. The reader interested only in the smooth case may perfectly ignore the word "sheaf" and replace it by the corresponding global objects. For technical reasons, it is then better to use "compactly supported" objects. More precisely:

1. In real differential geometry, sheaves can be ignored, and singular foliations on a manifold M will be defined as a locally finitely generated sub-$\mathcal{C}^\infty(M)$-module of compactly supported vector fields stable under Lie brackets.

2. In real analytic or holomorphic or algebraic settings, global objects may not exist, or it may be that are too few of them. One has to work with the sheaf of vector fields, and it does not make sense to consider compactly supported vector fields any more.

   However, the rings of germs of real analytic, holomorphic or regular functions being Noetherian, the "locally finitely generated" assumption is always satisfied and can therefore be omitted. A singular foliation is then simply a sub-sheaf of the sheaf of vector fields stable by multiplication under a function and stable under Lie bracket.

3. In smooth, real analytic or complex settings, singular foliations induce a partition of M into leaves which are smooth, real analytic or complex submanifolds respectively. This is not true anymore in algebraic geometry: the "leaves" are not algebraic sub-varieties. This is highly related to the well-known fact that the flow of a polynomial vector field is a real-analytic or holomorphic map but not a polynomial map in general.

Again, although we will deal with real analytic or holomorphic or algebraic settings, we will mostly take the smooth differential geometry point of view. Also, we will assume that the reader knows everything about differential geometry: classical or less-classical theorems about flows of vector fields will be mostly left to the reader, and only those specific to singular foliations shall be detailed.

Conventions

Throughout these whole notes: manifolds shall be separated and second countable, $\mathfrak{X}$ stands for the sheaf of vector fields on a manifold M. Compactly supported vector fields shall be denoted by $\mathfrak{X}_c(M)$. Smooth functions shall be denoted by $\mathcal{C}$. For holomorphic or polynomial functions, we shall use the symbol $\mathcal{O}$.

Also, for X a vector field on M or e a section of a vector bundle $E \to M$, we denote by $X_{|_m}$ and $e_{|_m}$ their value at a point $m \in M$.

Sections over an open subset $\mathcal{U} \subset M$ of a vector bundle $E \to M$ shall be denoted by $\Gamma_{\mathcal{U}}(E)$.

Restrictions to an open $\mathcal{U} \subset M$ or "any-mathematical-notion-N-that-restricts" will often be denoted by $\mathfrak{i}_{\mathcal{U}}^* N$.

Chapter 1

What is a singular foliation ?

1.1 Naive and less naive attempts of a definition of a singular foliation

In order to understand the geometric ideas behind the consensus definition of a singular foliation, let us make a list of definitions that are natural, but turned out to be mostly dead ends. This chapter is widely inspired by Sylvain Lavau's *"A short guide through integration theorems of generalized distributions"* [Lav18], and by Iakovos Androuilidakis and Marco Zambon's *"Stefan-Sussmann singular foliations, singular subalgebroids, and their associated sheaves"* [AZ16]. [].

1.1.1 Is a singular foliation simply a partition by smooth manifolds?

A regular foliation partitions a manifold into submanifolds, all of the same dimension. As a consequence, the most natural idea that comes in mind when trying to make up a definition of a singular foliation is to try to define them as being a a disjoint union of submanifolds called "leaves" - now of varying dimension. This perfectly makes sense, but let us give it an other name.

Definition 1.1.1: A first attempt to define singular foliations: partitionifolds

Let M be a manifold. A partitionifold of M is a partition of M into connected immersed submanifolds[a]*, called* leaves.

[a] From now on, "submanifold" means by default "immersed submanifolds".

Partitionifolds are such a general object that -as far as we know- nothing interesting can be said about them. Still, we will see that properly defined singular foliation induce a partitionifold of M, but it is certainly not its definition.

Notation 1.1.2: To a point, we associate its leaf

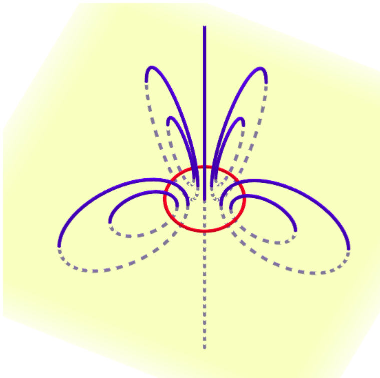
A partitionifold on a manifold M shall be denoted as a map:

$$\begin{array}{cccc} L_\bullet : & M & \to & \{\textit{Submanifolds of } M\} \\ & m & \mapsto & L_m \end{array}$$

that assigns a point $m \in M$ to the submanifold in the partition to which it belongs. Also, for all $m \in M$, L_m shall be called the leaf through m.

Below are two examples of partitionifolds that we not wish to allow as being decent "singular foliations".

Example 1.1.3. *The magnetic partition* is an example of partitionifold that behaves badly, although it is "regular" in the sense that all its leaves have the same dimension. It is given as follows:



These can be seen as being the lines of a magnetic field generated by an electric current in the red circle, to which the red circle itself is added. The problem with this partitionifold is that

1. all leaves have dimension 1 (they are all circles, except for one straight line),

2. but it is not a regular foliation in a neighborhood of the red circle.

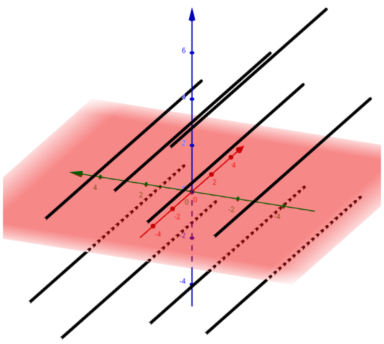
French speakers may also look at the Agrégation de Mathématiques of 1998, "Sujet de mathématiques générales": Its first part is dedicated to the construction of a partitionifold on $\mathbb{R}^3$ whose leaves are all circles.

$\square$

Example 1.1.4. *"Isolated lasagna in a dish of spaghetis".* Consider the partitionifold on $M = \mathbb{R}^3$ whose leaves are defined to be:

1. The plane $z = 0$ (the "isolated lasagna" in red). This is the only leaf of dimension 2.

2. All the straight lines (the "spaghettis" in black) parallel to the x-axis (in red) and not contained in the plane $z = 0$.



$\square$

To explain why partitionifolds are too large a class to deserve to be called singular foliations, let us introduce a very natural sub-space (in fact, sub-sheaf) of the sheaf of vector fields.

Notation 1.1.5: Vector fields tangent to every leaves

Let $L_\bullet$ be a partitionifold on M. We denote by $\mathfrak{T}(L_\bullet) \subset \mathfrak{X}(M)$ the sub-sheaf[a] *of vector fields tangent to all leaves, i.e. such that $X_{|_\ell} \in T_\ell L_\ell$ for all ℓ in the open space on which X is defined. We call such vector fields* tangent to $L_\bullet$ at all points *or simply* tangent to the partitionifold.

[a]The reader unfamiliar to sheaves could define instead $\mathfrak{T}(L_\bullet)$ to be the sub-spaces of compactly supported vector fields satisfying $X_{|_\ell} \in T_\ell L_\ell$ for all $\ell \in M$. Sheaves are only necessary while working within the framework of complex or real analytic geometry.

Paritionifolds are too general an object to satisfy many properties, but here is at least a result that they satisfy:

Proposition 1.1.6: Can not jump from leaves to leaves

Let $L_\bullet$ be a partitionifold on M. An integral curve $\gamma(t)$ of a vector field $X \in \mathfrak{T}(L_\bullet) \subset \mathfrak{X}(M)$ is always contained in one leaf.

Proof. There is a difficulty: a vector field X may be tangent to a submanifold $L \in M$ but its flow may not preserve it (for instance, consider the Euler vector field on $M = \mathbb{R}^n$, and let L be any open ball entered at 0).

In the present situation however, we are given a vector field X is tangent to $L_\bullet$ at all points. This implies that any integral curve $t \mapsto \gamma(t)$ of X is "locally lies in the same leaf", i.e. for any t_0 there is $\epsilon > 0$ such that $L_{\gamma(t)} = L_{\gamma(t_0)}$ if $|t - t_0| < \epsilon$. Since intervals are connected sets and $L_\bullet$ form a partition of M, the integral curve γ must be in one leaf on its full domain. □

Remark 1.1.7. It is clear that for any partitionifold $L_\bullet$ on M, and any open subset $\mathcal{U} \subset M$, a partitionifold on $\mathcal{U}$ is obtained by mapping $m \in \mathcal{U}$ to the connected component of m in $L_m \cap \mathcal{U}$.

We denote by $\mathrm{i}_U^* L^\bullet$ this partitionifold and call it *restriction to* $\mathcal{U}$ of $L_\bullet$.

Given partitionifolds $L_\bullet$ on M and $L'_\bullet$ on M', we call *isomorphism from $L_\bullet$ to $L'_\bullet$* a diffeomorphism $\phi\colon M \to M'$ such that $\phi(L_m) = \phi(L'_{\phi(m)})$ for all $m \in L$. When $M = M'$ and $L_\bullet = L'_\bullet$, we shall speak of a *symmetry of $L_\bullet$*.

Proposition 1.1.8: Flows are symmetries

Let M be a manifold equipped with a partitionifold $L_\bullet$. The flow at time t of a complete vector field $X \in \mathfrak{T}(L_\bullet)$ tangent to all leaves is a symmetry of $L_\bullet$.
More generally, for a maybe non-complete vector field $X \in \mathfrak{T}(L_\bullet)$ tangent to all leaves, its flow ϕ_t^X at time t, provided it is well-defined on some open subset $\mathcal{U} \subset L$, is a isomorphism from the restriction of $L_\bullet$ to $\mathcal{U}$ to the restriction of $L_\bullet$ to $\phi_t^X(\mathcal{U})$.

Proof. The first part of Proposition 1.1.8 is a consequence of the second one. We therefore only prove the second part and use notations of Remark 1.1.7. Consider two points $m_0, m_1 \in \mathcal{U}$ that are in the same leaf of $\mathrm{i}_U^* L_\bullet$, and therefore in the same leaf L of $L_\bullet$. There is a smooth path $m\colon [0,1] \to \mathcal{U}$ starting from m_0 and arriving at m_1 which is entirely contained in $L \cap \mathcal{U}$. Since integral curves can not jump from one leaf to an other one by Proposition 1.1.6, for every $u \in [0,1], s \in [0,t]$, non-finitely-many $\mapsto \phi_s^X(m(u))$ is valued in the leaf L. In particular, the curve

$$u \mapsto \phi_t^X(m(u))$$

is entirely contained in L. It is also contained in $\phi_t^X(\mathcal{U})$. Hence $\phi_t^X(m_0)$ and $\phi_t^X(m_1)$ are in the same leaf of $\mathrm{i}^*_{\phi_t^X(\mathcal{U})} L_\bullet$. This proves the claim. □

For $L_\bullet$ a partitionifold on L, and $S \subset M$ a submanifold, we can associate to every $s \in S$ the connected component $(L_s \cap S)_0$ of s in the intersection $L_s \cap S$. The map

$$\begin{array}{ccc} S & \to & \{\text{Connected subsets of } S\} \\ s & \mapsto & (L_s \cap S)_0 \end{array}$$

may not be a partitionifold: it is valued in connected subsets, but not in smooth manifolds. However,

Proposition 1.1.9: Restriction to a submanifold

Let M be a manifold equipped with a partitionifold $L_\bullet$. For every submanifold $S \cap M$ such that

$$T_sS + T_sL_s = T_sM, \tag{1.1}$$

the connected component $(L_s \cap S)_0$ of $s \in S$ in $L \cap S$ is a submanifold of S. In particular, $s \mapsto (L_s \cap S)_0$ is a partitionifold of S.

Proof. It is a classical result of differential geometry that $L_s \cap S$ is a submanifold in S provided that the tangent spaces of L_s and S add up to the tangent space of the ambiant manifold M at all points. □

Notation 1.1.10: How to denote a restriction?

Let M be a manifold equipped with a partitionifold $L_\bullet$. A manifold S satisfying (1.1) *shall be called* transverse to $L_\bullet$. *We denote by $\mathfrak{i}_S^* L_\bullet$ the partitionifold on S as in Proposition 1.1.9 and call it* restriction of $L_\bullet$ to M

Remark 1.1.11. Since open subsets of M are transverse to any smooth partitionifold $L_\bullet$ on M, the terminology and notations of Remark 1.1.7 match the previous conventions.

We now suggest a second notion that we claim could be the definition of a singular foliation: it is still not the consensus definition, but we are getting closer from a workable notion. It is in fact enough for certain purposes and is used in [Miy21, DLPR12, Sus73b].

Definition 1.1.12: A more subtle attempt: smooth partitionifolds

A partitionifold $L_\bullet$ is said to be smooth *if for every $\ell \in M$ and every tangent vector $u \in T_\ell L_\ell$, there exists a vector field X through u which is tangent to all leaves*[a].

[a] i.e. $X|_m \in T_m L_m$ for all $m \in M$

Said differently, a partitionifold is smooth if and only if, for every $\ell \in M$, the evaluation map

$$\begin{array}{ccc} \mathfrak{X}(L_\bullet) & \to & T_\ell L_\ell \\ X & \mapsto & X_{|_\ell} \end{array}$$

is a surjective linear map.

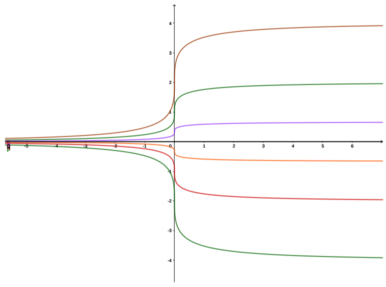
Let us start by a non-example.

Exercice 1.1.13. Show that neither the "magnetic partition" (Example 1.1.3) nor the "isolated lasagna in a spaghetti dish" (Example 1.1.4) are smooth partitionifolds.

□

Here is a second non-example.

Exercice 1.1.14. Consider the partitionifold on $\mathbb{R}^2$, with coordinates x, y whose leaves are the graph of the function $f_\lambda \colon x \mapsto \lambda\,(\mathrm{th}(\sqrt[3]{x}) + 1)$ with $\lambda \in \mathbb{R}$. For each value of λ, the graph of f_λ is a smooth[1] submanifold of dimension 1 in $\mathbb{R}^2$.



Show that this partitionifold of $\mathbb{R}^2$ is not smooth. Hint: consider a neighborhood of $(0,0)$ and use the fact that the tangent space of $L_{(0,y)}$ is for all $y \neq 0$ a vertical straight line.

□

Smooth partitionifolds behave much better than partitionifolds, as we will briefly show by giving several reasonable theorems that they satisfy.

The first interest of these results is that for a smooth partitionifold $L_\bullet$, along a given leaf, $L_\bullet$ "always looks the same".

Theorem 1.1.15: Along a leaf, landscape is always identical

Two points on the same leaf of a smooth partitionifold $L_\bullet$ have neighborhoods where the restrictions of $L_\bullet$ are isomorphic.

Theorem 1.1.15 is then an immediate consequence of Proposition 1.1.8 together with the following lemma, the proof of which is left to the reader.

Lemma 1.1.16. *Given any two points x, y on the same leaf L of an smooth partitionifold $L_\bullet$, there exists a finite number of complete vector fields $X_1, \ldots, X_n \in \mathfrak{T}(L_\bullet)$ (i.e. vector fields tangent to all leaves) such that if we apply successively the flows at time 1 of $X_1, \ldots, X_n$ to the point x, we obtain the point y.*

The following proposition means that singular leaves have smaller dimensions than regular ones:

Proposition 1.1.17: The dimension of the leaf is lower semi-continuous

Let M be a manifold equipped with an smooth partitionifold $L_\bullet$, the function:

$$\begin{array}{ccc} M & \to & \mathbb{N}_0 \\ m & \mapsto & \dim(L_m) \end{array}$$

is lower semi-continuous[a]*.*

[a] I.e. for all $k \in \mathbb{N}_0$, $\{m \in M \,|\, \dim(L_m) \geq k\}$ is an open subset in M

Proof. Let us choose a point $m_0 \in M$, let r be the dimension of the leaf through L_{m_0}, and let $(e_1, \ldots, e_r)$ be a basis of $T_{m_0} L_{m_0}$. By assumption, there exist r vector fields $X_1, \ldots, X_r$ through $(e_1, \ldots, e_r)$, defined in a neighborhood $\mathcal{U}$ of m_0 and tangent to all leaves. They are therefore independent at each point of a sub-neighborhood $\mathcal{U}' \subset \mathcal{U}$, so that $\dim(L_m) \geq r$ for all $m \in \mathcal{U}'$. □

Let L a leaf of a smooth partitionifold $L_\bullet$. A pointed submanifold (Σ, ℓ) that intersect L at ℓ (i.e. $\Sigma \subset M$ is a submanifold and $\ell \in \Sigma \cap L$) is said to be *transverse to L* if

$$T_\ell \Sigma \oplus T_\ell L = T_\ell M.$$

[1] Even if f_λ is not a smooth function at $x = 0$ for $\lambda \neq 0$, its graph is a smooth submanifold of dimension 1 in $\mathbb{R}^2$.

Lemma 1.1.18. *Any pointed submanifold (Σ, ℓ) transverse to a leaf L admits a neighborhood of ℓ on which is it transverse to $L_\bullet$.*

Proof. Let $X^1, \dots, X^k \in \mathfrak{T}(L_\bullet)$ be vector fields tangent to all leaves whose evaluations at ℓ form a basis of $T_\ell L$. There exists a neighborhood of ℓ is Σ into which

$$T_\sigma \Sigma \oplus \langle X^1_{|_\sigma}, \dots, X^k_{|_\sigma} \rangle = T_\sigma M.$$

This implies $T_\sigma \Sigma + T_\sigma L_\sigma = T_\sigma M$, which is precisely the definition of a transverse submanifold. □

In particular, for any pointed submanifold (Σ, ℓ) transverse to a leaf L, there exists a neighborhood $\mathcal{U}$ of ℓ in M such that the restriction $\mathrm{i}^*_{\Sigma \cap \mathcal{U}} L_\bullet$ is a smooth partitionifold, that we call a *transverse partitionifold* of the leaf L.

Corollary 1.1.19: The germ of a slice transverse to a leaf

Let M be a manifold equipped with an smooth partitionifold $L_\bullet$. Any two transverse partitionifolds of a given leaf L have neighborhoods on which their restrictions are isomorphic [a].

[a] More precisely, for any two pointed submanifolds (Σ_1, ℓ_1) and (Σ_2, ℓ_2) transverse to the same leaf L, there exists neighborhoods $\mathcal{U}_1 \subset \Sigma_1, \mathcal{U}_2 \subset \Sigma_2$ of ℓ_1, ℓ_2 and an isomorphism

$$\mathrm{i}^*_{\Sigma_1 \cap \mathcal{U}_1} L_\bullet \xrightarrow{\sim} \mathrm{i}^*_{\Sigma_2 \cap \mathcal{U}_2} L_\bullet \, .$$

This theorem implies that it makes sense to speak of the *local transverse model of a leaf L of a smooth partitionifold*.

We will have very similar theorems for singular foliations. Hence we simply decompose the proofs into exercises.

Again, we will see that singular foliations induce an smooth partitionifold, but this class is still too large. The following example illustrates two an oddity that we want to avoid.

Exercice 1.1.20. *"Vector fields tangent to the leaves are not finitely generated"*. On $M = \mathbb{R}$, consider the partitionifold whose 0-dimensional leaves are $\{1\}, \{\frac{1}{2}\}, \{\frac{1}{3}\}, \dots, \{\frac{1}{n}\}, \dots$ and $\{0\}$ and whose 1-dimensional leaves are the open intervals bounded by these points. Show that vector fields tangent to L are not a finitely generated module over $\mathcal{C}^\infty(M)$, and that there is no neighborhood $\mathcal{U}$ of 0 on which such vectors form a locally finitely generated $\mathcal{C}^\infty(\mathcal{U})$-module.

Question 1.1.21: Are smooth partitionifolds a good definition of singular foliations?

It is fine, but it not so widely used, and it has some limitations, see, e.g. exercice 1.1.20. However, the theory would not be so different from the one we will develop with the consensus definition.

As a consequence, we will try a new manner to define foliations with leaves of non-constant dimensions:

1.1.2 Is a singular foliation an involutive distribution?

A regular foliation may be defined as being an integrable sub-vector bundle $D \subset TM$. It is therefore tempting to allow the fibers of the vector bundle D to be of non-constant dimension, as long as its sections are closed under the Lie bracket of vector fields:

Definition 1.1.22: Integrable singular distributions

A singular distribution on a manifold M is a map $\mathfrak{D}$ associating to a point $m \in M$ a subspace $\mathfrak{D}_m \subset T_m M$. A singular distribution $\mathfrak{D}$ is said to be:

1. *involutive when $[\Gamma(\mathfrak{D}), \Gamma(\mathfrak{D})] \subset \Gamma(\mathfrak{D})$, where $\Gamma(\mathfrak{D}) \subset \mathfrak{X}(M)$ is the $\mathcal{C}^\infty(M)$-module of vector fields X such that $X_m \in \mathfrak{D}_m$ for all $m \in M$.*
2. *integrable when there exists a partitionifold $L_\bullet$ such that for all $m \in M$, $T_m L_m = \mathfrak{D}_m$.*

Exercice 1.1.23. Show that for any partitionifold $L_\bullet$ on M, the map

$$\mathfrak{D} : m \mapsto T_m L_m$$

is an involutive and integrable singular distribution.

Exercice 1.1.24. Let M be a manifold and m_0 a point. Show that the map

$$m \mapsto \begin{cases} T_{m_0}M & \text{if } m = m_0 \\ 0_{T_m M} & \text{if } m \neq m_0 \end{cases}$$

is an involutive but non-integrable singular distribution. (Here, 0_E stands for the zero element of a vector space E).

Let us have a discussion about leaves. There is a natural manner to define leaves for a singular distribution $\mathfrak{D}$, even if it is not integrable. Consider the equivalence relation on M generated by the relation $x_0 \sim x_1$ if there exists a path of class C^1 such that

$$x(0) = x_0, x(1) = x_1 \text{ and } \frac{d}{dt}x(t) \in \mathfrak{D}_{x(t)} \text{ and } \frac{d}{dt}x(t) \neq 0. \tag{1.2}$$

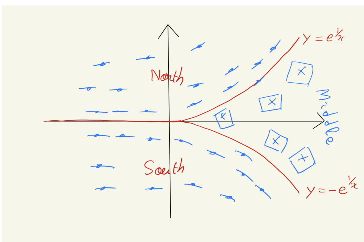
Equivalently, one could define an equivalence relation as follows: call integral submanifold of $\mathfrak{D}$ a submanifold Σ such that $T_\sigma\Sigma \subset \mathfrak{D}_\sigma$ for all $\sigma \in \Sigma$. We could then consider the equivalence relation generated by the relation $x_0 \sim x_1$ if there exists an integral submanifold containing both x_0 and x_1. The classes of this equivalence relation can not decently be called leaves, because they are not submanifolds, as seen in the following exercice.

Exercice 1.1.25. Here is an example (the "trumpet foliation") of an involutive singular distribution for which one class of the equivalence definition (1.2) is not a manifold. Take $M = \mathbb{R}^2$ with coordinates (x, y). Let $k(x) = e^{-1/x}$ for $x > 0$ and $k(x) = 0$ for $x \leq 0$. Divide $\mathbb{R}^2$ in three zones:

$$North := \{y \geq k(x)\}, Middle := \{x > 0 \text{ and } -k(x) < y < k(x)\}, South := \{y \leq -k(x)\}$$

Define a singular distribution by:

$$\mathfrak{D}_m = \begin{cases} \langle(1, k'(x))\rangle & \text{for } m \in North \\ T_m\mathbb{R}^2 & \text{for } m \in Middle \\ \langle(1, -k'(x))\rangle & \text{for } m \in South \end{cases}$$



1. Show that $\mathfrak{D}$ is involutive (but not integrable).
2. Show that the equivalence class of $(0, 0)$ is $\{y = 0\} \cup \overline{Middle}$.

It is clear that we have to avoid situations like the one in Exercice 1.1.24, as well as the tangent spaces of the partitionifolds of Examples 1.1.3 and 1.1.4 ("magnetic foliation" or "isolated lasagnas"). For that purpose, we will impose a second condition, similar to Definition 1.1.12 of smooth parititionifolds.

Definition 1.1.26: Smooth singular distributions

A singular distribution $\mathfrak{D}$ is said to be smooth if for every point $m \in M$ and $u \in \mathfrak{D}_m$, there exists a vector field $X \in \Gamma(\mathfrak{D})$ through u.

Exercice 1.1.27. Let $L_\bullet$ be a smooth partitionifold of M. Consider the singular distribution $\mathfrak{D}_L : m \mapsto T_m L_m$.

1. Show that it is integrable and involutive,
2. and smooth
3. and that the flow of any section in $\Gamma(\mathfrak{D}_L)$ preserves $\mathfrak{D}_L$.

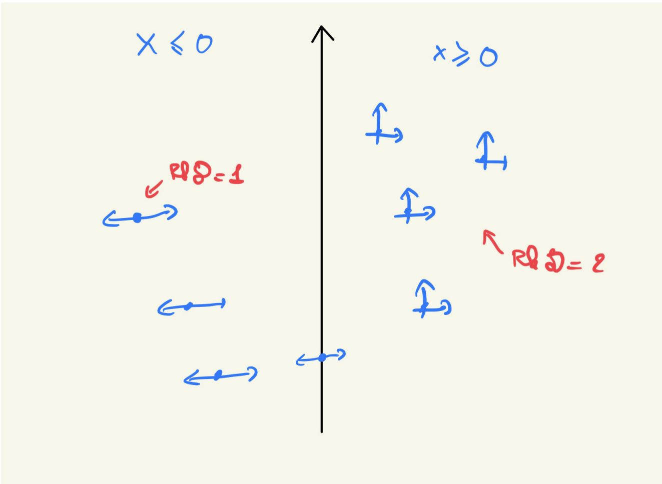
Exercice 1.1.28. For an involutive and integrable smooth distribution, show the classes of the equivalence relation (1.2) are precisely the leaves of $L_\bullet$

The two exercices above seem to indicate that smooth involutive singular distributions are a "good" notion.

There is however a type of counter-example which is quite annoying:

Exercice 1.1.29. Here is an integrable distribution, the "*infinite comb*", that will be a source of several counter-examples. Consider on $M = \mathbb{R}^2$ with variables (x, y) the singular distribution given by

$$\mathfrak{D}_{(x,y)} = \begin{cases} \langle \frac{\partial}{\partial x} \rangle & \text{if } x \leq 0 \quad \text{i.e. "Dimension 1 in the black zone - and horizontal."} \\ \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle & \text{if } x > 0 \quad \text{i.e. "Dimension 2 in the red zone."} \end{cases} \tag{1.3}$$



1. Show that the singular distribution $\mathfrak{D}$ is smooth.
2. Show that the singular distribution $\mathfrak{D}$ is involutive.
3. Show that any two points in $\mathbb{R}^2$ are in the same equivalence class of the relation 1.2.
4. Show that it is not integrable.

The last exercice shows that smooth involutive singular distributions may not be integrable. Another issue of the notion of smooth integrable distribution is that the flow of a complete vector field in $\Gamma(\mathfrak{D})$ may not be a symmetry of $\mathfrak{D}$. Again, the infinite comb is a counter-example:

Exercice 1.1.30. Show that the vector field $\frac{\partial}{\partial x}$ belongs to $\mathfrak{D}$ but that its flow does not preserve $\mathfrak{D}$.

Stefan-Sussmann theorems is a way out of these counter-examples.

Theorem 1.1.31: Stefan-Sussmann Theorem (1973) [Ste74]-[Ste80]-[Sus73a]-[Sus73b]-[DLPR12]

Let $\mathfrak{D}$ be an involutive smooth distribution on a manifold M. The following items are equivalent:

(i) $\mathfrak{D}$ is integrable.[a]

(ii) There exists a $\mathcal{C}^\infty(M)$-module $\mathcal{F}$ of vector fields such that:

(a) *$\mathcal{F}$ generates $\mathfrak{D}$.* [b]

(b) *The flow ϕ_t^X of any vector field $X \in \mathcal{F}$ preserves $\mathfrak{D}$.* [c]

[a] I.e. there exists a smooth partitionifold $L_\bullet$ such that $T_m L_m = \mathfrak{D}_m$ at all points $m \in M$.

[b] I.e. for all $m \in M$ and $u \in \mathfrak{D}_m$, there exists $X \in \mathcal{F}$ with $X_{|_m} = u$.

[c] I.e. $\phi_t^X(\mathfrak{D}_m) = \mathfrak{D}_{\phi_t^X(m)}$ for all $m \in M, X \in \mathcal{F}, t \in \mathbb{R}$ for which the flow is well-defined in a neighborhood of m.

It has been proven by [DLPR12] for any involutive smooth singular distribution, $\mathcal{F} \subset \Gamma(\mathfrak{D})$ in the above theorem can be assumed to be locally finitely generated over smooth functions. In the next section, we will see that "locally finitely generated" together with "closed under Lie bracket" are the conditions required for the modern "consensus definition" of what a singular foliation should be.

Let us conclude this section:

Question 1.1.32: Are involutive smooth singular distributions a good notion of singular foliations?

No, it is not!
Foliations should have leaves, and there is an issue with the notion of leaves. See discussion about infinite combs, which are not integrable.
However, with an additional condition on flows, Stefan-Sussmann Theorem 1.1.31 grants integrability. But this condition is hard to check in a concrete manner.

1.2 The consensus definition: Singular foliations through vector fields

Singular foliations may be seen as a subspace $\mathcal{F}$ of vector fields, subspace assumed to "behave like" vector fields tangent to the leaves of a partition of M by submanifolds. Notice that if what is given is a space of vector fields, satisfying some conditions, leaves are therefore not a priori given in the definition, and existence of leaves will be a theorem.

1.2.1 The smooth case

We now give the consensus consensus definition as it emerges nowadays in non-commutative geometry when the manifold is smooth. We denote by $\mathfrak{X}_c(M)$ the space of compactly supported vector fields on M. It is a module over the algebra $\mathcal{C}^\infty(M)$ of smooth functions.

Definition 1.2.1: The consensus definition of a smooth singular foliation

A singular foliation on a smooth manifold M is a subspace $\mathcal{F} \subset \mathfrak{X}_c(M)$ which

(α) *is involutive,*

(β) *is stable under multiplication under $\mathcal{C}^\infty(M)$,*

(γ) *is locally finitely generated.*

If only the first two conditions are satisfied, then we speak of an Lie-Rinehart subalegbra[a] *(of the Lie-Rinehart algebra of vector fields).*

[a] (In fact, generic Lie-Rinehart subalgebras of vector fields behave very badly in the smooth case and there is not much to say about them: We mainly need them for the complex case, and for pedagogical reasons in order to explain why we impose condition γ.)

Let us spell out Definition 1.2.1 item by item.

α "$\mathcal{F}$ is involutive" means

$$[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$$

where $[\cdot, \cdot]$ stands for the bracket of vector fields. In words, $\mathcal{F}$ is a sub-Lie algebra of the Lie algebra of compactly supported smooth vector fields on the manifold M.

β "$\mathcal{F}$ is stable under multiplication under $\mathcal{C}^\infty(M)$" means that for all $F \in \mathcal{C}^\infty(M), X \in \mathcal{F}$, $FX \in \mathcal{F}$. In algebraic terminology, it means that $\mathcal{F}$ is a $\mathcal{C}^\infty(M)$-sub-module of the $\mathcal{C}^\infty(M)$-module $\mathfrak{X}_c(M)$ of compactly supported vector fields.

γ The meaning of "$\mathcal{F}$ is locally finitely generated" has to be made very precise. It means that for any point $m \in M$, there exists a finite family $X^1, \ldots, X^r \in \mathcal{F}$ and an open neighborhood $\mathcal{U}$ such that for every $X \in \mathcal{F}$, there exists $f_1, \ldots, f_r \in \mathcal{C}^\infty(M)$ satisfying

$$X_{|_x} = \sum_{i=1}^{r} f_i(x)\ X^i_{|_x} \text{ for all } x \in \mathcal{U}.$$

Let us justify Definition 1.2.1 item by item. Now that we have explained the meaning of the three items, let us explain why one imposes this definition.

1. Why compactly supported vector fields?

   Let M be a non-compact manifold. The following $\mathcal{C}^\infty(M)$-modules:

   (a) compactly supported vector fields on M,

   (b) all smooth vector fields on M (compactly supported or not).

   are different as modules over $\mathcal{C}^\infty(M)$. But we do not wish to distinguish them. They obviously have the same leaf (M itself), and they have the same local behaviour. Hence, it is reasonable to impose all vector fields to be compactly supported. An other possible definition involves sheaves, as we will see later.

2. Why assuming α, i.e. $\mathcal{F}$ integrable ? If two vector fields X, Y are tangent to a submanifold L, so is its bracket. Since $\mathcal{F}$ must be thought of as being a replacement of vector fields tangent to the leaves of a smooth partitionifold $L_\bullet$, it makes sense to require $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$.

3. Why assuming α? Because, obviously, if X is tangent to all leaves in a partitionifold $L_\bullet$, so is FX for all smooth function F.

4. Why assuming γ, i.e. "locally finitely generated"? The idea is to avoid weird counter-examples as the infinite comb. Imposing locally finitely generated guaranties that leaves will make sense. This is the topic of a subsequent section.

The next exercice is crucial, for quite a few singular foliations are defined as families $X_1, \ldots, X_r$ that satisfy one the equivalent conditions listes there.

Exercice 1.2.2. Let M be a compact manifold, and let $X_1, \ldots, X_r \in \mathfrak{X}(M)$ be vector fields. Show that the following three items are equivalent:

(i) The $\mathcal{C}^\infty(M)$-module generated by $X_1, \ldots, X_r$ is a singular foliation,

(ii) There exists functions $c_{i,j}^k \in \mathcal{C}^\infty(M)$, with $i, j, k \in \{1, \ldots, r\}^3$, such that

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k$$

for all $i, j \in 1, \ldots, r$.

(iii) There exists functions $c^k_{i,j} \in \mathcal{C}^\infty(M)$, with $i,j,k \in \{1,\dots,r\}^3$ satisfying

$$c^k_{ij} = -c^k_{ji} \text{ and } [X_i, X_j] = \sum_{k=1}^n c^k_{ij} X_k$$

for all possible indices.

Exercice 1.2.3. For $\mathcal{F}$ a singular foliation on M and $\mathcal{V} \subset M$ an open subset, call $\mathcal{F}_\mathcal{V}$ the $\mathcal{C}^\infty(\mathcal{V})$-module generated by the restrictions to $\mathcal{V}$ of vector fields in $\mathcal{F}$. Show that the map $\mathcal{F}_\bullet : \mathcal{V} \mapsto \mathcal{F}_\mathcal{V}$

1. is a sheaf on M,
2. that coincides with the sheaf of vector fields that coincide, in a neighborhood of each point, with a vector field in $\mathcal{F}$.

Here is a lemma, whose proof is left to the reader:

Lemma 1.2.4. *Let $\mathcal{F}$ be a singular foliation. For every point $m \in M$, there exists $X^1, \dots, X^r \in \mathfrak{X}$ and an open neighborhood $\mathcal{U}$ of m in M and such that for any $\mathcal{V} \subset \mathcal{U}$, $\mathcal{F}_\mathcal{V}$ is generated over $\mathcal{C}^\infty(\mathcal{V})$ by the restriction to $\mathcal{V}$ of $X^1, \dots, X^r \in \mathfrak{X}(\mathcal{U})$.*

Proof. This is an immediate consequence of Exercice 1.2.3. □

1.2.2 Smooth singular foliations: a sheaf definition

The use of compactly supported global vector fields is conceptually easy, but some reader may prefer to use sheaves. Let us give a definition of singular foliations, equivalent to the previous one, and that uses the notion of sheaves. In this section, let us denote by

$$\mathfrak{X}_\bullet : \mathcal{U} \longrightarrow \mathfrak{X}(\mathcal{U}) \text{ and } \mathcal{C}^\infty_\bullet : \mathcal{U} \longrightarrow \mathcal{C}^\infty(\mathcal{U})$$

the sheaves of vector fields and smooth functions on the manifold M.

Definition 1.2.5: The consensus definition of a smooth singular foliation, version 2, with sheaves

A singular foliation on a smooth manifold M is a subsheaf

$$\mathcal{F}_\bullet : \mathcal{U} \mapsto \mathcal{F}_\mathcal{U}$$

of the sheaf $\mathfrak{X}_\bullet$ of vector fiels on M such that

(α) *$\mathcal{F}_\bullet$ is involutive,*[a]

(β) *is a sub-sheaf of $\mathcal{C}^\infty_\bullet$-modules,*[b]

(γ) *is locally finitely generated (See below).*

[a] I.e. $[\mathcal{F}_\mathcal{U}, \mathcal{F}_\mathcal{U}] \subset \mathcal{F}_\mathcal{U}$ for all open subset $\mathcal{U} \subset M$.
[b] I.e. $\mathcal{C}^\infty(\mathcal{U})\mathcal{F}_\mathcal{U} \subset \mathcal{F}_\mathcal{U}$ for all $\mathcal{U} \subset M$.

For sheaves of modules over functions, the meaning of locally finitely generated needs to be made more precise: here we mean that every point admits an open neighborhood $\mathcal{U}$ on which there exists $X_1, \dots, X_d \in \mathcal{F}_\mathcal{U}$ such that for every $\mathcal{V} \subset \mathcal{U}$, the restrictions of $X_1, \dots, X_d$ to $\mathcal{V}$ generate $F_\mathcal{V}$ as a $\mathcal{C}^\infty(\mathcal{V})$-module.

Let us compare it with

Proposition 1.2.6: No difference!

Let M be a smooth manifold. There is a one to one correspondence between:

(i) Singular foliations defined as in Definition 1.2.1.

(ii) Singular foliations defined as in Definition 1.2.9.

Proof. The correspondence $(i) \mapsto (ii)$ consists, given $\mathcal{F}$ as in Definition **??**, in considering the sheaf of vector fields that coincide locally with an element in $\mathcal{F}$, see Exercice 1.2.3.

The correspondence $(ii) \mapsto (i)$ consists in considering global compactly supported section of the sheaf in Definition 1.2.9.

The previously described maps are easily checked to be inverse one to the other. $\square$

1.2.3 Singular foliations on complex or real analytic manifolds

For complex manifolds, singular foliations have to be defined through sheaves - essentially because almost all differential geometric objects have to be defined through sheaves, since there is no or few globally defined functions, vector fields and so on.

In this section, we fix M a complex or real analytic manifold, and denote by $\mathcal{O}_\bullet$ its sheaf of holomorphic or real analytic functions. For the reader not used to sheaves, it means that for any open subset $\mathcal{U} \subset M$, we denote by $\mathcal{O}_\mathcal{U}$ the $\mathbb{C}$-algebra of holomorphic $\mathbb{C}$-valued functions on $\mathcal{U}$.

In view of the smooth case, it would be tempting to define a complex singular foliation on the complex manifold M to be a sub-sheaf $\mathcal{F}_\bullet$ of the sheaf $\mathfrak{X}_\bullet$ of holomorphic vector fields on M which is is involutive, stable under multiplication by $\mathcal{O}_\bullet$ and locally finitely generated as a module over $\mathcal{O}_\bullet$. This definition is perfectly correct, but the last assumption can in fact be dropped, in view of the following classical theorem:

Theorem 1.2.7. [Tou68] Germs of holomorphic (resp. real analytic) functions near $O \in \mathbb{C}^n$ (resp. $\mathbb{R}^n$) form a Noetherian ring.

In a chart neighborhood $\mathcal{U}$ of a point $m \in M$, with coordinates $z_1, \ldots, z_d$, holomorphic vector fields decompose as sums

$$\sum_{i=1}^{d} f_i(z_1, \ldots, z_d) \frac{\partial}{\partial z_i}$$

with $f_1, \ldots, f_d$ being $\mathbb{C}$-valued holomorphic functions on $\mathcal{U}$. This means that, as a module over holomorphic functions, holomorphic vector fields of $\mathcal{U}$ decomposes as

$$\mathfrak{X}_\mathcal{U} \simeq \underbrace{\mathcal{O}_\mathcal{U} \oplus \cdots \oplus \mathcal{O}_\mathcal{U}}_{d \text{ terms}}$$

(with d the dimension of the manifold). Consider now the germs $\mathcal{O}_m$ of holomorphic functions at m. Considering the germs in

$$\underbrace{\mathcal{O}_m \oplus \cdots \oplus \mathcal{O}_m}_{d \text{ terms}}$$

of all elements in $X \in \mathcal{F}_\mathcal{V}$ for $\mathcal{V} \subset \mathcal{U}$. The henceforth obtained sub-module is finitely generated over $\mathcal{O}_m$ by Theorem 1.2.7. This does not imply that any submodule $\mathcal{F}_\mathcal{U} \subset \mathfrak{X}_\mathcal{U}$ is finitely generated. But this implies that there is a second neighborhood $\mathcal{V} \subset \mathcal{U}$ on which there exist r holomorphic vector fields $X_1, \ldots, X_r$ such that any $X \in \mathcal{F}_\mathcal{U}$ is a linear combinaison of them in a neighborhood of m. As a consequence, the assumption "locally finitely generated" can be removed.

Definition 1.2.8: The consensus definition of a complex singular foliation

A singular foliation on a complex (or real analytic) manifold M is a subsheaf $\mathcal{F}$ of the sheaf $\mathfrak{X}(M)$ which

(α) *involutive,*

(β) *stable under multiplication under $\mathcal{O}_\bullet$.*

This definition can be adapted immediately to the real analytic setting - we leave it to the reader.

1.2.4 Singular foliation on an affine complex variety

Consider a (maybe non-irreducible) affine variety $W \subset \mathbb{C}^n$. Let denote by $\mathcal{O}_n$ the algebra of polynomial functions in n-variables, by $\mathcal{I}_W \subset \mathcal{O}_n$ the ideal if functions vanishing on W. We call *functions on* W the quotient ring $\mathcal{O}_W := \frac{\mathcal{O}_n}{\mathcal{I}_W}$. We call *vector fields on* W and denote by $\mathfrak{X}_W$ the $\mathcal{O}_W$-module of derivations of $\mathcal{O}_W$. It is equipped with the commutator as a Lie bracket.

Again, $\mathcal{O}_W$ is a Noetherian ring, and $\mathfrak{X}_W$ is a $\mathcal{O}_W$-module of finite rank, so that any sub $\mathcal{O}_W$-module is finitely generated. The assumption "locally finitely generated" is therefore useless in that context, and we suggest the following definition.

Definition 1.2.9: The consensus definition of a algebraic singular foliation

A singular foliation on an affine variety W is a sub-$\mathcal{O}_W$-module $\mathcal{F}$ of the sub-$\mathcal{O}_W$-module of $\mathfrak{X}_W$ which

(α) *is involutive,*

(β) *is stable under multiplication under $\mathcal{O}_W$.*

Notice that the definition does not make reference to the ambient space.

For schemes, or affino-projective varieties, again, the use of sheaves will be necessary, but the assumption of "locally finitely generated" can be dropped, since affine varieties are the local model.

Exercice 1.2.10. Write the definition of a singular foliation on a scheme.

1.2.5 Abstract singular foliation on a commutative algebra.

There is purely algebraic definition of what a singular foliation is. Let $\mathcal{O}$ be a commutative unital algebra (which may be thought of as being an algebra of "functions" - whatever it means).

Definition 1.2.11: Algebraic singular foliations

A sub-$\mathcal{O}$-module $\mathcal{F}$ of $\mathrm{Der}(\mathcal{O})$ *is said to be an algebraic singular foliation if:*

1. *$\mathcal{F}$ is a stable under the Lie bracket of* $\mathrm{Der}(\mathcal{O})$*,*
2. *and is finitely generated as an $\mathcal{O}$-module.*

It is said to be an involutive $\mathcal{O}$-module if it only satisfies the first condition.

Exercice 1.2.12. If the algebra $\mathcal{O}$ is *(i)* finitely generated and *(ii)* Noetherian, then every involutive $\mathcal{O}$-sub-module of derivations of $\mathcal{O}$ is a singular foliation.

The definition above makes sense, in particular, in algebraic geometry, in order to define algebraic singular foliations on $\mathbb{K}^n$ or on an affine variety of $\mathbb{K}^n$. They can even be used to make sense of singular foliation on any ideal (not attached to an affine variety).

Definition 1.2.13

A singular foliation *on an ideal* $\mathcal{I} \subset \mathcal{O}$ *is a locally finitely generated sub-*$\mathcal{O}/\mathcal{I}$*-module* $\mathcal{F}$ *of the sub-*$\mathcal{O}/\mathcal{I}$*-module of* $\mathrm{Der}(\mathcal{O}/\mathcal{I})$ *which*

(α) *is involutive,*

(β) *is stable under multiplication under* $\mathcal{O}/\mathcal{I}$.

1.2.6 Globally finitely generated singular foliations

There has been a lot of discussions about the limits and sense of the "locally finitely generated" condition. But quite a few singular foliations are in fact globally finitely generated.

Let M be a smooth, complex or real analytic manifold induced by an affino-projective variety. Let $U \mapsto \mathcal{O}_U$ be the relevant ring of functions in any of these contexts.

Although we needed adaptations to define singular foliations, globally finitely generated can be defined in the same manner in all contexts.

Definition 1.2.14: A common definition

A singular foliation $\mathcal{F}$ *on a manifold* M *is said to be* finitely generated *if there exists vector fields* $X_1, \dots, X_r$ *such that for every open subset* $\mathcal{U} \subset M$, $\mathcal{F}_{\mathcal{U}}$ *is the* $\mathcal{O}_U$ *module generated by the restrictions to* $\mathcal{U}$ *of* $X_1, \dots, X_r$.

Remark 1.2.15. Notice that we do not assume, in the smooth case, $X_1, \dots, X_r$ to be in $\mathcal{F}$ since they may not be compactly generated. When singular foliations are seen as sheaves, $X_1, \dots, X_n$ belongs to $\mathcal{F}_M$.

Remark 1.2.16. Let $\mathcal{F}$ be a singular foliation on M. Every point $m \in M$ has a neighborhood on which it is finitely generated.

For a globally finitely generated singular foliation $\mathcal{F}$, and any choice of generators $X_1, \dots, X_r$, we call *a choice of Christoffel symbols of* $\mathcal{F}$ *with respect to* $X_1, \dots, X_r$ a family $(c_{ij}^k)_{i,j,k=1}^r$ such that

$$[X_i, X_j] = \sum_{k=1}^{r} c_{ij}^k X_k.$$

Since there are, in general, relations between the generators $X_1, \dots, X_k$, the Christoffel symbols c_{ij}^k are not unique.

Exercice 1.2.17. Let $X_1, \dots, X_r$ be generators of a finitely generated singular foliation $\mathcal{F}$, and $(c_{ij}^k)_{i,j,k=1}^r$ a choice of Christoffel symbols of $\mathcal{F}$ with respect to $X_1, \dots, X_r$.

1. Show that
$$\left(\frac{c_{ij}^k - c_{ji}^k}{2}\right)_{i,j,k=1}^r$$
is again a choice of Christoffel symbols of $\mathcal{F}$ with respect to $X_1, \dots, X_r$.

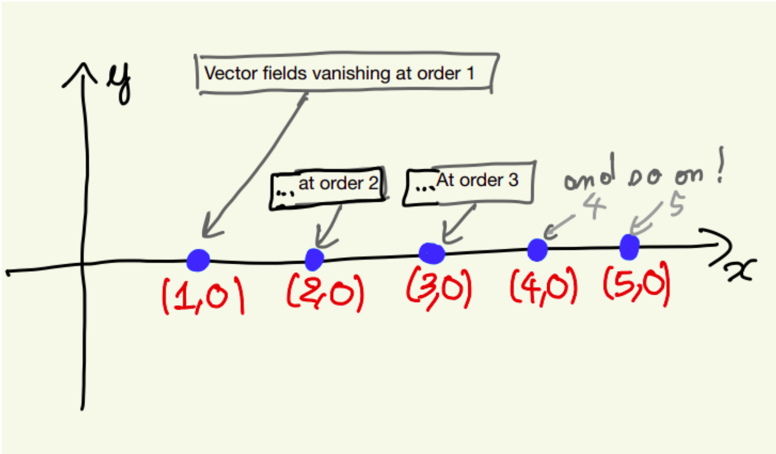
2. Show that, without any loss of generality, Christoffel symbols of $\mathcal{F}$ with respect to $X_1, \dots, X_r$ can be assumed to satisfy $c_{ji}^k = -c_{ij}^k$ for all possible indices.

Exercice 1.2.18. The *"non-finitely-many"* singular foliation - an example due to Iakovos Androulidakis and Marco Zambon. On $M = \mathbb{R}^2$, call $\mathcal{F}$ the space of all vector fields $X \in \mathfrak{X}(\mathbb{R}^2)$ that vanish at order n at the point of coordinates $(n, 0)$. I.e. vector fields of the form:

$$X = f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y}$$

such that for all $a, b, n \in \mathbb{N}_0$ such that $a + b \leq n$:

$$\left.\frac{\partial^{a+b} f}{\partial x^a \partial y^b}\right|_{(0,n)} = \left.\frac{\partial^{a+b} g}{\partial x^a \partial y^b}\right|_{(0,n)} = 0.$$



A representation of the "non-finitely-many" singular foliation.

Show that

1. $\mathcal{F}$ is an integrable distribution.
2. $\mathcal{F}$ is locally finitely generated.
3. $\mathcal{F}$ is not globally finitely generated.
4. $\mathcal{F}$ is not the image through the anchor map of a Lie algebroid on $\mathbb{R}^2$.

1.2.7 The rank at a point of a singular foliation

Given a singular foliation on a smooth, real analytic or complex manifold M, there are two notions that must not be confused: the rank at that point and the dimension of the tangent space at that point.

The *rank* of an $\mathcal{O}$-module $\mathcal{A}$ is the minimal number of its generators. It is denoted by $rk_{\mathcal{O}}(\mathcal{A})$ and takes values in $\mathbb{N} \cup \{+\infty\}$.

Let m be a point in a (smooth, complex, or real analytic) manifold. We say that a sequence $(\mathcal{U}_i)_{i\geq 0}$ of open neighborhoods of m *converges to* m if for any open neighborhood $\mathcal{V}$ of m, there exists i_0 such that for all $i \geq i_0$, we have $\mathcal{U}_i \subset \mathcal{V}$.

Proposition 1.2.19: The rank at a point is well-defined

Let $\mathcal{F}$ be a singular foliation on a smooth, complex, or real analytic manifold M. The sequence

$$n \mapsto rk_{\mathcal{O}_{\mathcal{U}_n}}(\mathcal{F}_{\mathcal{U}_n}).$$

is constant and finite after a certain rank, and this constant does not depend on the choice of a sequence of open neighborhoods converging to m.
It is therefore an integer that depends only on m and $\mathcal{F}$. It is called the rank of $\mathcal{F}$ at m, *and denoted by* $\mathrm{rk}_m(\mathcal{F})$.

Proof. The proofs will be different in the smooth or real analytic / complex cases. We leave it to the reader. ☐

Exercice 1.2.20. (Difficult!) We now work in the smooth case. Let $\mathcal{F}$ be a singular foliation of rank less than or equal to r at every point of the manifold M. Prove that it is finitely generated.

Hint: start by proving that the $\mathcal{C}^\infty(M)$-module of sections of a vector bundle is always finitely generated.

1.2.8 The tangent space of a singular foliation, and its dimension

Let $\mathcal{F}$ be a singular foliation on a complex, real analytic or smooth manifold M.

We call *tangent space of* $\mathcal{F}$ *at* $m \in M$ the subspace of T_mM, denoted by $T_m\mathcal{F}$), obtained by evaluating at m all vector fields in $\mathcal{F}$, defined in any open neighborhood $\mathcal{U}$ of m in M.

Remark 1.2.21. In the smooth case, if a singular foliation $\mathcal{F}$ is defined through compactly supported vector fields, then:

$$T_m\mathcal{F} := \{X_{|_m} \,|\, X \in \mathcal{F}\}.$$

If it is defined as a sub-sheaf $\mathcal{F}_\bullet$ of the sheaf $\mathfrak{X}$ of vector fields, it is defined by:

$$T_m\mathcal{F} := \cup_{\mathcal{U}\in\mathfrak{V}_m}\{X_{|_m} \,|\, X \in \mathcal{F}_\mathcal{U}\}$$

where $\mathfrak{V}_m$ stands for the set of all open neighborhoods of m in M. However, since singular foliation are locally finitely generated, it is then a theorem that there exists an open neighborhood $\mathcal{U}$ such that:

$$T_m\mathcal{F} := \{X_{|_m} \,|\, X \in \mathcal{F}_\mathcal{U}\}.$$

Lemma 1.2.22. *For every point $m \in M$ in a manifold M equipped with a singular foliation $\mathcal{F}$, the dimension of the tangent space at m is less or equal than the rank of $\mathcal{F}$ at m. In equation:*

$$\dim(T_m\mathcal{F}) \leq \mathrm{rk}_m(\mathcal{F})$$

Proof. This follows from the discussion in Remark 1.2.21. □

1.2.9 The regular part of a singular foliation

Let $\mathcal{F}$ be a singular foliation on a manifold M. The map:

$$m \mapsto T_m\mathcal{F}$$

is a singular distribution. It is smooth by construction. We denote it by $\mathfrak{F}$.

Remark 1.2.23. It is not obvious that $\mathfrak{F}$ is involutive. It happens to be true, because we will see later on that it is integrable, but this point will come after a long discussion.

Definition 1.2.24: Regular point

A regular point *of a singular foliation $\mathcal{F}$ on a smooth, complex or real analytic manifold M is a point m in a neighborhood of which* $\dim(T_m\mathcal{F})$ *is constant.*

Remark 1.2.25. Since it is a singular distribution, the map

$$\begin{array}{ccc} M & \to & \mathbb{N} \\ m & \mapsto & \dim(T_m\mathcal{F}) \end{array}$$

is lower upper continuous. This implies that it reaches a local maximum if and only if it is locally constant.

In the complex or real analytic case, if M is connected, it implies that a point m is regular if and only if $\dim(T_m\mathcal{F})$ reaches its maximal value.

The subset of all regular point of a singular foliation $\mathcal{F}$ on a smooth, complex or real analytic manifold M is an open subset. We call it the *regular part of* $\mathcal{F}$ and denoted by M_{reg} (at least when there is no ambiguity on the singular foliation that we consider). By upper semi-continuity, it is also a dense subset.

Proposition 1.2.26

The regular part M_{reg} of a singular foliation $\mathcal{F}$ is a dense open subset of M. Moreover, the restriction of $\mathcal{F}$ to M_{reg} is a regular foliation.

As a consequence, for any singular foliation, there exists a dense open subset on which is it simply a "good old" regular foliation.

1.2.10 Some conventions

From now on:

1. We will call *foliated manifolds* pairs made of a manifold equipped with a singular foliation.
2. We will not make any more notation distinction, in the smooth case, between $\mathcal{F}$ and $\mathcal{F}_\bullet$ (I.e. between singular foliations seen as sub-modules of compactly supported vector fields or seen as sheaves).

1.3 Examples of singular foliations

Let us give an ordered list of examples of singular foliations.

1.3.1 Regular foliations

Although it seems grammatically problematic, regular foliations are examples of singular foliations. More precisely:

Proposition 1.3.1: Good old regular foliations are singular foliations

A singular foliation on a complex, real analytic or smooth connected manifold is regular if and only if the map $m \mapsto \dim(T_m\mathcal{F})$ is constant.

We leave the proof to the reader.

1.3.2 Singular foliations and Lie algebroids

Recall that a *Lie algebroid over M* is a triple $(A, \rho, [\cdot,\cdot])$ with A a vector bundle over M, $\rho\colon A \to TM$ a vector bundle morphism over the identity of M called *anchor map* and $[\cdot,\cdot]$ a Lie bracket on the sheaf of sections of A such that the so-called *Leibniz identity* holds for all $a, b \in \Gamma(A), f \in \mathcal{O}_M$:

$$[a, fb] = f[a,b] + \rho(a)[f]\, b.$$

The smooth case

Let us consider that singular foliations on a smooth manifold are defined as in Definition 1.2.9, through compactly supported vector fields.

Proposition 1.3.2: Image through anchor map of Lie algebroids: smooth case

The image through the anchor map of compactly supported sections of Lie algebroid over M is a singular foliation on M.

The complex or real-analytic case

Proposition 1.3.2 can not be extended immediately from the smooth context to the complex or real analytic contexts altogether[2]. We denote by $\mathcal{O}_{\mathcal{U}}$ the relevant sheaf of functions. For $A \to M$ a vector bundle, we denote by $\Gamma_{\mathcal{U}}(A)$ the sections of A over an open subset $\mathcal{U}$. Of course, $\Gamma_{\mathcal{U}}(A)$ is a $\mathcal{O}_{\mathcal{U}}$-module, and, assigning to an open subset the sections over it

$$\mathcal{U} \mapsto \Gamma_{\mathcal{U}}(A),$$

one defines a sheaf of $\mathcal{O}$-modules over M.

The technical difficulty is that

$$\mathcal{U} \mapsto \rho(\Gamma_{\mathcal{U}}(A))$$

is not a sheaf on M (and therefore not a sub-sheaf of the sheaf $\mathfrak{X}_\bullet$ of vector fields on M). It is only a pre-sheaf. To turn it into a sheaf, one has to "sheafify" it, i.e. to map an open subset $\mathcal{U} \subset M$ to the sub-$\mathcal{O}_{\mathcal{U}}$-module of vector fields $X \in \mathfrak{X}(\mathcal{U})$ on $\mathcal{U}$ such that every $m \in \mathcal{U}$ admits a neighborhood $\mathcal{V}$ on which there exists $a \in \Gamma_{\mathcal{V}}(A)$ with $\rho(a) = X$ (on $\mathcal{V}$). This defines a sheaf of $\mathcal{O}$-module $\underline{\rho(\Gamma(A))}$ that we call the *image of the Lie algebroid* $(A, \rho, [\cdot,\cdot])$ *through its anchor map.*

Proposition 1.3.3: Image through anchor map of Lie algebroids: complex case

Let $(A, \rho, [\cdot,\cdot])$ *be a Lie algebroid over a complex or real analytic manifold* M*. The sheaf* $\underline{\rho(\Gamma(A))}$ *(=image of the Lie algebroid* $(A, \rho, [\cdot,\cdot])$ *through its anchor map) is a singular foliation on* M*.*

Examples

Exercice 1.3.4. Using the relevant Lie algebroid, prove that for any smooth, complex or real-analytic manifold M, the following $\mathcal{C}^\infty(M)$-modules are singular foliations, that come from a Lie algebroid.

1. "Tangent Lie algebroid". Compactly supported vector fields on M form a smooth singular foliation.

2. "Transformation Lie algebroid". Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g} \to \mathfrak{X}(M)$ be a Lie algebra morphism, denoted $x \mapsto \underline{x}$. The $\mathcal{C}^\infty(M)_c$ module generated by $\{\underline{x}, x \in \mathfrak{g}\}$ is a singular foliation.

3. For any Poisson manifold (M, π), the $\mathcal{C}^\infty(M)$-module generated by Hamiltonian vector fields is a singular foliation on M. The corresponding Lie algebroid is T^*M with anchor $\pi^\#$ and the bracket $[\cdot,\cdot]$. See [CFM21] for a recent introduction to the subject, or [CDW87] for an excellent and now classical article on the subject.

Exercice 1.3.5. [LGL22] Let $\mathcal{F}$ be a singular foliation and $\varphi \in \mathcal{C}^\infty(M)$ a function. Check that that

$$\varphi\mathcal{F} := \{\varphi X, X \in \mathcal{F}\}$$

is a singular foliation again. Show that if $\mathcal{F}$ is the image through the anchor map of a Lie algebroid, so is $\varphi\mathcal{F}$.

Projective singular foliations

Here is an important class of singular foliations that come from a Lie algebroid. We will state the results in the smooth case, and leave the generalisation to the reader for the complex or real analytic setting.

[2]The presentation we do here also essentially works on affine varieties or scheme, but we will leave it to the interested reader to adapt

Definition 1.3.6: Generators and no relations

[Deb01] We say that a singular foliation on a smooth manifold M is Debord *if $\mathcal{F}$ is a projective $\mathcal{C}^\infty(M)$-module.*

In a concrete manner, Debord foliations are those which admit, in a neighborhood $\mathcal{U}$ of every point, generators $X_1, \dots, X_r$ between which there is no relation. I.e. if

$$\sum_{i=1}^{r} f_i X_i = 0$$

when each one of the functions $f_1, \dots, f_r$ are zero.

Remark 1.3.7. Equivalently, we could use Definition 1.2.9. A singular foliation $\mathcal{F}$ is then Debord if for every open subset $\mathcal{U}$, $\mathcal{F}_\mathcal{U}$ is a $\mathcal{C}^\infty(\mathcal{U})$-module which is projective in the category of $\mathcal{C}^\infty(\mathcal{U})$-modules.

By the smooth Serre-Swan theorem [Nes20], there exists a vector bundle $A \to M$ and a $\mathcal{C}^\infty(M)$-module isomorphism

$$\Gamma_c(A) \simeq \mathcal{F}.$$

Composing this isomorphism with the inclusion

$$\Gamma_c(A) \simeq \mathcal{F} \hookrightarrow \mathfrak{X}_c(M),$$

we obtain a morphism of $\mathcal{C}^\infty(M)$-modules $\Gamma_c(A) \hookrightarrow \mathfrak{X}_c(M)$, which has to be given by a vector bundle morphism:

$$\rho\colon A \to TM.$$

The vector bundle morphism ρ does not need to be injective at all points, but only at the level of sections. More precisely:

Proposition 1.3.8: Debord algebroids

A singular foliation on a smooth manifold M is Debord if and only if it is the image of a Lie algebroid whose anchor map is injective on a dense open subset.

Exercice 1.3.9. Show that compactly supported vector fields on a manifold M vanishing on a codimension 1 submanifold form a Debord singular foliation.

1.3.3 Singular foliations attached to a submanifold (I) the affine variety case

We now work within complex algebraic geometry. Let $\mathcal{O}$ be the algebra of polynomial functions on an affine variety M. The reader not familiar with algebraic geometry can assume $M = \mathbb{C}^n$ so that $\mathcal{O} = \mathbb{C}[x_1, \dots, x_n]$ is the algebra of polynomials in n variables.

By definition, vector fields on M are the $\mathcal{O}$-module $\mathfrak{X}(M)$ of derivations of $\mathcal{O}$. For $M = \mathbb{C}^n$, vector fields are simply expressions of the form

$$\sum_{i=1}^{n} P_i(x_1, \dots, x_n) \frac{\partial}{\partial x_i},$$

and are uniquely determined by the polynomial functions $(P_i(x_1, \dots, x_n))_{i=1,\dots,n}$.

Let $W \subset M$ be an affine variety. Let $\mathcal{I}_W \subset \mathcal{O}$ be the ideal of polynomial functions vanishing on W. Since $\mathcal{O}$ is Noetherian, this ideal has finitely many generators $\varphi_\bullet = (\varphi_1, \dots, \varphi_k)$.

A difficulty with algebraic geometry is that geometrical properties have to be translated in a purely algebraic language. For instance, for X a vector field on M, we have the following correspondence.

Geometry		Algebra
X vanishes at all points in W	$\Leftrightarrow$	$X \in \mathcal{I}_W \mathfrak{X}(W)$
X is tangent to W	$\Leftrightarrow$	$X[\mathcal{I}_W] \subset \mathcal{I}_W$

We will take the right column as a definition of the left column:

Proposition 1.3.10: Two foliations associated to an affine variety

Let $W \subset M$ be an affine variety. Vector fields tangent to W and vector fields vanishing on W are singular foliations.

The proof is based on the more general result.

Lemma 1.3.11. *Let $\varphi_\bullet = (\varphi_1, \dots, \varphi_k)$ be polynomial functions on M. The following families are singular foliations (in the sense of algebraic geometry[3]).*

1. *The $\mathcal{O}$-module $\mathfrak{X}_{\varphi_\bullet=0}$ of all vector fields $X \in \mathfrak{X}(\mathbb{C}^n)$ such that $X[\mathcal{I}_{\varphi_\bullet}] \subset \mathcal{I}_{\varphi_\bullet}$, with $\mathcal{I}_{\varphi_\bullet}$ the ideal generated by $\varphi_\bullet$.*
2. *$I_{\varphi_\bullet}\mathfrak{X}(\mathbb{C}^n)$, i.e. vector fields $X \in \mathfrak{X}(\mathbb{C}^n)$ of the form $\sum_{i=1}^k \varphi_i X_i$ with $X_1, \dots, X_k \in \mathfrak{X}(\mathbb{C}^n)$.*

Remark 1.3.12. The $\mathcal{O}$-module $d\varphi_\bullet^\perp$ of vector fields $X \in \mathfrak{X}(M)$ such that $X[\varphi_1] = \cdots = X[\varphi_k] = 0$ is also a singular foliation. We invite the reader to see as the singular foliation of all vector fields tangent to the fibers of $\varphi_\bullet \colon M \to \mathbb{C}^k$.

Remark 1.3.13. When M is a smooth manifold, a singular foliation $\mathcal{F}_{pol}$ over the algebra of polynomial functions may be seen as a complex singular foliation: it suffices to consider the sheaf of all vector fields which are linear combinations, with coefficients in holomorphic functions, of vector fields in $\mathcal{F}_{pol}$. In short, it suffices to take the tensor product with holomorphic functions.

1.3.4 Vector fields vanishing at a point at prescribed order

We can also construct singular foliations by playing with order of vanishing at certain points. Let $\mathcal{F}_1$ be the space of all smooth vector fields on $\mathbb{R}^n$ vanishing at 0.

Exercice 1.3.14. Show that $\mathcal{F}_1$ is a singular foliation generated by the finite family of vector fields

$$\left(x_i \frac{\partial}{\partial x_j} \right)_{i,j=1}^n .$$

Hint: use the "Hadamard's lemma", i.e. the fact that any compactly supported smooth function F on $\mathbb{R}^n$ vanishing at 0 decomposes as

$$F = \sum_{i=1}^n x_i F_i$$

for some compactly supported smooth functions $F_1, \dots, F_n \in \mathcal{C}^\infty(\mathbb{R}^n)$.

What about vector fields whose coefficients vanish at order 2 at the origin? Those are vector fields

$$X = \sum_{i=1}^n F_i(x_1, \dots, x_n) \frac{\partial}{\partial x_i}$$

such that

$$F_i(0, \dots, 0) = 0 \text{ and } \frac{\partial F_i}{\partial x_j}(0, \dots, 0) = 0 \text{ for all } i, j = 1, \dots, n.$$

It is a classical result that a smooth function F on $\mathbb{R}^n$ vanishes at $0 \in \mathbb{R}^n$ if and only if it decomposes as

$$F = \sum_{i,j=1}^n x_i x_j F_{i,j}$$

for some smooth functions $F_{i,j} \in \mathcal{C}^\infty(\mathbb{R}^n)$. As a consequence, vector fields whose coefficients vanish at order 2 at the origin are generated, as a $\mathcal{C}_c^\infty(\mathbb{R}^n)$-module by the family

$$\left\{ x_i x_j \frac{\partial}{\partial k} \,\middle|\, 1 \leq i \leq j \leq n \text{ and } k = 1, \dots, n \right\}.$$

Since this module is obviously stable under Lie bracket, this space forms a singular foliation.

More generally, let $\mathcal{F}_k$ be smooth vector fields on $\mathbb{R}^n$ that vansih at 0 together with all their partial derivatives or order $i \leq k-1$. The following exercice shows that it is a singular foliation.

[3] I.e. it is a finitely generated sub-$\mathcal{O}$-module or vector fields on M (=derivations of $\mathcal{O}$) stable under Lie bracket.

Exercice 1.3.15.

1. Show that $\mathcal{F}_k$ is a $\mathcal{C}^\infty(\mathbb{R}^n)$-module stable under Lie bracket.
2. Show that $\mathcal{F}_k = \mathcal{I}_0^k \mathcal{F}$ where $\mathcal{I}_0$ is the ideal of smooth functions on $\mathbb{R}^n$ vanishing at the origin.
3. Find an explicit family of generators of $\mathcal{F}_k$ over $\mathcal{C}^\infty(\mathbb{R}^n)$.
4. For the case of $\mathbb{R}^2$ and $k = 2$ find all possible relations between these generators.

This example could be also seen as a complex, real analytic or algebraic singular foliation. In all these contexts: $\mathcal{F}_k = \mathcal{I}_0^k \mathcal{F}$ where $\mathcal{I}_0$ is the ideal of relevant (sheaf of) functions vanishing at the origin.

Of course, this can be enlarged to any point in a manifold. In conclusion:

Proposition 1.3.16: An example

Let M be a smooth, real analytic or complex manifold. For every point $m \in M$, and every choice of an integer $k \geq 1$, compactly supported vector fields on M vanishing together with their k first derivatives is a singular foliation on M.

More sophisticated examples

This example, (inspired by Grabowska and Grabowski [GG20]), appeared in [LGR21]. We present it as a real analytic singular foliation on $\mathbb{R}^n$ (we could see of course also see it as a complex singular foliation on $\mathbb{C}^n$).

On $M = \mathbb{R}^n$, we attribute to the canonical coordinates $(x_1, \dots, x_n)$ the strictly positive weights $(i_1, \dots, i_n)$. Equipped with this weight, the ring $\mathcal{A}$ real analytic functions on M become a graded algebra.

$$\mathcal{A} = \sum_{i=0}^{\infty} \mathcal{A}_i.$$

It is also a filtered algebra, with respect to the filtration:

$$\mathcal{A}^{\geq k} = \sum_{i=k}^{\infty} \mathcal{A}_i.$$

Example 1.3.17. Assume $i_1 = 1, i_2 = 2$ and so on. The weight of $x_1^3 x_3^2 x_5$ is $1 \times 3 + 3 \times 2 + 1 \times 5 = 14$, so that $x_1^3 x_3^2 x_5 \in \mathcal{A}_{14}$.

Let k be a non-negative integer. The space of real analytic vector fields X such that:

$$X[\mathcal{A}^{\geq n}] \subset \mathcal{A}_{n+k} \text{ for all } n \in \mathbb{N}$$

is a module, that we denote by $\mathcal{F}_k$, over real analytic functions. It is stable under Lie bracket. It is generated by the family

$$\left\{ x_1^{j_1} \dots x_n^{j_n} \frac{\partial}{\partial x_a} \middle| i_1 j_1 + i_2 j_2 + \dots + i_n j_n \geq j_a + k \right\}.$$

If $(j_1, \dots, j_n)$ satisfies the above condition, so does $(j_1', \dots, j_n')$ as long as $j_i' \geq j_i$ for all indices $i = 1, \dots, n$. This implies that the generating family can be chosen to be finite. Therefore, $\mathcal{F}_k$ is finitely generated, and is a real analytic singular foliation.

1.3.5 Singular foliations attached to a submanifold (II) the smooth or complex case

This section makes sense in the smooth, real analytic or complex contexts indifferently.

Proposition 1.3.18: Vector fields tangent to L of vanishing along L

Let L be a submanifold of M, and $k \in \mathbb{N}$ an integer.

1. *Vector fields tangent to L,*
2. *vector fields vanishing at order k at all point in L*
3. *vector fields X such that $X[\mathcal{I}_L] \subset \mathcal{I}_L^k$*

are singular foliations.
Here $\mathcal{I}_L$ stands for the ideal of functions vanishing on L.

Proof. The proof consists in

1. Checking that the space $\mathfrak{X}_L(M)$ of all vector fields on M tangent to the sub-manifold L,
   (a) is a module on functions
   (b) stable under Lie bracket,
   (c) and that in any local coordinates $(x_1, \ldots, x_a, y_1, \ldots, y_b)$ where L is given by $0 = y_1 = \cdots = y_b$, is generated by
   $$\left\{ \frac{\partial}{\partial x_i}, y_j \frac{\partial}{\partial y_k} \middle| 1 \leq i \leq a \text{ and } 1 \leq j, k \leq b \right\}$$
   (This verification is only necessary in the smooth case)
2. Then in checking that the second and third spaces are algebraically described by $\mathcal{I}_L^k \mathfrak{X}(M)$ and $\mathcal{I}_L^{k-1} \mathfrak{X}_L(M)$ respectively. Since the ideal $\mathcal{I}_L$ is locally finitely generated, this completes the proof.

□

Exercice 1.3.19. Let $L_1, L_2 \subset M$ be submanifolds of M that intersect transversally, i.e. such that:

$$T_x L_1 + T_x L_2 = T_x M \qquad \forall x \in L_1 \cap L_2.$$

Consider the space of all vector fields on M tangent to both L_1 and L_2. Show that it is a singular foliation.

1.3.6 Linear singular foliations

A faithful finite-dimensional representation of a Lie algebra may be seen as singular foliation: it suffices to consider the singular foliation associated to its transformation Lie algebroid. Let us be more precise.

Notice that for every vector space V of finite dimension, there is a Lie algebra morphism $X \mapsto \hat{X}$ mapping a linear endomorphism of $X \in \mathrm{End}(V)$ to the vector field $\hat{X}$ on V such that $\hat{X}[\alpha] = X^*(\alpha)$ for any $\alpha \in V^*$ (seen as a function on V).

Remark 1.3.20. Upon choosing a basis $(e_1, \ldots, e_d)$ of V, and the corresponding coordinates $(x_1, \ldots, x_d)$, this morphism maps a matrix $(a_{i,j})_{i=1}^d$ to the vector field $\sum_{i,j=1}^d a_{i,j} x_i \frac{\partial}{\partial x_j}$.

Let $\mathfrak{g}$ be a Lie algebra, and V be a finite-dimensional representation of $\mathfrak{g}$, described by a Lie algebra morphism $\eta \colon \mathfrak{g} \to \mathrm{End}(V)$. Consider the $\mathcal{O}_V$-module[4] $\mathcal{F}^\theta$ generated by the vector fields $\{\widehat{\eta(x)}, x \in \mathfrak{g}\}$.

Proposition 1.3.21. *Let (V, η) be a representation of a Lie algebra $\mathfrak{g}$. Then $\mathcal{F}^\theta$ is a singular foliation on V.*

The exercise supposes that the notion of leaves is already familiar to the reader. It also assumes the notion of "isotropy Lie algebra at a point". It explains how the initial representation can be deduced from the induced singular foliation in the faithful case.

[4]With $\mathcal{O}_V$ being smooth, holomorphic, or polynomial functions depending on whether the base field is $\mathbb{R}$ or $\mathbb{C}$, and depending on the preferences of the reader.

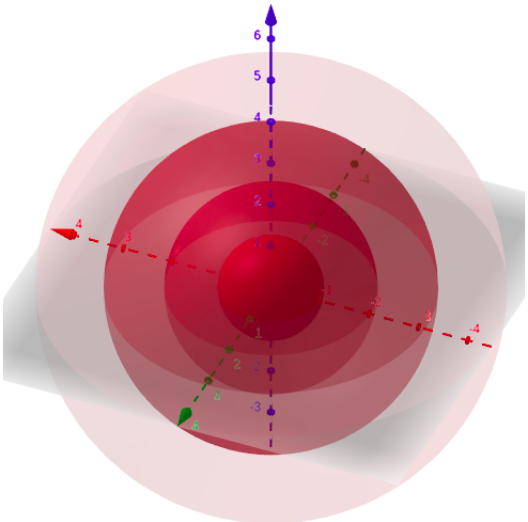
Exercice 1.3.22. Let $(\mathfrak{g}, V, \theta, \mathcal{F}^\theta)$ be as in Proposition 1.3.21.

1. Show that the leaves of $\mathcal{F}^\theta$ are the orbits for the Lie group action $G \to \mathrm{GL}(V)$ integrating θ.
2. This question supposes that the notion of isotropy Lie algebra at a point is known. Show that the isotropy Lie algebra of $\mathcal{F}^\theta$ at $0 \in V$ is $\frac{\mathfrak{g}}{\ker(\theta)}$.
3. Is the following statement correct: "Two faithful representations (V, θ) and $V', \theta')$ are isomorphic if and only if their induced singular foliations $\mathcal{F}^\theta$ and $\mathcal{F}^{\theta'}$ are diffeomorphic.
4. Is it true that the isotropy Lie algebra of $\mathcal{F}^\theta$ at a point v is coincides with the stabilizer of v?
5. Is it at least true for a faithful representation?

Example 1.3.23. The singular foliation by concentric spheres, i.e. the singular foliation on $\mathbb{R}^n$ generated by the vector fields

$$\left\{ x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \middle| 1 \leq i < j \leq n \right\}$$

comes from the action of $\mathfrak{so}(n)$ on $\mathbb{R}^n$. Its leaves are by concentric spheres.



Concentric spheres in three dimensions.

1.3.7 Miscellaneous examples

We list as exercises several instances of singular foliations (or at least Lie-Rinehart algebras) that do not enter in any of the previous categories.

Exercice 1.3.24. The double tangent bundle $\mathbb{T}M := TM \oplus T^*M$ can be equipped with a Leibniz bracket, due to Irene Dorfman and a non-degenerate bilinear form, that, altogether, form what is called a Courant Lie algebroid, see []. Let $D \subset \mathbb{T}M$ be a Dirac structure. Show that:

1. $\{X \in \mathfrak{X} | \exists \alpha \in \Omega^1(M) \text{ s.t. } (X, \alpha) \in \Gamma(D)\}$ is a singular foliation.
2. $\{X \in \mathfrak{X} | (X, 0) \in \Gamma(D)\}$ is a Lie-Rinehart subalgebra of vector fields (and therefore a singular foliation in the real analytic of complex cases).

The singular foliation of the second item is included into the singular foliation of the first item.

As a particular case, it can be shown that, for ω a closed real analytic or holomorphic form on a manifold M,

$$\{X \in \mathfrak{X} | \mathrm{i}_X \omega = 0\}$$

is a singular foliation on M. In the smooth case, the result holds also provided it is locally finitely generated.

Exercice 1.3.25. Yahya Turki [Tur15] introduced the following notion: we say that a bivector field $\pi \in \Gamma(\wedge^2 TM)$ is *foliated* if $\pi^\sharp(\Omega^1(M))$ is closed under the Lie bracket, i.e. if is a singular foliation.

1. Show that for any twisted Poisson structure (π, Ω) (also, called "Poisson structures with background" or "WSW-structures", see [KS05]-[KS02] for a definition) on a manifold M, π is a foliated bi-vector field.
2. In the neighborhood of a regular point of π, there exists a closed 3-form Ω such that the pair (π, Ω) is a twisted Poisson structure.
3. Give an example of a foliated bivector field which is not a twisted Poisson structure.

(Hint: this is done in [Tur15]!).

1.4 New constructions from old ones

In the present section, we work indifferently in the context of smooth, complex or real analytic geometry. Most arguments presented here, however, make no sense in algebraic geometry, and have to be adapted. Conversely, some of them only make sense in algebraic geometry. We will be more precise in due time.

Here is a first exercice to train on these notions.

Exercice 1.4.1. Let $\mathcal{F}$ be a smooth singular foliation on M and $\varphi \in \mathcal{O}_M$ be a function. Show that

$$\varphi\mathcal{F} := \{\varphi X, X \in \mathcal{F}\}$$

is a singular foliation again. State and show the corresponding result in the real analytic, complex and algebraic settings.

1.4.1 Direct products of singular foliations

For X_1, X_2 vector fields on M_1, M_2 respectively, we shall denote by (X_1, X_2) the vector field on $M_1 \times M_2$ whose valued at $(m_1, m_2) \in M_1 \times M_2$ is $(X_1|_{m_1}, X_2|_{m_2}) \simeq T_{(m_1,m_2)} M_1 \times M_2$.

For $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ foliated manifolds, the product manifold $M_1 \times M_2$ can be equipped with the direct product both foliations.

Definition 1.4.2: Direct product of singular foliations

The direct product *of two singular foliations $\mathcal{F}_1, \mathcal{F}_2$ on M_1 and M_2 is the singular foliation $\mathcal{F}_1 \times \mathcal{F}_2$ on $M_1 \times M_2$ such that, for every open subset $\mathcal{U}_1 \subset M_1, \mathcal{U}_2 \subset M_2$, $\mathcal{F}_1 \times \mathcal{F}_2$ is the $\mathcal{O}_{\mathcal{U}_1 \times \mathcal{U}_2}$-module generated by vector fields of the form (X_1, X_2) with $X_1 \in \mathcal{F}_1$ and $X_2 \in \mathcal{F}_2$.*

This definition has to be justified.

Exercice 1.4.3. Show that the direct product of finitely generated singular foliations is a finitely generated singular foliation. Compare their ranks.

1.4.2 Pull-back (version 1)

Let us give the easiest version of the pull-back of a singular foliation: the pull-back through surjective submersion. We will come back to this notion later on, using a more general definition due to Androulidakis and Skandalis [AS09].

Proposition 1.4.4: Pull-back of singular foliations

Let $\mathcal{F}$ be a singular foliation on M and let $\psi : P \to M$ be a surjective submersion. We call pull-pack of $\mathcal{F}$ by ψ *and denote by $\psi^{-1}(\mathcal{F})$ the singular foliation generated, as an $\mathcal{O}_P$-module, by* *vector fields ψ-compatible to a vector field in $\mathcal{F}$.*

Remark 1.4.5. In particular, all vector fields tangent to the fibers of ψ are contained in $\psi^{-1}(\mathcal{F})$.

Remark 1.4.6. In the smooth case, there were two manners to define singular foliations: one with compactly supported vector fields and one with sheaves. There is a difficulty if we use compactly supported vector fields: the pull-back singular foliation $\psi^{-1}(\mathcal{F})$ can not be defined as being the $\mathcal{C}^\infty(M)$-module generated by compactly supported vector fields on P which are ψ-compatible to a compactly supported vector field in $\mathcal{F}$. Indeed, if the fibers of ψ are not compact, there is no compactly supported vector field on P which is ψ-compatible to a non-zero vector field on M.

Exercice 1.4.7. A *horizontal distribution on the surjective submersion* $\psi\colon P \to N$, is a regular distribution $p \mapsto \mathcal{H}_p$ on P such that[5]

$$\mathcal{H}_p \oplus \ker(T_p\psi) = T_pP \text{ for all } p \in P \ .$$

We call *horizontal lift* of $X \in \mathfrak{X}(M)$ and denote $\mathcal{H}(X)$ the unique section of $\mathcal{H}$ such that $T\psi(\mathcal{H}(X)_{|_p}) = X_{|_{\psi(p)}}$ for all $p \in P$. Show that $\psi^{-1}(\mathcal{F})$ is generated, as a sheaf of $\mathcal{C}^\infty(P)$-module, horizontal lifts of vector fields in $\mathcal{F}$ and vertical vector fields (= vector fields tangent to the fibers of ψ).

Exercice 1.4.8. Let $\phi\colon M \to N$ be a surjective submersion with connected fibers. For $\mathcal{F}_M$ a singular foliation on M, show that the following are equivalent:

1. There exists a singular foliation $\mathcal{F}_N$ on N such that $\phi^{-1}(\mathcal{F}_N) = \mathcal{F}_M$.
2. Vector fields tangent to the fibers of ϕ belong to $\mathcal{F}_M$.

1.4.3 The suspension of a singular foliation

We call *suspension* of the manifold M with respect to a diffeomorphism $\phi\colon M \to M$ the quotient of $M \times \mathbb{R}$ by the action of the additive group $\mathbb{Z}$ by:

$$k \cdot (m,t) \sim (\phi^k(m), t+k)$$

for all $k \in \mathbb{Z}, m \in M, t \in \mathbb{R}$. Since the action of $\mathbb{Z}$ is discrete and proper, the quotient is a manifold that we call *suspension of M by $\mathbb{Z}$* and denote by $\frac{M\times\mathbb{R}}{\mathbb{Z}_\phi}$.

Let us assume now that M comes equipped with a singular foliation $\mathcal{F}$ and that $\phi\colon M \to M$ is a symmetry of $\mathcal{F}$, i.e. that $\phi(\mathcal{F}) = \mathcal{F}$.

Proposition 1.4.9: Suspension

Let $\phi : M \to M$ be a symmetry for a singular foliation $\mathcal{F}$. Then there exists a unique singular foliation on the suspension $\frac{M\times\mathbb{R}}{\mathbb{Z}_\phi}$ whose pull-back on $M \times \mathbb{R}$ is the direct product singular foliation $\mathcal{F} \times \mathfrak{X}(\mathbb{R})$.
We call this singular foliation the suspension of $\mathcal{F}$ by the symmetry ϕ *and denote it by* $\frac{\mathcal{F}\times\mathfrak{X}(\mathbb{R})}{\mathbb{Z}_\phi}$.

Proof. The proof simply relies on the fact that for all $k \in \mathbb{Z}$:

$$(m,t) \mapsto (\phi^k(m), t+k)$$

is a symmetry of the direct product singular foliation $\mathcal{F} \times \mathfrak{X}(\mathbb{R})$. ☐

Let us now recall a classical result of differential geometry about suspensions of diffeomorphisms:

Lemma 1.4.10. *If a diffeomorphism ϕ is the time 1 flow of a complete vector field X, then the suspension $\frac{M\times\mathbb{R}}{\mathbb{Z}_\phi}$ is diffeomorphic to the direct product $M \times S^1$ (= the suspension $\frac{M\times\mathbb{R}}{\mathbb{Z}_{id_M}}$ of the identity of M).*

In view of Lemma 1.4.10 following proposition is therefore not a surprise.

[5] Those are also called Ehresmann connection. They exist for any surjective submersion.

Proposition 1.4.11: Inner symmetry have trivial suspension

If a symmetry ϕ of a singular foliation $\mathcal{F}$ on M is the time 1 flow of a complete vector field in $\mathcal{F}$, then its suspension $\frac{\mathcal{F}\times\mathfrak{X}(\mathbb{R})}{\mathbb{Z}_\phi}$ on $\frac{M\times\mathbb{R}}{\mathbb{Z}_\phi}$ is isomorphic to the direct product singular foliation $\mathcal{F}\times\mathfrak{X}(S^1)$ on $M\times S^1$.

1.4.4 Restriction of a singular foliation to a transverse submanifold

Let $\mathcal{F}$ be a singular foliation on a smooth manifold M, and let $S\subset M$ be a sub-manifold. We would like to restrict the singular foliation $\mathcal{F}$ to S.

The next exercise presents a naive idea - which works, but has to be made more precise.

Exercice 1.4.12. Let $\mathcal{F}$ be a singular foliation on a singular foliation M. Consider $\mathfrak{i}_S^*\mathcal{F}_{naive}$ to be the sub-space of all vector fields on a smooth embedded submanifold S obtained by restricting to S vector fields in $\mathcal{F}$ that are tangent to S. Show that $\mathfrak{i}_S^*\mathcal{F}_{naive}$

1. is a sub-Lie algebra of $\mathfrak{X}(S)$,
2. is a sub-$\mathcal{C}^\infty(S)$-module of $\mathfrak{X}(S)$.

It is therefore a Lie-Rinehart subalgebras of vector fields on S, but may not be locally finitely generated.

1. Give an example where $\mathfrak{i}_S^*\mathcal{F}_{naive}$ is not locally finitely generated.

Not being locally finitely generated is not the only problem with $\mathfrak{i}_S^*\mathcal{F}_{naive}$:

1. The sheaf description is not so clear.
2. If S is only immersed and not embedded, it is not a $\mathcal{C}^\infty(S)$-module.

So we have to be more precise. Let $S\subset M$ be a (maybe only immersed) submanifold of M: we now work in the smooth real analytic or complex settings altogether. We denote by $\mathfrak{i}\colon \hookrightarrow M$ the canonical inclusion.

We have to define the sheaf $\mathfrak{i}_S^*\mathcal{F}\subset\mathfrak{X}(S)$ as a subsheaf of the sheaf $\mathcal{U}\mapsto\mathfrak{X}(S)_\mathcal{U}$ of vector fields on S. We proceed as follows. To every $\mathcal{U}\subset S$, we associate the space of all vector fields $Y\in\mathfrak{X}(S)_\mathcal{U}$ such that for every $s\in\mathcal{U}$, there exists $X\in\mathcal{F}_\mathcal{W}$ (for some open subset $\mathcal{W}$ containing $\mathfrak{i}(s)$) such that

$$T_s\mathfrak{i}(Y_{|_{s'}}) = X_{|_{\mathfrak{i}(s')}}$$

for every s' is a neighborhood of s in S. It is easily checked that the previous assignment

1. is a sheaf of $\mathcal{C}^\infty(S)$-modules
2. is closed under Lie bracket.

This settles the real analytic and complex cases.

Proposition 1.4.13: Restriction: the complex case.

et $\mathcal{F}$ be a singular foliation on a complex or real analytic manifold M. For every submanifold $S\subset M$, $\mathfrak{i}_S^*\mathcal{F}$ is a singular foliation on S. In the smooth case, it is a sub-Lie-Rinehart algebra of vector fields on S.
It is called the restriction of $\mathcal{F}$ to S

In the smooth case, the situation is more involved, as seen through the following example.

Exercice 1.4.14. Here is an example of a sub-manifold S in a foliated manifold $(\mathcal{F},S)$ for which $\mathfrak{i}_S^*\mathcal{F}$ is not finitely generated, hence is not a singular foliation.

Consider the foliation of $\mathbb{R}^2$ by horizontal lines, i.e. $\{$ is generated by $\frac{\partial}{\partial x}$. Let f be a function which has support $[0,\infty)$. Then the graph of f yields an embedded submanifold. Show that $\mathfrak{i}_S^*\mathcal{F}$ is not finitely generated. (Instead consider $\mathfrak{i}:\mathbb{R}\to\mathbb{R}^2, t\mapsto(t,f(t))$ and show that $\mathfrak{i}^*\mathcal{F}$ is exactly the space of vector fields which are supported in $(-\infty,0]\subset\mathbb{R}$. This space of vector fields is not finitely generated.

However, even in the smooth case, there is a situation where the restriction yields a locally finitely generated module, and therefore a singular foliation. We will have to make use of some transversality condition.

Definition 1.4.15. We say that a submanifold S of a foliated manifold $(M, \mathcal{F})$ is *transverse* to $\mathcal{F}$ if $T_s S + T_s \mathcal{F} = T_s M$ for all $s \in S$.

Transversality is enough to be guaranty that $\mathfrak{i}_S^* \mathcal{F}$ is locally finitely generated.

Proposition 1.4.16: Transverse submanifolds

Let $S \subset M$ be a submanifold transverse to a smooth singular foliation $\mathcal{F}$. Then $\mathfrak{i}_S^* \mathcal{F}$ is a singular foliation on S.
It is called the restriction of the singular foliation to S.

The following exercises describe this structure more precisely.

Exercice 1.4.17. Let S be a transverse manifold in $(M, \mathcal{F})$, and let $\mathfrak{i}_S^* \mathcal{F}$ be its induced singular foliation.

1. Show that the rank of $\mathfrak{i}_S^* \mathcal{F}$ at a point s is $\mathrm{rk}_s \mathcal{F} - \mathrm{codim}(S)$.
2. Show that $T_s \mathfrak{i}_S^* \mathcal{F} = T_s \mathcal{F} \cap T_s S$
3. (Supposes that the notion of leaf is known.) Show that the leaf of $\mathfrak{i}_S^* \mathcal{F}$ through s is the connected component containing s of S with the leaf through s of $\mathcal{F}$.
4. (Supposes that the notion of isotropy Lie algebra is known.) Show that the isotropy Lie algebra of $\mathcal{F}$ and $\mathfrak{i}_S^* \mathcal{F}$ coincide at any point $s \in S$.

Exercice 1.4.18. The goal of this exercise is to show that there is a neighborhood of a transverse submanifold S in a foliated manifold $(M, \mathcal{F})$ on which $\mathcal{F}$ coincides with a neighborhood of the zero section in the normal bundle $N_S := TL/TS \xrightarrow{p} S$, equipped with the pull-pack singular foliation $p^* \mathfrak{i}_S^* \mathcal{F}$.

1. Show the "tubular neighborhood theorem", i.e. that there is a neighborhood $\mathcal{U}$ of S in L diffeomorphic to a neighborhood of the zero section in the normal bundle $N_S := TL/TS \xrightarrow{p} S$, through a diffeomorphism which is the identity on S.
2. Show that the tubular neighborhood in the previous item can be chosen such that vector fields tangent to the fibers of $p \colon N_S \to S$ are included in $\mathcal{F}$.
3. Conclude that $\mathcal{F}$ is isomorphic to $p^* \mathfrak{i}_S^* \mathcal{F}$ (Hont, see Exercice 1.4.8.)

Exercice 1.4.19. This exercise requires the notion of almost Lie algebroid. Let $(A, [\cdot, \cdot], \rho)$ be an almost Lie algebroid over a singular foliation $\mathcal{F}$. Let $S \subset M$ a a submanifld Show that $\rho^{-1}(TS) \subset \mathfrak{i}_S^* A$ is an almost Lie algebroid over $\mathcal{F}_S$. Here $\mathfrak{i}_S^* A$ stands for the restriction of the vector bundle A to S.

1.4.5 Blow-up of a singular foliation along a leaf

In this section, we work in the realm of complex algebraic geometry over $\mathbb{C}$. We could work over $\mathbb{R}$, dealing with smooth objects: indeed, this is the context in which Debord and Skandalis [Deb13a] introduced the notion of Blow-up of a singular foliation.

Blow-up at a point

Recall that for any $d \in \mathbb{N}$, the set $\mathbb{P}^d$ (or $\mathbb{P}^d_{\mathbb{C}}$) of all straight lines through the origin of $\mathbb{C}^{d+1}$ is a complex manifold of dimension d over $\mathbb{C}$, called the *d-dimensional projective space*. Formally, it is defined as the equivalence classes of relation on the quotient $\mathbb{C}^{d+1} \setminus \{(0, \ldots, 0)\}$ under the equivalence relation:

$$u = (u_0, u_1, \ldots, u_d) \sim v = (v_0, v_1, \ldots, v_d) \Longleftrightarrow \exists \lambda \in \mathbb{C} \setminus \{0\} \text{ such that } u = \lambda v.$$

Equivalently, it can be defined as the quotient manifold

$$\mathbb{P}^d := \mathbb{C}^{d+1} \setminus \{(0, \ldots, 0)\} / \mathbb{C} \setminus \{0\},$$

where the group $\mathbb{C} \setminus \{0\}$ acts by diagonal multiplication on $\mathbb{C}^{d+1}$. In particular, elements in $\mathbb{P}^d$ shall be denoted as $d+1$-tuples of elements not all equal to zero and defined up to a non-zero constant, and denoted by $[x_1, \ldots, x_{d+1}]$.

Lemma 1.4.20. *The projective space $\mathbb{P}^d$ is a complex manifold of dimension d. It is given by the $d+1$ following charts:*

$$\psi_i \colon (x_1, \ldots, x_d) \mapsto [x_1, \ldots x_{i-1}, \underbrace{1}_{i^{th}\ term}, x_{i+1} \ldots x_{d+1}].$$

The idea of the blow-up at the origin consists in replacing $\mathbb{C}^{d+1}$, by pairs made of straight lines through the origin (=elements of $\mathbb{P}^d$) and a point on that straight line.

Definition 1.4.21. The blow-up $Bl_0(\mathbb{C}^{d+1})$ of $\mathbb{C}^{d+1}$ at the origin consists of all pairs $(D, z) \in \mathbb{P}^d \times \mathbb{C}^{d+1}$ such that $z \in D$.

Given coordinates $[x_1, \ldots, x_{d+1}]$ and $(z_1, \ldots, z_d)$ on $\mathbb{P}^d$ and $\mathbb{C}^{d+1}$ respectively, we can describe $Bl_0(\mathbb{C}^{d+1})$ in terms of coordinates:

$$\mathrm{Bl}_0(\mathbb{C}^{d+1}) = \{(x, z) \in \mathbb{P}^d \times \mathbb{C}^{d+1} \mid z_i x_j = z_j x_i,\ i, j = 0, \ldots, d\}.$$

These equations make sense, because multiplying all the x_i be a non-zero factor leave them invariant.

Lemma 1.4.22. $\mathrm{Bl}_0(\mathbb{C}^{d+1})$ *is a complex manifold of dimension $d+1$. It is given by the $d+1$ following charts:*

$$\phi_i \colon (x_1, \ldots, x_{d+1}) \mapsto ([x_1, \ldots x_{i-1}, \underbrace{1}_{i^{th}\ term}, x_{i+1} \ldots x_{d+1}], (x_i x_1, \ldots x_i x_{i-1}, \underbrace{x_i}_{i^{th}\ term}, x_i x_{i+1}, \ldots, x_i x_{d+1})).$$

The projection on the second factor

$$\sigma \colon \mathrm{Bl}_0(\mathbb{C}^{d+1}) \longrightarrow \mathbb{C}^{d+1}$$

is a smooth map. For $z \neq 0$ the pre-image $\sigma^{-1}(z)$ is pair (D, z) with D being the unique line $D \in \mathbb{P}^d$ passing through $z \in \mathbb{C}^{d+1}$. But $\sigma^{-1}(0) = \mathbb{P}^d$. In particular:

$$\sigma \colon \mathrm{Bl}_0(\mathbb{C}^d) \backslash \sigma^{-1}(\{0\}) \longrightarrow \mathbb{C}^{d+1} \backslash \{0\} \tag{1.4}$$

is a biholomorphism.

Proposition 1.4.23: Blow-up of a vector field

For a holomorphic vector field X of $\mathbb{C}^{d+1}$, the following two points are equivalent:

(i) X vanishes at 0

(ii) there exists a vector field $\tilde{X}$ on $\mathrm{Bl}_0(\mathbb{C}^d)$ *such that $\sigma(\tilde{X}) = X$.*

If it exists, then the vector field in item (ii) is unique.

Proof. The set $\mathrm{Bl}_0(\mathbb{C}^{d+1})$ is a complex manifold of dimension $d+1$. It is given on the i-th of the $d+1$ following charts of Lemma 1.4.22 by:

$$(x_1, \ldots, x_{d+1}) \mapsto (x_i x_1, \ldots x_i x_{i-1}, x_i, x_i x_{i+1}, \ldots, x_i x_{d+1}).$$

It is then a direct computation to check that the pull-back of the coordinate functions $(z_1, \ldots, z_{d+1})$ of $\mathbb{C}^{d+1}$ are given by

$$\sigma^*(z_j) = \begin{cases} x_i x_j & j \neq i \\ x_i & j = i \end{cases}$$

This implies that the unique vector field X_j on that chart such that

$$\sigma_*(X_j) = \frac{\partial}{\partial z_j}$$

is

$$X_j = \begin{cases} \frac{1}{z_i}\frac{\partial}{\partial z_j} & j \neq i \\ \frac{1}{z_i}\frac{\partial}{\partial z_i} - \sum_{j\neq i} \frac{z_j}{z_i}\frac{\partial}{\partial z_j} & j = i \end{cases}$$

In turn, this implies that for every vector field $X = \sum_{i=1}^{d+1} P_i(z_1, \ldots, z_d)\frac{\partial}{\partial z_j}$ the unique vector field on the i-th chart such that $\sigma_*(Z) = X$ is

$$Z = \sum_{j\neq i} \left(\frac{P_j(z_1 z_i, \ldots, z_i, z_i z_d)}{z_i} - \frac{z_j P_i(z_1 z_i, \ldots, z_i, z_i z_d)}{z_i} \right) \frac{\partial}{\partial z_j} + \frac{P_i(z_1 z_i, \ldots, z_i, z_i z_d)}{z_i}\frac{\partial}{\partial z_i}$$

This vector field is well-defined on the whole chart if and only if the functions $P_1, \ldots, P_{d+1}$ vanish at the origin. This proves the claim. ☐

Proposition 1.4.24: Blow-up of a singular foliation at the origin

Let $\mathcal{F}$ be a complex or algebraic singular foliation on $\mathbb{C}^{d+1}$. Assume all vector fields on $\mathbb{C}^{d+1}$ vanish at 0. Then there exists an unique singular foliation $\widetilde{\mathcal{F}}$ on $\mathrm{Bl}_0(\mathbb{C}^d)$ such that (1.4) is an isomorphism of foliated manifolds.
We call $\widetilde{\mathcal{F}}$ the blow-up of $\mathcal{F}$ at the origin.

Exercice 1.4.25. We call $\sigma^{-1}(0)$ the exceptional divisor of the blow-up: its points are canonically identified with straight lines through the origin. Let $\mathcal{F}$ be a singular foliation on $\mathbb{C}^{d+1}$ made of vector fields vanishing at 0, and let $\tilde{\mathcal{F}}$ be its blow up.

1. Let $X \in \mathfrak{X}(\mathbb{C}^d)$ be a vector field vanishing at 0 and $\tilde{X} \in \mathfrak{X}(\mathrm{Bl}_0(\mathbb{C}^d))$ such that $\sigma_*(\tilde{X}) = X$. Show that $\tilde{X}$ vanishes at every point of the exceptional divisor if and only if

$$X = \lambda \sum_{i=1}^{d+1} z_i \frac{\partial}{\partial z_i} + \text{ quadratic terms}$$

for some $\lambda \in \mathbb{C}$ where "quadratic terms" means vector fields vanishing at least quadratically at zero.

2. Show that some point D in the exceptional divisor is a point-leaf[6] if and only if D (seen now as a straight line) is an eigenvector for all the linearisations of all vector fields in $\mathcal{F}$.

Blow-up along a smooth submanifold

This construction can be extended considerably. For $E \to N$, we denote by $\mathbb{P}(E)$ the complex manifold obtained by taking the projective space of all the fibers of E. Also, we denote by $\mathcal{N}_{N/M}$ the normal bundle $TM|_N/TN$ of a submanifold $N \subset M$. Morally, $\mathbb{P}(\mathcal{N}_{N/M})$ stands for the projective space of directions normal to N in M.

Proposition 1.4.26 (Definition)**.** *Let N be a complex submanifold of M. There exists a complex manifold $\widetilde{M} = \mathrm{Bl}_N(M)$, called* blow-up of M along N, *and a holomorphic map $\sigma\colon \widetilde{M} \longrightarrow M$ such that restriction $\widetilde{M} \setminus \sigma^{-1}(N) \longrightarrow M \setminus N$ is an isomorphism and $\sigma^{-1}(N) \simeq \mathbb{P}(\mathcal{N}_{N/M})$.*

The hypersurface $\sigma^{-1}(N) \simeq \mathbb{P}(\mathcal{N}_{N/M}) \subset \mathrm{Bl}_N(M)$ is called the *exceptional divisor* of the blow-up $\sigma\colon \mathrm{Bl}_N(M) \longrightarrow M$.

[6] I.e, a leaf reduced to a point - equivalently a point where the tangent space of the foliation is zero.

Example 1.4.27. Let $L = \mathbb{C}^m \subset \mathbb{C}^d$ be the submanifold defined by the equations $z_{m+1} = \cdots = z_d = 0$. The blow-up of $\mathbb{C}^d$ along $\mathbb{C}^m$ is the complex manifold

$$\mathrm{Bl}_{\mathbb{C}^m}(\mathbb{C}^d) := \{(x, z) \in \mathbb{P}^{d-m-1} \times \mathbb{C}^d \mid z_i x_j = z_j x_i,\ i, j = m+1, \ldots, d\},$$

together with the projection

$$\sigma \colon \mathrm{Bl}_{\mathbb{C}^m}(\mathbb{C}^d) \longrightarrow \mathbb{C}^d.$$

It is easily checked that σ is a biholomorphism from $\mathbb{C}^d \setminus \mathbb{C}^m$ and $\sigma^{-1}(\mathbb{C}^m) \simeq \mathbb{P}(\mathcal{N}_{\mathbb{C}^m/\mathbb{C}^d})$, where $\mathcal{N}_{\mathbb{C}^m/\mathbb{C}^d}$ is the normal bundle.

The next statement has in fact be first proven (in the smooth case) by Debord an Skandalis [Deb13a].

Proposition 1.4.28: Blow-up of a singular foliation, general case

Let $\mathcal{F}$ be a singular foliation on M and $L \subset M$ a leaf.
There exists an unique singular foliation $\tilde{\mathcal{F}}$ on the blow-up $\mathrm{Bl}_L(M)$ of M along L such that σ is an isomorphism from $\mathrm{Bl}_L(M) \backslash \sigma^{-1}(L)$ to $M \backslash L$.
It is called the the blow-up of $\mathcal{F}$ along L.

Remark 1.4.29. In fact, we not need to take L to be a leaf: the construction would work for any submanifold to which all vector fields in $\mathcal{F}$ are tangent?. The proof is based on a lemma: a vector field $X \in (M)$ reads $X = \sigma(\tilde{X})$ for $\tilde{X}$ a vector field in $\mathrm{Bl}_L(M)$ if and only if X is tangent to L.

Remark 1.4.30. The hypersurface $\sigma^{-1}(N) \simeq \mathbb{P}(\mathcal{N}_{N/M}) \subset \mathrm{Bl}_N(M)$ is called the *exceptional divisor* of the blow-up $\sigma \colon \mathrm{Bl}_N(M) \longrightarrow M$. If $\mathcal{F}$ is made of vector fields tangent to L, its blow-up is made of vector fields tangent to the exceptional divisor.

1.4.6 Pull-back (version 2)

We have already defined pull-back through surjective submersions, but also restriction on some submanifolds. Let us unify these constructions, following an idea of Androulidakis and Skandalis [[AS09]].

We restrict ourself to the case of smooth manifolds: the complex or real analytic cases are similar.

Let M, B be manifolds together with a smooth map $p \colon M \to B$, and $\mathcal{F}_B$ be a singular foliation on B.

Let p^*TB be the pull-back through p of the tangent bundle TB:

$$p^*TB := \{(m, u) \in M \times TB \mid u \in T_{p(m)}B\}.$$

There are natural maps:

$$\begin{tikzcd} \mathfrak{X}(M) \arrow[dr, "Tp"] & & \mathfrak{X}(B) \arrow[dl, "p^*"'] \\ & \Gamma(p^*TB) & \end{tikzcd}$$

defined as follows:

1. Any vector field X on B gives a section p^*X of p^*TB defined by

   $$m \mapsto (m, X_{p(m)}).$$

   called the pull-back of X.

2. There is a natural vector bundle morphism:

   $$\begin{array}{llll} Tp: & TM & \to & p^*TB \\ & u & \mapsto & (m, Tp(u)) \end{array}$$

   At the level of sections, it induces a map $\mathfrak{X}(M) \to \Gamma(p^*B)$.

We call $p^*\mathcal{F}_B$ the $\mathcal{C}^\infty(M)$-module generated by $\{p^*X | X \in \mathcal{F}_B\}$.

Exercice 1.4.31. 1. Show that the submodule of $\Gamma(TM)$ defined by

$$p^{-1}(\mathcal{F}_B) := \{X \in \mathfrak{X}(M) \mid Tp(X) \in p^*\mathcal{F}_B\} \subset \mathfrak{X}(M)$$

is involutive.

2. Assume the map, $p^*(\mathcal{F}_B) \oplus \mathfrak{X}(M) \to \Gamma(p^*(TM))$, $(\alpha, \beta) \mapsto \alpha + Tp(\beta)$ is surjective. (In this case we say that p is *transverse to* $\mathcal{F}_B$.) Show that $p^{-1}(\mathcal{F}_B)$ is a singular foliation on M.

3. Show that p is transverse to $\mathcal{F}_B$ if and only if, for all $m \in M$, we have $T_{p(m)}\mathcal{F}_B + T_m p(T_m M) = T_{p(m)}B$.

Definition 1.4.32: Pull-back w.r.t. a transverse map

Let $(B, \mathcal{F}_B)$ be a foliated manifold, and $p : M \to B$ be a smooth map transverse to $\mathcal{F}_B$. We call the singular foliation $p^{-1}(\mathcal{F}_B)$ the pull-back of $\mathcal{F}_B$ through p.

Exercice 1.4.33. Explain why this notion "unifies" (= i.e. admits as particular cases) pull-back with respect to surjective submersions seen in Section 1.4.2 and restrictions to transverse submanifolds see in Section 1.4.4.

Pull-back of surjective submersion: some more points

Most of the coming lines uses notions that will be defined much later in the text. Let us recall some important facts and definitions on vector fields

Definition 1.4.34. A vector field $X \in \mathfrak{X}(M)$ said to be p -related to a vector field $\widetilde{X}$ on B if for all $m \in M$,

$$(\mathrm{d}p)_m(X_m) = \widetilde{X}_{p(m)} \tag{1.5}$$

Equivalenty, for any $f \in \mathcal{C}^\infty(M)$, $X[f \circ p] = \widetilde{X}[f] \circ p$, or, equivalently, s.t. the following diagram commutes:

$$\begin{tikzcd}
\mathcal{C}^\infty(B) \arrow[r, "\tilde{X}"] \arrow[d, "p^*"'] & \mathcal{C}^\infty(B) \arrow[d, "p^*"] \\
\mathcal{C}^\infty(M) \arrow[r, "X"'] & \mathcal{C}^\infty(M)
\end{tikzcd}$$

Definition 1.4.35. A vector field $X \in \mathfrak{X}(M)$ said to be p-projetable on B if it p-related to a vector field $\widetilde{X} \in \mathfrak{X}(B)$.

Suppose that $p\colon M \to B$ is a surjective submersion.

A vector field on M is said to be a *p-vertical vector field* if it takes values in $\ker(\mathrm{d}_m p) \subset T_m M$ for all $m \in M$. Equivalently, vertical vector fields are vector fields p-related to $0 \in \mathfrak{X}(B)$.

Exercice 1.4.36. Show that the p-vertical vector fields are contained in $p^*\mathcal{F}_B$.

The pull-back of a singular foliation through a surjective submersion is quite easy to describe.

Lemma 1.4.37. *Let $(B, \mathcal{F}_B)$ be a foliated manifold, and $p\colon M \to B$ be a surjective submersion. Then p is transverse to $\mathcal{F}_B$.*

Moreover, for every $m \in M$, and local generators $\tilde{X}_1, \dots, \tilde{X}_r$ of $\mathcal{F}_B$, the pull-back singular foliation $p^{-1}(\mathcal{F}_B)$, in a neighborhood of m, is generated by vector fields $X_1, \dots, X_r, Y_1, \dots, Y_k$, where $Y_1, \dots, Y_k$ are vertical vector fields that form a local trivialization of $\ker(Tp)$*, and $X_1, \dots, X_r$ are p-related to $\tilde{X}_1, \dots, \tilde{X}_r$.*

Exercice 1.4.38. Assume that the fibers of p are connected. Show that the leaves of $p^*\mathcal{F}$ are the inverse images through p of the leaves of $\mathcal{F}_B$.

Exercice 1.4.39. Show that the isotropy Lie algebra of $p^*\mathcal{F}_B$ at a point m is canonically isomorphic to the isotropy Lie algebra of $\mathcal{F}_B$ at $p(m)$.

Exercice 1.4.40. Show that the transverse singular foliation of $p^*\mathcal{F}_B$ at a point $m \in M$ is canonically isomorphic to the transverse singular foliation of $p^*\mathcal{F}_B$ at the point $p(m) \in B$.

Proposition 1.4.41. *Let $p\colon M \to B$ be a surjective submersion with connected fibers. Let $\mathcal{F}$ be a singular foliation on M. Then the following are equivalent:*

(i) There exists a singular foliation $\mathcal{F}_B$ on B such that $\mathcal{F} = p^*\mathcal{F}_B$.

(ii) Each fiber of p is contained in a leaf of L.

(iii) For every $m \in M$, we have $\ker(T_m p) \subset T_m\mathcal{F}$.

Remark 1.4.42. This is not an obvious statement, for it will use the assumption "locally finitely generated". It is wrong, for instance, for general involutive distributions. For instance, for the "infinite comb" of Example, the projection $(x, y) \mapsto y$ onto the horizontal axis satisfies *(ii)* and *(iii)* but does not satisfy *(i)*.

1.4.7 Push-forward

Let $p\colon M \to N$ be a smooth, complex or real analytic map, depending on the context. We will assume that p is a surjevtive submersion.

The *push-forward* $T_m p : T_m M \to T_{p(m)}N$ does not extend in general to vector fields: for X a vector field on M and $n = p(m) = p(m')$ with $m \neq m' \in M$, then X_m and $X_{m'}$ are both pushed forward to tangent vectors at $b \in B$, but in general $T_m p(X_m) \neq T_{m'}p(X_{m'})$. When this happens, we denote this vector field by $p_*(X)$ and we call it the *push-forward of X through p.*

Let us introduce a notation: for $p\colon M \to B$ a surjective submersion, we denote by $\mathfrak{X}(M)_p$ the space of vector fields X on p which are p-compatible to a vector field on B, that we denote by $p_*(X)$.

Assume that we now given a singular foliation $\mathcal{F}$ on M. Then $\mathcal{F} \cap \mathfrak{X}(M)_p$ is both a $\mathcal{C}^\infty(B)$-module and stable under Lie bracket, and so is

$$p_*(\mathcal{F} \cap \mathfrak{X}(M)_p) \subset \mathfrak{X}(B).$$

When the latter is finitely generated (which always happens in the complex case), it is a singular foliation that we call *push-forward* singular foliation and denote by $p_*(\mathcal{F})$.

Exercice 1.4.43. Here are examples where $p_*(\mathcal{F} \cap \mathfrak{X}(M)_p)$ is not finitely generated.

For Androulidakis-Zambon's "non-finitely-many" singular foliation of exercice 1.2.18, consider $p\colon (x, y) \mapsto y$ the projection onto the horizontal axis.

1. Show that for every p-projectable vector field X on $\mathbb{R}^2$ whose derivatives vanish at order n at the point of coordinates $(n, 0)$, its projection $p_*(X) \in \mathfrak{X}(\mathbb{R})$ is a vector field that vanishes at order n at 0.

2. Show that $p_*(\mathcal{F} \cap \mathfrak{X}(M)_p)$ coincides with the space of vector fields on $\mathbb{R}$ vanishing at 0 with all their derivatives.

3. Conclude.

1.4.8 New constructions from old ones in algebraic geometry

In this section, we define singular foliations as in Definition 1.2.11, with $\mathcal{O}$ being the algebra of polynomial functions on $\mathbb{K}^d$. By purely algebraic methods, we can define new singular foliations out of old ones. When $\mathcal{O}$ is the algebra of functions on an affine variety W, and $\mathcal{I}$ is the ideal of functions vanishing on a sub-affine variety $S \subset W$, these constructions have a geometric meaning that we detail.

Restriction

Consider an algebraic singular foliation $\mathcal{F}$ over $\mathcal{O}$. For every *foliated ideal* $\mathcal{I} \subset \mathcal{O}$, i.e. any ideal such that

$$\mathcal{F}[\mathcal{I}] \subset \mathcal{I}.$$

The quotient space $\mathcal{F}/\mathcal{I}\mathcal{F}$ inherits a natural algebraic singular foliation structure over $\mathcal{O}/\mathcal{I}$. We call the latter algebraic singular foliation the *restriction w.r.t the ideal* $\mathcal{I}$. In the context of affine varieties, when $\mathcal{I}$ is the ideal of functions vanishing on an affine subvariety W, it is a foliated ideal if and only if all vector fields in $\mathcal{F}$ are tangent to W, and the previous construction corresponds to the restriction of $\mathcal{F}$ to W.

Algebra Extension

Let $\mathcal{F} \subset \mathrm{Der}(\mathcal{O})$ be an algebraic singular foliation. Assume that the algebra $\mathcal{O}$ has no zero divisor, and let $\mathbb{O}$ be its field of fractions. For any subalgebra $\tilde{\mathcal{O}}$ with $\mathcal{O} \subset \tilde{\mathcal{O}} \subset \mathbb{O}$ such that every derivation $X \in \mathcal{F}$ valued in derivations of $\mathbb{O}$ preserves $\tilde{\mathcal{O}}$, there is natural algebraic singular foliation over $\tilde{\mathcal{O}}$ given as $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{F}$.

It is not easy to give this construction a geometric meaning. The blow-up is a construction of that type on any affine chart.

Localization

In algebra and in particular in algebraic geometry localization is a common construction that is usually done. Let us recall the definition for the algebra $\mathcal{O}$.

Definition 1.4.44. A subset $S \subset \mathcal{O}$ is called *multiplicatively closed* if $1 \in S$, and if S is stable under multiplication. For a multiplicative set $S \subset \mathcal{O}$, the *localization* of $\mathcal{O}$ at S is the algebra

$$S^{-1}\mathcal{O} := \left\{ \frac{f}{s} \middle| f \in \mathcal{O},\, s \in S \right\} \diagup \sim \tag{1.6}$$

- under the equivalence relation on $\mathcal{O} \times S$ defined by, $(f,s) \sim (g,t)$: there is an element $u \in S$ such that $u(ft - gs) = 0$.

- the set of all equivalence classes is an algebra together with the addition and multiplication given by
$$\frac{f}{s} + \frac{g}{t} := \frac{ft + gs}{st} \quad \text{and} \quad \frac{f}{s}.\frac{g}{t} := \frac{fg}{st} \tag{1.7}$$

Remark 1.4.45. The algebra $\mathcal{O}$ is a subalgebra of $S^{-1}\mathcal{O}$ via the homomorphism $\mathcal{O} \hookrightarrow S^{-1}\mathcal{O}$, $f \mapsto \frac{f}{1}$.

Definition 1.4.46. Let $\mathcal{F} \subset \mathrm{Der}(\mathcal{O})$ be an algebraic singular foliation and $S \subset O$ be a multiplicative subset. The *localization of* $\mathcal{F}$ *at* S is the algebraic singular foliation $S^{-1}\mathcal{F} \subset \mathrm{Der}(S^{-1}\mathcal{O})$ made of derivations of type $S^{-1}X$ for some $X \in \mathrm{Der}(\mathcal{O})$. Where for every $(f,s) \in \mathcal{O} \times S$,

$$\begin{aligned} S^{-1}X\left(\frac{f}{s}\right) : S^{-1}\mathcal{O} &\longrightarrow S^{-1}\mathcal{O} \\ \frac{f}{s} &\longmapsto \left(\frac{X[f]s - fX[s]}{s^2}\right). \end{aligned}$$

The Lie bracket of $S^{-1}\mathcal{F}$ is given as follows:

$$\forall\, X, Y \in \mathcal{F},\, \forall\, (s,t) \in S^2,\ \left[\frac{1}{s}X, \frac{1}{t}Y\right] = \frac{1}{st}\,[X,Y] + \frac{Y[s]}{s^2 t}\,X - \frac{X[t]}{st^2}Y. \tag{1.8}$$

Geometrically, localisation corresponds to restriction to (Zariski) open subsets.

Germification

Let $W \subseteq \mathbb{C}^N$ be an affine variety and $\mathcal{O}_W$ its coordinates ring. We recall that for $U \subseteq W$ an open subset, a function $f\colon U \longrightarrow \mathbb{C}$ is said to be *regular* at a point $x_0 \in U$ if there exists polynomial functions $g, h \in \mathcal{O}_W$ on W with $h(x_0) \neq 0$ such that $f = \frac{g}{h}$ in a neighborhood of x_0, namely if there exists an open set $V \subset U$ that contains x_0 such that $f|_V = \frac{g}{h}|_V$.

Definition 1.4.47. A *function germ at a point* $x_0 \in W$ is an equivalence class of pairs (U, f) with $x_0 \in U \subset W$ an open subset containing x_0, and $f\colon U \longrightarrow \mathbb{C}$ is regular at x_0, under the relation equivalence: $(U, f) \sim (V, g)$ if $f|_U = g|_V$ on an open subset of $U \cap V$. The *germs of regular functions at* x_0 is the algebra given on the set of equivalence classes of the above equivalence relation and it is denoted by $\mathcal{O}_{W,x_0}$.

Remark 1.4.48. It is important to notice that $\mathcal{O}_{W,x_0}$ is a local ring. Also, since $\mathcal{O}_{W,x_0} \simeq (\mathcal{O}_W)_{\mathfrak{m}_{x_0}}$ where $\mathfrak{m}_{x_0} = \{f \in \mathcal{O}_W \mid f(x_0) = 0\}$ and $(\mathcal{O}_W)_{\mathfrak{m}_{x_0}}$ is the localization w.r.t the complement of $\mathfrak{m}_{x_0}$.

Definition 1.4.49. Let $\mathcal{F} \subset \mathrm{Der}(\mathcal{O}_W)$ be an algebraic singular foliation. For a given point $x_0 \in W$, the *germs of* $\mathcal{F}$ *at* x_0 is the localization of the algebraic singular foliation $\mathcal{F}$ at the complementary of the maximal ideal $\mathfrak{m}_{x_0}$.

Remark 1.4.50. This construction can be applied in smooth or complex differential geometry as well.

1.5 Morphisms of singular foliations

Isomorphisms of singular foliations are easily defined; they are diffeomorphisms (and biholomorphisms in the complex case) that intertwine their respective singular foliations. General morphisms are more tricky. There is a case, however, for which the definition is easy and unambiguous: surjective submersions.

Definition 1.5.1: Morphisms of singular foliations: the submersion case

Let $\mathcal{F}, \mathcal{G}$ be singular foliations on P and M respectively. A submersion $\Phi\colon P \to M$ is said to be a morphism of singular foliation if $\mathcal{G} \subset \varphi^{-1}(\mathcal{F})$.

However, it is embarrassing to define morphisms only for submersions. For instance, we would like that a transverse sub-manifold, the inclusion map should be also a sort of morphism. A more general notion has been introduced by Androulidakis and Skandalis [AS09], see Section 1.4.6. Recall from that section that we say that a smooth map $\phi : P \to M$ is transverse to $\mathcal{F}$ if for all $p \in P$

$$T_{\phi(p)}\mathcal{F} + T_p\phi(T_pP) = T_{\phi(p)}M.$$

This is enough to define the pull-back $\varphi^{-1}(\mathcal{F})$.

Definition 1.5.2: Morphisms of singular foliations: the transverse case

Let $\mathcal{F}, \mathcal{G}$ be singular foliations on P and M respectively. A maps $\varphi\colon P \to M$ is said to be a morphism of singular foliation if

1. *φ is a transverse to $\mathcal{F}$,*
2. *$\mathcal{G} \subset \varphi^{-1}(\mathcal{F})$.*

Exercice 1.5.3. Show that the inclusion of a transverse submanifold S in a foliated manifold $(M, \mathcal{F})$ is a morphism in the previous sense for every sub-singular foliation of the restriction $\mathfrak{i}_S^*\mathcal{F}$.

Exercice 1.5.4. (This exercice supposes the notion of leaves). Is it true that a morphism of singular foliation maps two points in the same leaf to two points in the same leaf?

Exercice 1.5.5. This exercise requires the notion of Lie algebroid morphism [Mac05]. Assume that the base map of a Lie algebroid morphism is a submersion: is it a morphism of singular foliations?

Even more general definitions can be found in [GV21], where they are defined as sheaf morphisms compatible with the Lie bracket.

1.6 Singular foliations do admit leaves (i.e. induce a partitionifold)

We show in this section that to any singular foliation on M is attached a smooth partitionifold.

1.6.1 What is a leaf ?

Let $\mathcal{F} \subset \mathfrak{X}(M)$ be an involutive sub-$\mathcal{C}^\infty(M)$-module[7] of compactly supported smooth vector fields on M.

Question 1.6.1: What is a leaf?

What are the leaves of $\mathcal{F}$? And do they exist?

There are two natural notions of leaves, two different notions that deserve to be called "leaves".

1. The first idea that leaves are "reachable points". That is, we will define an equivalence relation on M by pairing two points in M such that one can be reached one to the other by following the flow of vector fields in $\mathcal{F}$.

2. One will use the tangent space of $\mathcal{F}$. A leaf should be submanifold (by definition !) whose tangent space at a point is the tangent distribution at that point.

Here is a formal definition.

Definition 1.6.2. Let $\mathcal{F} \subset \mathfrak{X}_c(M)$ be an involutive sub-$\mathcal{C}^\infty(M)$-module. We say that a point $y \in M$ is *reachable from a point* $x \in M$ if there exists:

1. a finite sequence $x_0, \ldots, x_N$ of points in M with $x_0 = x$ and $x_N = y$

2. time-dependent vector fields $(X_t^{(i)})_{t\in\mathbb{R}} \in \mathcal{F}$ for $i = 0, \ldots, N-1$, with X_i being defined in a neighborhood of x_i and x_{i+1},

   ... such that for all indices $i = 0, \ldots, N-1$, the integral curve starting at x_i at time $t = 0$ of $X_t^{(i)}$ reaches x_{i+1} at time $t = 1$.

Proposition 1.6.3. *The relation on M defined by $x \sim y$ if y is reachable from x is an equivalence relation.*

We call *reachable leaves* or *R-leaves* for short the equivalence classes of the previous relation.

Definition 1.6.4. A *tangent-leaf*, or *T-leaf* for short, is a connected submanifold $L \subset M$ such that for every $\ell \in L$,

$$T_\ell L = T_\ell \mathcal{F},$$

and which is maximal among connected sub-submanifolds that satisfy the same property[8].

[7]Not a singular foliation yet.
So not finitely generated yet !

[8]I.e. it cannot be strictly included in a submanifold that satisfy the same property.

Exercice 1.6.5. "The infinite comb (revisited)" Let $M := \mathbb{R}^2$ be the Cartesian plane with coordinates (x, y). Let $\mathcal{I}_- \subset \mathcal{C}^\infty(\mathbb{R}^2)$ be the ideal of functions vanishing identically on $\mathbb{R}_- \times \mathbb{R}$.

Consider all vector fields of the form

$$\mathcal{F}_{comb} = \left\{ f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y} \middle| g \in \mathcal{C}^\infty(\mathbb{R}^2), f \in \mathcal{I}_- \right\}$$

1. Show that $\mathcal{F}_{comb}$:
   (a) is stable under multiplication by $\mathcal{C}^\infty(\mathbb{R}^2)$,
   (b) is involutive, i.e. is closed under the Lie bracket of vector fields:
   $$[\mathcal{F}_{comb}, \mathcal{F}_{comb}] \subset \mathcal{F}_{comb}.$$
2. Draw what vector fields in $\mathcal{F}_{comb}$ look like.
3. Draw what $T_{x,y}\mathcal{F}_{comb}$ looks like, depending on the sign of x.
4. Show that any point in $M = \mathbb{R}^2$ is reachable from any point in $M = \mathbb{R}^2$. How many R-leaves exists?
5. Does $\mathcal{F}_{comb}$ admits T-leaves ?
6. If yes, does T-leaves and R-leaves match?

This example shows that, in order to have a good definition of leaves, i.e. to have leaves that satisfy both *(i)* and *(ii)* in Definition, we need to assume $\mathcal{F}$ is finitely generated. Otherwise, we may end up with the problem encountered with $\mathcal{F}_{comb}$: there are points where the tangent point of the set of reachable point is bigger than the tangent space of the foliation at that point.

1. Prove that $\mathcal{I}_-$ is not finitely generated $\mathcal{C}^\infty(\mathbb{R}^2)$-module.
2. Can we find an ideal $\mathcal{I} \subset \mathcal{C}^\infty(\mathbb{R})$ which satisfies the following two conditions.
   (a) its zero locus is exactly $\mathbb{R}^-$,
   (b) it is stable under $\frac{\partial}{\partial x}$
   (c) it is finitely generated over $\mathcal{C}^\infty(X)$.
   Hint: Consider the singular foliation generated by $\frac{\partial}{\partial x}$ and $f\frac{\partial}{\partial y}$ with $f \in \mathcal{I}$.

Definition 1.6.6: Definition of leaves

A leaf *of an involutive distribution $\mathcal{F} \subset \mathfrak{X}_c(M)$ is a submanifold $L \subset M$ which:*

(i) is a T-leaf,

(ii) but is also a R-leaf.

Here is the main result of this section, which is attributed to Hermann [Her62].

Theorem 1.6.7: Hermann: Singular foliations do admit leaves!

Every singular foliation on a smooth manifold M partitions M into leaves.

Said otherwise, every singular foliation $\mathcal{F}$ induces a smooth partitionifold $L_\bullet$, such that for avery $m \in M$, L_m is the set of reachable points from m, and such that $T_m\mathcal{F} = T_mL_m$. Here is an even more precise statement that we will indeed prove, and immediately implies the previous one.

Theorem 1.6.8: Second version

Let $\mathcal{F}$ be a singular foliation on a smooth manifold M. Every R-leaf L is a (maybe immersed) submanifold of M, whose tangent space $T_\ell L$ coincides with $T_\ell \mathcal{F}$ at every $\ell \in R$.

We rest of the present section is dedicated to the proof of this statement.

Let us summarize this discussion:

What is given: Let $\mathcal{F} \subset \mathfrak{X}(M)$ be a involutive sub-$\mathcal{C}^\infty(M)$-module, cf Def. **??**

Not a singular foliation yet.
So not finitely generated yet !

Let $T\mathcal{F}$ be its singular distribution, cf Def 3.1.1

Question : What is a leaf ?

Answer 1: Reachable points !
The R-leaf through $x \in M$ is the set of points one can reach starting from x and following flows of vector fields in $\mathcal{F}$
See Def.

Answer 2: be tangent to distribution !
The T-leaf through $x \in M$ is a submanifold L containing x whose tangent space at any $\ell \in L$ coincides with $T_\ell \mathcal{F}$,
See Definition 3.1.1.

Good point: Reachable-leaves exists and partitions M
Bad point: may not be a sub-manifold

Good point: Tangent leaves are submanifolds (by definition)
Bad point: may not exist

Problem: T-leaf $\neq$ R-leaf for involutive $\mathcal{C}^\infty(M)$-modules, see Exercice xxx.

(Theorem 3.1.1.) T-leaf = R-leaf for singular foliations.

We can call them simply **leaves** (and forget R- and T-leaves).

That is, if we assume also that $\mathcal{F}$ is locally finitely generated.

Cor. 3.1.1: Leaves partition M.

Cor. 3.1.1: Leaves are submanifolds.

1.6.2 A singular foliation is a symmetry of itself

The first step to prove Theorems 1.6.7 and 1.6.8 is to prove that vector fields in a singular foliation have flow which are infinitesimal symmetries of themselves. The arguments presented in this section are elementary, but quite complicated. Much better conceptual arguments proving the same results will be given using the notion of anchored bundle and almost Lie algebroids .

This is actually a particular instance of the following more general statement.

Proposition 1.6.9: The flow of an infinitesimal symmetry is a symmetry

Let $Y \in \mathfrak{X}(M)$ be a vector field whose time t-flow $\varphi_t^Y : M \to M$ exists. If[a] $[Y, \mathcal{F}] \subset F$, *then* $(\varphi_t^Y)_*$ *is a symmetry of* $\mathcal{F}$.

[a] That is, if Y is an "infinitesimal symmetry" of $\mathcal{F}$

In fact we are going to prove a more general result.

Proposition 1.6.10. *Let Y be vector field such that $[Y,\mathcal{F}] \subset \mathcal{F}$. For every open neighborhood $\mathcal{U}$ on which $\mathcal{F}$ is generated by vector fields $X_1, \dots, X_r$, and any $\mathcal{V} \subset \mathcal{U}$ an open subset such that $\phi_t^Y(x)$ exists and is in $\mathcal{U}$ for all $x \in \mathcal{V}$ and $|t| \leq \epsilon$, there exists a matrix $\mathbf{A}(t,x)$, whose coefficients are functions on $\mathcal{V}$ depending on t such that for all $i = 1, \dots, r$:*

$$(\phi_t^Y)_* \begin{pmatrix} X_1 \\ \vdots \\ X_r \end{pmatrix} = \begin{pmatrix} & & \\ & \mathbf{A}(t,x) & \\ & & \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_r \end{pmatrix}$$

Moreover, we can assume that

$$\mathbf{A}(s, \phi_t^Y(x)) \circ \mathbf{A}(t,x) = \mathbf{A}(t+s, x) \tag{1.9}$$

for all $s,t \in \mathbb{R}, x \in \mathcal{V}$ for which $\phi_t^Y(x) \in \mathcal{V}$, and $|t|, |t+s| \leq \epsilon$.

Proof. Consider an open neighborhood $\mathcal{U}$ of a point $m \in M$ on which $\mathcal{F}$ is generated by $X_1, ..., X_n$. Let us chose $\epsilon > 0$ and a smaller neighborhood $\mathcal{V} \subset U$ such that if $|t| \leq \epsilon$, $\phi_t^Y(\mathcal{V}) \subset \mathcal{U}$. By definition of a symmetry of a singular foliation, there exists smooth function $b_i^j \in \mathcal{C}^\infty(\mathcal{U})$, such that $[Y, X_i] = \sum_{j=1}^r b_i^j X_j$. Let us write this expression as a matrix:

$$\mathrm{ad}_Y \begin{pmatrix} X_1 \\ \vdots \\ X_r \end{pmatrix} = \begin{pmatrix} & & \\ & \mathbf{ad_Y}(x) & \\ & & \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_r \end{pmatrix} \tag{1.10}$$

with $\mathbf{ad_Y}$ being a shorthand for the matrix of functions on $\mathcal{U}$ whose i-th line and j-th column is b_j^i. For any diffeomorphism $\phi \colon \mathcal{V} \to \phi(\mathcal{V})$, the push-forward map $\phi_* \colon \mathfrak{X}(\phi(\mathcal{V})) \simeq \mathfrak{X}(\mathcal{V})$ is defined by $\phi_*(X)|_m = T_{\phi(m)}\phi^{-1}(X_{\phi(m)})$. It satisfies for all $F \in \mathcal{C}^\infty(\phi(\mathcal{V}))$ and $X \in \mathfrak{X}(\phi(\mathcal{V}))$ the relation:

$$\phi_*(FX) = \phi^* F \ \phi_*(X) \tag{1.11}$$

Also, if $\phi = \phi_t^Y$ is the flow of Y at time t:

$$\frac{\partial}{\partial t} (\phi_t^Y)_* X = (\phi_t^Y)_* [Y, X] = [Y, (\phi_t^Y)_* X] \tag{1.12}$$

We want to show that there exist time-dependant functions $A_i^j(t,x)$ on $\mathcal{V}$ such that

$$(\phi_t^Y)_*(X_i) = \sum_{j=1}^r A_i^j(t) \, X_j \tag{1.13}$$

where $(\phi_t^Y)_*(X_i)$ is to be understood as the image through the push-forward map of the restriction of X_i to $\phi_t^Y(\mathcal{V})$. We also want the matrix of functions $(A_i^j(t,x))$ to be invertible for all t, x.

Again, let us write the expression we wish to obtain in a matrix form. Below, both sides are column vectors of vector fields on $\mathcal{V}$:

$$(\phi_t^Y)_* \begin{pmatrix} X_1 \\ \vdots \\ X_r \end{pmatrix} = \begin{pmatrix} & & \\ & \mathbf{A}(t,x) & \\ & & \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_r \end{pmatrix} \tag{1.14}$$

with $\mathbf{A}(t,x)$ being a shorthand for the matrix of functions on $\mathcal{V}$ whose i-th line and j-th column is $A_j^i(t,x)$. Consider the initial value problem with parameters $x \in \mathcal{V}$:

$$\frac{\partial A_i^j(t,x)}{\partial t} = \sum_{k=1}^r b_i^k(\phi_t^X(x)) \, A_k^j$$

with initial conditions $A_i^j(0,x) = \delta_{i,j}$. Or, equivalently, consider the initial value problem on the vector space $r \times r$ matrices:

$$\frac{\partial}{\partial t}\left(\quad \mathbf{A}(t,x) \quad \right) = \left(\quad \mathbf{ad_Y}\left(\phi_t^Y(x)\right) \quad \right)\left(\quad \mathbf{A}(t,x) \quad \right) \tag{1.15}$$

with initial condition $\mathbf{A}(0,x) = \mathrm{id}$. The initial value problem have solutions for all $x \in \mathcal{V}$ and $|t| \leq \epsilon$, upon changing $\mathcal{V}$ for a smaller neighborhood that we still call $\mathcal{V}$ if necessary. Those solutions depend smoothly on the parameters $x \in \mathcal{V}$. Also, the matrix $\mathbf{A}(t,x)$ is invertible for all $|t| \leq \epsilon$ and $x \in \mathcal{V}$. Last, as any differential equation, it satisfies (1.9).

We claim that Equation (1.14) holds. To show it, let us introduce the column vector whose components are vector fields on $\mathcal{V}$:

$$R(t,x) = \left(\quad \mathbf{A}(t,x) \quad \right)^{-1} \circ \quad (\phi_t^Y)_* \begin{pmatrix} X_1 \\ \vdots \\ X_r \end{pmatrix}$$

An easy computation gives (we now abbreviate the matrix notations, also $(X_\bullet)$ stands for the column vector $X_1, \dots, X_r$):

$$\begin{aligned}
\frac{\partial R(t,x)}{\partial t} &= -\mathbf{A}^{-1} \circ \frac{\partial \mathbf{A}}{\partial t} \circ \mathbf{A}^{-1} \circ (\phi_t^Y)_*(X_\bullet) + \mathbf{A}^{-1} \circ \underbrace{(\phi_t^Y)_* \circ \mathrm{ad}_Y}_{\text{by Eq. (1.12)}} (X_\bullet) \\
&= -\mathbf{A}^{-1} \circ \frac{\partial \mathbf{A}}{\partial t} \circ \mathbf{A}^{-1} \circ (\phi_t^Y)_*(X_\bullet) + \mathbf{A}^{-1} \circ (\phi_t^Y)_* \circ \underbrace{\mathbf{ad_Y}(x)}_{\text{by Eq. (1.10)}} (X_\bullet) \\
&= -\mathbf{A}^{-1} \circ \frac{\partial \mathbf{A}}{\partial t} \circ \mathbf{A}^{-1} \circ (\phi_t^Y)_*(X_\bullet) + \mathbf{A}^{-1} \circ \underbrace{\mathbf{ad_Y}(\phi_t^Y(x)) \circ (\phi_t^Y)_*}_{\text{by Eq. (1.11)}}(X_\bullet) \\
&= \mathbf{A}^{-1} \circ \underbrace{\left(-\frac{\partial \mathbf{A}}{\partial t} \circ \mathbf{A}^{-1} + \mathbf{ad_Y}(\phi_t^Y(x)) \right)}_{= 0 \text{ by Eq. (1.15)}} \circ (\phi_t^Y)_*(X_\bullet) \\
&= 0
\end{aligned}$$

Since $R(0,x) = (X_\bullet)$, we have $R(t,x) = (X_\bullet)$ for all $t \leq \epsilon$ and the (1.14) follows. This implies that the push-forward of any vector field in $\mathcal{F}$ under the flow of Y is a vector field in $\mathcal{F}$ at least for t small enough. Composing such push-forward maps, we obtain that it is still true for all t such that the flow of Y is well-defined. ☐

Let us restate Proposition 1.6.10 in a different manner. We call *infinitesimal symmetry of $\mathcal{F}$* a vector field Y such that $[Y, \mathcal{F}] \subset \mathcal{F}$ (in contrast with *symmetry of $\mathcal{F}$* which are diffeomorphisms such that $\phi_*(\mathcal{F}) = \mathcal{F}$).

Proposition 1.6.11: Symmetries and infinitesimal symmetries

When the flow of an infinitesimal symmetry of $\mathcal{F}$ exists, it is a symmetry of $\mathcal{F}$.

This proposition has an immediate and very important corollary. Again, a much better proof will be given using the notion of anchored bundle and almost Lie algebroids.

Remark 1.6.12. The results of this section can be easily extended to the real analytic or complex settings.

Proposition 1.6.13. *Let $X \in \mathcal{F}$ be a vector field whose time t-flow $\varphi_t^X : M \to M$ exists. Then $(\varphi_t^X)_*$ is a symmetry of $\mathcal{F}$.*

Remark 1.6.14. It deserves to be noticed that the conclusion of the corollary is not true for the infinite comb. Its proof indeed made an intense use of the assumptions "locally finitely generated".

1.6.3 The local splitting theorem

The second step in the proof of Theorems 1.6.7 and 1.6.8 is an equivalent of Weinstein's splitting theorem in Poisson geometry. Let us state this theorem first.

The statements

The results of this section are valid in the smooth, complex (upon replacing $\mathbb{R}$ by $\mathbb{C}$ in the statements below), or real analytic cases. They are *not* true in algebraic geometry.

Theorem 1.6.15: Local splitting, version 1

Consider $\mathcal{F}$ a singular foliation on a manifold M of dimension d.
Any $m \in M$ a point admits a neighborhood on which $\mathcal{F}$ is isomorphic to the direct product of

1. *the singular foliation of all vector fields on an open ball in $\mathbb{R}^l$,*
2. *a singular foliation $\mathcal{T}$ on an open ball in $\mathbb{R}^{d-l}$, only made of vector fields vanishing at its center, where the rank is $r-l$.*

Above, $d = \dim(M)$, $l = \dim(T_m\mathcal{F})$ and $r = \mathrm{rk}_m(\mathcal{F})$.

Alternatively, it can be practical to state this result in local coordinates.

Theorem 1.6.16: Local spliting, version 2

Let $\mathcal{F}$ be a singular foliation on a smooth, complex or real analytic singular foliation M. Every point $m \in M$ admits a chart $\mathcal{U}_m$ with local coordinates $(x_1, \dots, x_l, y_1, \dots, y_{n-l})$, centered at m, on which the restriction of $\mathcal{F}$ admits the following generators:

a) the l vector fields $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_l}$,

b) and $r-l$ vector fields of the form

$$f_1(y_1, \dots, y_{n-l})\frac{\partial}{\partial y_1} + \dots + f_{n-l}(y_1, \dots, y_{n-l})\frac{\partial}{\partial y_{n-l}}$$

with $f_1(0, \dots, 0) = \dots = f_{n-l}(0, \dots, 0) = 0$.

Above, $l = \dim(T_m\mathcal{F})$ and $r = \mathrm{rk}_m(\mathcal{F})$.

Here is a third version of the local splitting theorem. Notice that there is no equivalent statement for the Weinstein splitting theorem in Poisson geometry.

Theorem 1.6.17: Local spliting, version 3

Let $\mathcal{F}$ be a singular foliation on a smooth, complex or real analytic singular foliation M of dimension d. For every $m \in M$, there exists

1. *an open neighborhood $\mathcal{U}$ of m in M*
2. *a singular foliation $\mathcal{T}$ of rank $\mathrm{rk}_m(\mathcal{F}) - l$ on an open neighborhood $\mathcal{V}$ of 0 in $\mathbb{K}^{d-l}$, with $l = \dim(T_m\mathcal{F})$, admitting $\{0\}$ as a leaf,*
3. *a surjective submersion $\phi \colon \mathcal{U} \to \mathcal{V}$,*

such that the restriction of $\mathcal{F}$ to $\mathcal{U}$ coincides with the pull-back singular foliation $\phi^{-1}(\mathcal{T})$.

Recall the following lemma:

Lemma 1.6.18. *If a vector field X is not zero at some point $m \in M$, then there exists a local chart $\mathcal{U}$ with coordinates $(x, y_1, \dots, y_{d-1})$, centered at m, such that, on $\mathcal{U}$, we have $X = \frac{\partial}{\partial x}$.*

Proof. It is obvious that all three versions of the local splitting theorem are equivalent. We will prove the version 2. Our proof is by recursion: since the statement is local by nature, it suffices to consider the following recursion assumption

$\mathcal{H}_l$ = "The statement is proved at $m = 0$ for any singular foliation $\mathcal{F}$ on an open ball in a finite dimensional vector space such that $\dim(T_0\mathcal{F}) \leq l$".

For $l = 0$, $\mathcal{H}_0$ is automatically true and there is nothing to prove. Assume now $\mathcal{H}_l$ is valid, and let us prove $\mathcal{H}_{l+1}$.

Let $X^1, \dots, X^r$ be generators of a singular foliation $\mathcal{F}$ defined in open neighborhood of $0 \in \mathbb{K}^d$. Without any loss of generality, one can assume $X^r|_m \neq 0$. By the Hadamard lemma 1.6.18, there exists local coordinates $(x, y_1, \dots, y_{d-1})$ centered at 0, such that, on $\mathcal{U}$, in which $X^r = \frac{\partial}{\partial x}$. In these coordinates, the remaining generators read as:

$$X^j = \sum_{i=1}^{r-1} F_i^j(x, y_1, \dots, y_{d-1}) \frac{\partial}{\partial y_i} + g^j(x, y_1, \dots, y_{d-1}) \frac{\partial}{\partial x}.$$

Since $X^r = \frac{\partial}{\partial x}$ belongs to $\mathcal{F}$, there is a second family of generators of $\mathcal{F}$ giver by X^r together with the $r - 1$ vector fields:

$$\widehat{X}^j := X^j - g^j(x, y_1, \dots, y_{d-1}) \frac{\partial}{\partial x} = \sum_{i=1}^{r-1} F_i^j(x, y_1, \dots, y_{d-1}) \frac{\partial}{\partial y_i}.$$

Let $\mathcal{G}$ be the module generated by $\widehat{X}^1, \dots, \widehat{X}^{r-1}$. This module has the following description: $\mathcal{G}$ is the intersection of $\mathcal{F}$ with vector fields on the fiber of the map

$$\Pi \colon (x, y_1, \dots, y_{d-1}) \mapsto x.$$

In equation:

$$\mathcal{G} = \mathcal{F} \cap \{\Pi - vertical\}$$

Since both $\mathcal{F}$ and Π-vertical vector fields are closed under Lie bracket, it defines, in particular, a singular foliation of rank $d - 1$ on some neighborhood of 0.

Now, $[X^r, \mathcal{G}] \subset \mathcal{F}$, since $X^r \in \mathcal{F}$ and $\mathcal{G} \subset \mathcal{F}$. Also, $\mathcal{G}$ being vertical with respect to Π while X^r is Π-compatible on the vector field $\frac{\partial}{\partial x}$ on $\mathbb{R}$, the Lie bracket $[X^r, \mathcal{G}]$ is valued in Π-vertical vector fields, so that

$$[X^r, \mathcal{G}] \subset \mathcal{F} \cap \{\Pi - vertical\} = \mathcal{G}.$$

Said otherwise, X^r is an infinitesimal symmetry of $\mathcal{G}$. By Proposition , it implies that its flow is a symmetry of $\mathcal{G}$. Concretely, it means that for all $(x, y_1, \dots, y_{d-1})$ and all $t \in \mathbb{R}$ such that $(x + t, y_1, \dots, y_{d-1})$ is still in the considered open subset,

$$\left(\phi_t^{X^r}\right)_* \begin{pmatrix} \widehat{X}_1(x, y) \\ \vdots \\ \widehat{X}_{r-1}(x, y) \end{pmatrix} = \begin{pmatrix} & & \\ & \mathbf{A}(t, x, y) & \\ & & \end{pmatrix} \begin{pmatrix} \widehat{X}_1(x, y) \\ \vdots \\ \widehat{X}_{r-1}(x, y) \end{pmatrix},$$

where $\mathbf{A}(t, x, y)$ is an invertible matrix that satisfies:

$$\mathbf{A}\left(s, \phi_t^{X^r}(x, y)\right) \circ \mathbf{A}(t, x, y) = \mathbf{A}(t + s, x, y)$$

Since the flow at time t of X^r reads

$$\phi_t^{X^r} \colon (x, y_1, \dots, y_{d-1}) \longrightarrow (x + t, y_1, \dots, y_{d-1})$$

it means that there exists an invertible matrix $\mathbf{A}(t,x,y)$ such that:

$$\begin{pmatrix} \widehat{Y}_1(x+t,y) \\ \vdots \\ \widehat{Y}_{r-1}(x+t,y) \end{pmatrix} = \begin{pmatrix} & & \\ & \mathbf{A}(t,x,y) & \\ & & \end{pmatrix} \begin{pmatrix} \widehat{Y}_1(x,y) \\ \vdots \\ \widehat{Y}_{r-1}(x,y) \end{pmatrix},$$

where the invertible matrix $\mathbf{A}(t,x,y)$ satisfies:

$$\mathbf{A}\left(s, \phi_t^{X^r}(x,y)\right) \circ \mathbf{A}(t,x,y) = \mathbf{A}(t+s,x,y)$$

In particular, the vector fields

$$\begin{pmatrix} Z_1(x,y) \\ \vdots \\ Z_{r-1}(x,y) \end{pmatrix} = \begin{pmatrix} & & \\ & \mathbf{A}(x,0,y) & \\ & & \end{pmatrix}^{-1} \begin{pmatrix} \widehat{Y}_1(0,y) \\ \vdots \\ \widehat{Y}_{r-1}(0,y) \end{pmatrix}.$$

are well-defined in a neighborhood of 0 and satisfy the following two properties:

1. they are local generators of $\mathcal{G}$ (since the matrix $\mathbf{A}(t,x,y)$ is invertible for t,x,y small enough),
2. they are invariant under the flow of X^r, since

In coordinates, it means that they are of the form:

$$Z^i = \sum_{i=1}^{d-1} f_j^i(y_1,\ldots,y_{d-1}) \frac{\partial}{\partial y_j}.$$

They therefore define a singular foliation $\mathcal{H}$ of rank $r-1$ on $\mathbb{K}^{d-1}$. By construction, the dimension of $T_0\mathcal{G}$ is $l-1$. We can then apply the recursion hypothesis, and we obtain the existence of coordinates $(x_1,\ldots,x_l,y_1',\ldots,y_{d-l-1}')$ on which $\mathcal{H}$ is of the form (). These variables, together with $x_{l+1} := x$ form a system of coordinates on which $\mathcal{F}$ is of the form (). □

1.6.4 Leaves are manifolds

We will use the following property of immersed submanifolds:

Proposition 1.6.19. *If a connected subset $L \subset M$ satisfies that every $m \in L$ has a neighborhood $\mathcal{U}$ such that the connected component of m in $L \cap \mathcal{U}$ is a submanifold of dimension k, then it is a (maybe immersed) submanifold of dimension k.*

Choose a R-leaf L. An immediate consequence of the local splitting theorem is that every point $m \in L$ admits a neighborhood $\mathcal{U} \subset M$ admitting the following property: For the restriction $\mathfrak{i}_{\mathcal{U}}^*\mathcal{F}$ the set of reachable points $L_m^{\mathcal{U}}$ is the submanifold $y_1 = \cdots = y_{d-\ell} = 0$ in some local coordinates $(x_1,\ldots,x_d,y_1,\cdots = y_{d-\ell})$ on which $m = (0,\ldots,0)$. Said otherwise, the connected component of m in $L \cap \mathcal{U}$ is a submanifold. It therefore satisfies the assumptions of Proposition 1.6.19 and is an immersed submanifold. It is therefore also a T-leaf. This concludes the proof of Theorems 1.6.7 and 1.6.8.

Remark 1.6.20. Notice that the functions $x_1,\ldots,x_{d-l}$ that appear in the local splitting theorem define a diffeomorphism $\Phi_m^{\mathcal{U}}$ from the submanifold $L_m^{\mathcal{U}}$ to an open neighborhood of $\mathbb{K}^{d-l}$. The families $(L_m^{\mathcal{U}},\Phi_m^{\mathcal{U}})_{m\in M}$ form an atlas for L.

1.6.5 The transverse singular foliation of a leaf

The next corollary is an immediate consequence of the two following facts:

1. leaves are R-leaves, i.e. two points in the same leaf
2. flow of vector fields in $\mathcal{F}$ are symmetries of $\mathcal{F}$.

Corollary 1.6.21: Along a leaf, landscape is always the same

Any two points in the same leaf of a singular foliation have open neighbourhoods on which the singular foliations are isomorphic

This result has a natural consequence about transverse singular foliations. Recall that for we say that a pointed sub-manifold $S \subset M$, with a distinguished point $m \in S$ is transverse to the leaf L through m if

$$T_m S \oplus T_m L = T_m M,$$

Recall also that for any transverse submanifold (S, m) there exists a neighborhood of m in S on which $\mathcal{F}$ is transverse to $\mathcal{F}$ so that a restricted singular foliation $\mathcal{T}_S := \mathfrak{i}_S \mathcal{F}$ can be defined.

Theorem 1.6.22: Transverse of a leaf

Let L be a leaf of a smooth, real analytic or complex singular foliation. For any two submanifolds transverse to L, the restricted singular foliation are isomorphic in neighborhoods of their intersections with L.

In the smooth case, all these results can be proven using the notion of $\mathcal{F}$-connection, introduced in [LGR19] see also [LGR21].

Definition 1.6.23: $\mathcal{F}$-connection

Let L be a leaf of a singular foliation $\mathcal{F}$. We say that a triple $(\mathcal{U}_L, p, H)$ where:

1. *$\mathcal{U}_L$ is n open neighborhood of L in M,*
2. *$p\colon \mathcal{U}_L \longrightarrow L$ is a submersion (whose restriction to L is the identity)*
3. *H is an Ehresmann distribution with respect to p, i.e. a distribution such that*

$$H_m \oplus \ker(T_m p) = T_m M$$

   for all $m \in \mathcal{U}_L$,

*is an $\mathcal{F}$-*connection *if sections of H are included in $\mathcal{F}$.*

Remark 1.6.24. The fibers $p^{-1}(m)$ of $p\colon \mathcal{U}_L \longrightarrow L$ are submanifolds transverse to L. In particular, they admit an restricted singular foliation that we denote by $\mathcal{T}_m$.

Remark 1.6.25. For any $\mathcal{F}$-connection $(\mathcal{U}_L, p, H)$, we have $H_m \subset T_m\mathcal{F}$ for all $m \in M$. This condition is however not sufficient.

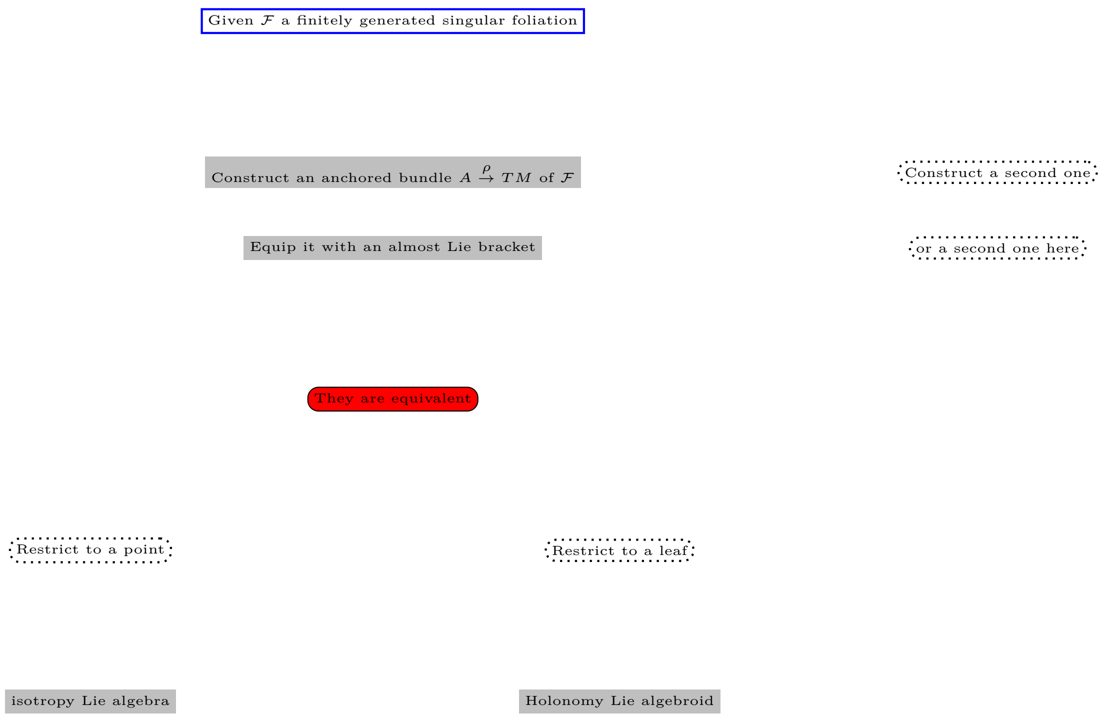
Remark 1.6.26. [LGR19, LGR21] In the smooth case, the existence of a flat $\mathcal{F}$-connection is equivalent to the existence, near L, a a regular foliation whose leaves admitting L as a leaf and included in $\mathcal{F}$.

It has been proven in [LGR19, LGR21] that:

1. In the smooth setting, every leaf admits an $\mathcal{F}$-connection.
2. That parallel transportation w.r.t an $\mathcal{F}$-connection along a path on L from m to n induces, as long as it is defined, an isomorphism of singular foliation from a neighborhood of m in $(p^{-1}(m), \mathcal{T}_m)$ to a neighborhood of n in $(p^{-1}(n), \mathcal{T}_n)$, with $\mathcal{T}_m$ as in Remark 1.6.24.
3. This notion can be used to define the equivalent of the "first return map", or holonomy, or monodromy of a singular leaf. We refer to [LGR19, LGR21] for more details.

Chapter 2

The canonical Lie and groupoid structures hidden behind a singular foliation



2.1 The almost Lie algebroids of a singular foliation

In this section M is a smooth or complex manifold, (or a Zarisky open subset of $\mathbb{C}^d$). Also, $\mathcal{O}(M)$ stands for the corresponding sheaf of functions.

2.1.1 Anchored bundles

As we will see, the smooth setting will be considerably simpler, and will have much better properties. However, we will consider all settings, as much as we can.

Remark 2.1.1. The reader interested only in the smooth setting is invited to skip the next lines, where we recall the meaning of $\rho(\Gamma(A))$ for $\rho\colon A \to TM$ a vector bundle morphism: As we saw in Section 3.1.1,

$$\mathcal{U} \mapsto \rho(\Gamma_{\mathcal{U}}(A))$$

is a pre-sheaf in the complex and real analytic cases, but it can be sheafified (in the smooth case, it is a sheaf, so sheafification is useless). We denote by $\rho(\Gamma(A))$ this sheaf and call it the image of $\Gamma(A)$ through ρ.

that it will be still a fields?

Definition 2.1.2: Anchored bundle

An anchored vector bundle *is a pair* (A,ρ) *made of a vector bundle* $A \to M$*, and a vector bundle morphism called its* anchor map*.*

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[d] & TM \arrow[d] \\ M \arrow[r, equal] & M \end{tikzcd}$$

By construction, $\rho(\Gamma(A)) \subseteq \mathfrak{X}(M)$ is a sub-sheaf of $\mathcal{O}(M)$-module which is locally finitely generated. Moreover, it is generated, locally, by less at most $\mathrm{rk}(A)$ generators.

Definition 2.1.3: Anchored bundle over $\mathcal{F}$

Let $\mathcal{F}$ be a singular foliation on M. We say that an anchored bundle (A,ρ)

1. terminates inside $\mathcal{F}$ *if* $\rho(\Gamma(A)) \subset \mathcal{F}$
2. *and* is over $\mathcal{F}$ *if* $\rho(\Gamma(A)) = \mathcal{F}$.

Notice that anchored bundle over $\mathcal{F}$ could be defined for any locally finitely generated module over functions. We have not used $[\mathcal{F},\mathcal{F}] \subset \mathcal{F}$ at this point. This section is dedicated to answer the following question:

Question 2.1.4: Behind a singular foliation?

Let $\mathcal{F}$ be a singular foliation on M.

1. *Does there always exist an anchored bundle (A,ρ) over $\mathcal{F}$?*
2. *If yes, how unique (= canonical) are they?*
3. *If yes, what properties and additional structures do they have?*

Remark 2.1.5. For Debord foliations, an anchored bundle exists, by Serre-Swann theorem.

Does there always exist an anchored bundle (A,ρ) over $\mathcal{F}$?

Here is an answer to the first part of question.

Proposition 2.1.6: Answer to the first part of Question ??

Let $\mathcal{F}$ be a singular foliation on M.

1. *If $\mathcal{F}$ is finitely generated, then there exists an anchored bundle (A,ρ) over $\mathcal{F}$, and A can be chosen to be a trivial vector bundle. In particular, any singular foliation is of this type in a neighborhood of any point.*
2. *In the smooth setting, the following points are equivalent.*

(i) $\mathcal{F}$ is finitely generated.

(ii) There exists a anchored bundle (A, ρ) over $\mathcal{F}$.

Proof. Assume that $\mathcal{F}$ is finitely generated, and let $X_1, \ldots, X_r$ be generators. Let A be the trivial vector bundle of rank r, i.e.

$$A = \mathbb{K}^r \times M \Longrightarrow M.$$

Denote the canonical trivialisation of E by $e_1 \ldots, e_r$ and define the anchor map by $\rho(e_i) = X_i$ for all $i = 1, \ldots, r$. We have $\rho(\Gamma(A)) = \mathcal{F}$ by construction. This proves the first item of the statement. It also proves the implication (i) $\implies$ (ii). Let us show that (ii) $\implies$ (i). Let (A, ρ) be as in (i). It is a classical theorem in smooth differential geometry that there exists a vector bundle $B \to M$ such that $A \oplus B$ is a trivial vector bundle $E \to M$. Define an anchor map on that trivial vector bundle by

$$\rho_E : \; E \xrightarrow{pr_A} A \xrightarrow{\rho} TM$$

where pr_A is the projection onto A with respect to B. The pair (E, ρ_E) is a trivial vector bundle such that $\rho_E(\Gamma(E)) = \mathcal{F}$. In particular, $\mathcal{F}$ has $\mathrm{rk}(E)$ generators. This concludes the proof. $\square$

Are two anchored bundles over $\mathcal{F}$ really different?

Let us define morphisms of anchored bundles - and add an equivalence class of them.

Definition 2.1.7: Morphisms and Equivalences

Let $(A_1 \to M_1, \rho_1)$ and $(A_2 \to M_2, \rho_2)$ be two anchored bundles.

1. *We call* morphism of anchored bundles *any vector bundle morphism $\Phi \colon A_1 \longrightarrow A_2$ over a map $\phi \colon M_1 \to M_2$ making the following diagram commutative:*

$$\begin{tikzcd}
A_1 \arrow[rrr, "\Phi"] \arrow[dr] \arrow[dd, "\rho_2"'] & & & A_2 \arrow[dl] \arrow[dd, "\rho_1"] \\
& M_1 \arrow[r, "\phi"] & M_2 & \\
TM_1 \arrow[ur] \arrow[rrr, "T\phi"] & & & TM_2 \arrow[ul]
\end{tikzcd} \tag{2.1}$$

We speak of an isomorphism of anchored bundle *when Φ is an isomorphism of vector bundles.*

2. *Two morphisms of anchored bundles Φ, Φ' as in item 1 are said to be* equivalent *if $\rho \circ (\Phi - \Phi') = 0$.*

3. *An* equivalence of anchored bundles *is a pair of anchored bundle morphisms*[a]

$$\begin{tikzcd}
A_1 \arrow[r, shift left, "\Phi"] & A_2 \arrow[l, shift left, "\Psi"]
\end{tikzcd} \tag{2.2}$$

such that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are equivalent to the identities of A_1 and A_2.

It is easily checked that both equivalences above are an equivalence relation on the class of anchored bundles and their sets of morphisms.

[a] In the complex or real analytic settings, an equivalence of vector bundle morphisms shall be a a covering $(\mathcal{U}_i)_{i \in I}$ of M and an equivalence (Φ_i, Ψ_i) on each one of the open sets $\mathcal{U}_i$. We also assume Φ_i, Φ_j and Ψ_i, Ψ_j to be equivalent on $\mathcal{U}_i \cap \mathcal{U}_j$.

For a good understanding of the next theorem, recall that an anchored bundle is said to terminate inside a singular foliation $\mathcal{F}$ if $\rho(\Gamma(A)) \subset \mathcal{F}$ and to be over $\mathcal{F}$ if $\rho(\Gamma(A)) = \mathcal{F}$.

Proposition 2.1.8: The unique up to equivalence anchored bundles

Any two anchored bundles over the same singular foliation are equivalent.

Proof. Let $U \subset M$ be an open subset of M and fix a local trivialisation of A_1. Since $\rho_1(\Gamma(A_1)) = \rho_2(\Gamma(A_2))$, we can define a $\mathcal{O}(U)$-linear map

$$\varphi_U \colon \Gamma(A_1)_U \longrightarrow \Gamma(A_2)_U$$

such that $\rho_1(a) = \rho_2(\varphi(a))$ for every $a \in \Gamma(A_1)_U$. Likewise, we have a map

$$\psi_U \colon \Gamma(A_2)_U \longrightarrow \Gamma(A_1)_U$$

In the smooth case, we use partition of unity to glue these local maps to a global one. □

Leaves of anchored bundles

So far, we have simply used the fact that $\mathcal{F}$ is a locally finitely generated module over functions. Stability under Lie bracket has not been used at this point. Here is, however, a first result that makes use of leaves.

Theorem 2.1.9: Along a leaf, always the same landscape

Let $(A, \rho, [\cdot,\cdot]_A)$ be an anchored bundle over a singular foliation $\mathcal{F}$. Any two points in the same leaf have neighborhoods on which the restrictions of (A, ρ) are isomorphic.

We delay the proof to after the introduction of almost Lie algebroid structures.

2.1.2 Almost Lie algebroids: definition and existence

Now that we have clarified the existence and the (relative) unicity of an anchored bundle, comes the second point in Question 2.1.4, what structure does it have? Here is a candidate.

Definition 2.1.10: Almost Lie algebroids

Let (A, ρ) be an anchored vector bundle over M. We call almost-Lie algebroid bracket *a skew-symmetric bilinear (over $\mathbb{K}$) map*

$$[\cdot\,,\cdot]_A : \Gamma(A) \wedge \Gamma(A) \longrightarrow \Gamma(A)$$

that satisfies the Leibniz identity*,*

$$[x, fy]_A = \rho(x)[f]y + f[x,y]_A, \quad \textit{for all } x, y \in \Gamma(A), f \in \mathcal{O}(M) \tag{2.3}$$

and the anchor condition*:*

$$\rho([x,y]_A) = [\rho(x), \rho(y)], \quad \textit{for all } x, y \in \Gamma(A). \tag{2.4}$$

Remark 2.1.11. In the definition of an almost Lie algebroid, we do not assume $[\cdot\,,\cdot]_A$ to satisfy the Jacobi identity. When it does, we have in fact a Lie algebroid whose image through the anchor map is $\mathcal{F}$. It however satisfies that for any sections $x, y, z \in \Gamma(A)$:

$$\rho\left([x,[y,z]_A]_A + [y,[z,x]_A]_A + [z,[x,y]_A]_A\right) = 0.$$

The following Lemma makes almost Lie algebroids a good candidate to answer item 3 in Question 2.1.4.

Lemma 2.1.12. *For every almost-Lie algebroid on $(A \to M, \rho, [\cdot,\cdot]_A)$, the image of the anchor map $\rho(\Gamma(A)) \subseteq \mathfrak{X}(M)$ is a singular foliation on M.*

Proof. It is an immediate consequence of the anchor condition. □

We can now answer the second point of Question 2.1.4. We learnt from Marco Zambon the following result:

Proposition 2.1.13: Almost Lie algebroids

Every finitely generated foliation on M is the image under the anchor map of an almost-Lie algebroid.
In the smooth case, moreover,

1. *Every anchored vector bundle (A, ρ) over M such that $\rho(\Gamma(A)) = \mathcal{F}$ can be endowed with an almost-Lie algebroid bracket.*
2. *A singular foliation is the image under the anchor map of an almost-Lie algebroid if and only if it is finitely generated.*

Proof. Let $X_1, \ldots, X_r$ be generators of $\mathcal{F}$. There exist functions c_{ij}^k such that

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k \text{ and } c_{ji}^k = -c_{ij}^k.$$

Let A be the trivial vector bundle over M with fibers $\mathbb{K}^r$, and let $e_1, \ldots, e_r$ be the canonical trivialisation of this bundle. We define the almost Lie algebroid anchors and brackets on generators by

$$\begin{cases} \rho(e_i) &= X_i \\ [x, y]_A &= \sum_{k=1}^{r} c_{ij}^k e_k \end{cases}$$

and extend them using linearity (for the anchor) or Leibniz identity (for the bracket). This is easily checked to be an almost Lie algebroid.

The second part of the statement (i.e. the smooth case) comes from the observation that almost Lie algebroid brackets on a given anchored bundle (A, ρ) can be glued using a partition of unity. More precisely, given an anchored bundle $(A \to M, \rho)$, a partition of unity $(\chi_i)_{i\in I}$ relative to an open cover $(\mathcal{U}_i)_{i\in I}$, and almost Lie algebroid brackets $[\cdot,\cdot]_i$ (relative to ρ) on $\mathcal{U}_i$ for all $i \in I$, the following expression:

$$[\cdot,\cdot] = \sum_{i\in I} \chi_i \, [\cdot,\cdot]_i$$

is an almost Lie algebroid bracket on $\cup_{i\ \in I}\mathcal{U}_i$ - relative to the anchor ρ. □

Now that we have defined the object "almost Lie algebroid that terminates in $\mathcal{F}$", let us make it a category by defining morphisms. In fact, we will only deal with morphisms over the identity of M, which are much simpler. The subtlety is that we do not assume morphisms of almost Lie algebroid strictures to be compatible with the bracket, but only to be compatible with the anchor! This is absolutely counter-intuitive, but makes perfect sense having Lie ∞-algebroids in mind.

Definition 2.1.14: Morphisms and Equivalences

Let M be a manifold.

1. *We call* morphism of almost Lie algebroids *a morphisms of anchored bundles - forgetting the almost Lie algebroid bracket.*

2. Two such morphisms are equivalent if and only if they are equivalent as anchored bundle morphisms.

3. In particular, an equivalence between almost Lie ∞-algebroids is simply an equivalence of their underlying anchored bundles.

This deserves justification: why did we not require that "morphisms" respect the almost Lie algebroid brackets? The answer comes from the following proposition:

Proposition 2.1.15. *Let $(A_1, [\cdot\,,\cdot]_{A_1}, \rho_1)$ and $(A_2, [\cdot\,,\cdot]_{A_2}, \rho_2)$ be almost Lie algebroids that terminate inside the same singular foliation $\mathcal{F}$. For any morphism Φ from the first one to the second one:*

$$[\Phi(a), \Phi(b)]_{A_2} - \Phi([a, b]) \in \ker(\rho_2)$$

Proof. By definition of an almost Lie algebroid:

$$\begin{aligned}\rho_2([\Phi(a), \Phi(b)]_{A_2} - \Phi([a, b])) &= [\rho_2 \circ \Phi(a), \rho_2 \circ \Phi(b)]_{A_2} - \rho_2 \circ \Phi([a, b])) \\ &= [\rho_1(a), \rho_1(b)]_{A_2} - \rho_1([a, b])) \\ &= 0\end{aligned}$$

This proves the claim. □

Let us conclude this section by a theorem that follows from Proposition 2.1.13, and Theorem 2.1.16.

Theorem 2.1.16: The class of almost Lie algebroids over $\mathcal{F}$.

Any two almost Lie algebroids over a finitely generated singular foliation are equivalent.

2.1.3 An alternative proof of Proposition 1.6.10

We use the notion of almost Lie algebroid to give a much simpler proof of a result that was crucial to establish the existence of leaves: Proposition 1.6.10. This statement asserts that the flows of vector fields in $\mathcal{F}$ are symmetries of $\mathcal{F}$. It follows in fact from the following proposition, that we shall now prove, using two important tools: linear vector fields and, as we said, almost Lie algebroids.

Proposition 2.1.17. *Let $\mathcal{F}$ be a singular foliation on M. Choose $X \in \mathcal{F}$ whose time 1-flow ϕ_1^X of X is a well-defined diffeomorphism from $\mathcal{U}$ to $\mathcal{V}$. Then the restrictions of any anchored bundle (A, ρ) over M to $\mathcal{U}$ and $\mathcal{V}$ are isomorphic.*

These isomorphisms can be seen, when $\mathcal{U} = \mathcal{V} = M$ and A is a trivial bundle, as families of invertible matrices as in Proposition 1.6.10. They form therefore an alternative proof to those results.

The proof is based on the notion of linear vector field. A vector field Y on a vector bundle $E \xrightarrow{p} M$ is said to be linear if one of the following equivalent conditions holds:

(i) For any function f on E whose restriction to any fiber of $p\colon E \to M$ is a polynomial of degree $\leq k$, $Y[f]$ is a polynomial of degree $\leq k$. $Y[p^*\mathcal{O}] \subset p^*\mathcal{O}$ and $Y[\Gamma(A^*)] \subset \Gamma(A^*)$, with the understanding that $\Gamma(A^*)$ must be considered as a function on A.

(iii) In any local coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_m)$, with x_i the coordinates on the base, and y_j linear coordinates on the fibers, the vector field Y is of the form:

$$Y = \sum_{i=1}^{n} A_i(x_1, \ldots, x_n) \frac{\partial}{\partial x_i} + \sum_{i,j=1}^{n} B_{ij}(x_1, \ldots, x_n) y_i \frac{\partial}{\partial y_j}.$$

Here is a lemma that describes a practical manner to define a linear vector field:

Lemma 2.1.18. *Let $E \stackrel{p}{\to} M$ be a vector bundle and $X \in \mathfrak{X}(M)$ be a vector field. For any linear map:*

$$\delta_X : \Gamma(E^*) \longrightarrow \Gamma(E^*)$$

such that for every function $f \in \mathcal{O}_M$ and every $\epsilon \in \Gamma(E^*)$:

$$\delta_X(f\epsilon) = f\delta_X(\epsilon) + X[f]\,\epsilon \tag{2.5}$$

there exists a unique linear vector field on E that projects on X and whose restriction to fiberwise linear functions on E is δ_X.

Here is a lemma about flow of linear vector fields.

Lemma 2.1.19. *Let Y be a linear vector field on a vector bundle $E \stackrel{p}{\to} M$. There exists a vector field $p_*(Y)$ on M to which Y is p-related. The flow ϕ_t^Y at time t of a linear vector field is defined if and only the flow at time t of its projection $p_*(Y)$ on M is defined. In that case, it is vector bundle isomorphism*

$$\begin{tikzcd} E \arrow[r, "\phi_t^Y"] \arrow[d] & E \arrow[d] \\ M \arrow[r, "\phi_t^{p_*(Y)}"] & M \end{tikzcd}$$

Now we can prove Proposition. Choose a vector field $\tilde{X}$ on 2.1.17.

Proof. Let us choose $X \in \mathcal{F}$ and $a \in \Gamma(A)$ such that $\rho(a) = X$. The following two items construct a linear vector field on TM and a linear vector field on A that project on X.

1. The map:

$$\begin{array}{rccc} \delta_X^{TM}: & \Gamma(T^*M) & \to & \Gamma(T^*M) \\ & \alpha & \mapsto & L_X\alpha \end{array}$$

   satisfies Condition (2.5) and therefore defines a linear vector field $\widehat{X}$ on $TM \stackrel{p}{\to} M$ which is p-compatible to X.

2. Let $a \in \Gamma(A)$ be any section such that $\rho(a) = X$. Choose an almost Lie algebroid bracket $[\cdot,\cdot]_A$ on (A, ρ). We define the Lie derivative $\mathcal{L}_a$ of a section $\alpha \in \Gamma(A^*)$ by

$$\langle \mathcal{L}_a\alpha, b\rangle = \rho(a)\,[\langle \alpha, b\rangle] - \langle \alpha, [a, b]_A\rangle.$$

   The Leibniz condition implies that the right hand side of the previous expression is linear over functions, and is therefore a well-defined section of $\Gamma(A^*)$. Moreover

$$\alpha \mapsto \mathcal{L}_a\alpha$$

   satisfies condition (2.5) and defines therefore a linear vector field $\widehat{a}$ on A which is p-related to $\rho(a) = X$.

Since the vector fields $\widehat{a}$ and $\widehat{X}$ on $A \stackrel{p}{\to} M$ and $TM \stackrel{p}{\to} M$ are p-related to X, the pair $(\widehat{a}, \widehat{X}) \in (A \times TM)$ is tangent to the fibered product $A \times_{p,M,p} TM$ and its restriction is a linear vector field on $A \oplus TM \stackrel{p}{\to} M$ which is again p-related to X. We denote it by $\widehat{a, X}$

Now, the graph of ρ:

$$\{(b, \rho(b))|b \in A\} \subset A \oplus TM$$

is a submanifold of $A \oplus TM$ that we denote by $Gr(\rho)$.

We claim that $\widehat{a, X}$ is tangent to that submanifold. Let us check it: the submanifold $Gr(\rho)$ is the zero locus of the functions[1]

$$\begin{array}{rccc} f_\alpha: & A \oplus TM & \to & \mathbb{R} \\ & (b, u) & \mapsto & \alpha(\rho(b) - u) \end{array}$$

with $\alpha \in \Gamma(T^*M)$. This comes from the easily checked identity:

$$\widehat{a, X}\,[f_\alpha]\ (b, u) \mapsto \mathcal{L}_X\alpha\,(\rho(b) - u). \tag{2.6}$$

[1]The proof could be repeated almost word by word to the holomorphic setting.

Remark 2.1.20. Equation 2.6 is precisely the point where we use the anchor condition of the almost Lie algebroid is used - so far, only the Leibniz identity was used.

Assume now that the flow at time t of $X \in \mathcal{F}$ exists on $\mathcal{U}$. By Lemma 2.1.19, the flow of $\widehat{a}$ is a vector bundle morphism $\Phi_t^a : A \to A$ over $\phi_t^X : M \to M$. The flow of $\widehat{a, X}$ is also a linear vector field, given by

$$\begin{array}{ccc} (A \oplus TM)_{|\mathcal{U}} & \to & (A \oplus TM)_{|\mathcal{V}} \\ (b, u) & \mapsto & (\Phi_t^a(b), T\phi_t^X(u)). \end{array}$$

Now, since $\widehat{a, X}$ is tangent to $Gr(\rho)$, its flow must preserve it, which precisely means that Diagram (2.1) commutes, and therefore means that Φ_t^a is an isomorphism of anchored bundles. This proves the claim. □

2.2 Isotropy Lie algebra and holonomy Lie algebroids

Let us use the almost Lie algebroids associated to a singular foliation in the previous section to associate a Lie algebra to any point of a singular foliation.

2.2.1 Kernel and Strong-kernel of a vector bundle morphism

Consider a vector bundle morphism over the identity of M:

$$\begin{tikzcd} B \arrow[r, "\Phi"] \arrow[d] & C \arrow[d] \\ M \arrow[r, equal] & M \end{tikzcd}$$

Choose a point $m \in M$. There are two subspaces in B_m that deserve to be called "kernels".

1. the usual kernel $\ker(\Phi_m)$, i.e.
$$\{u \in B_m \,|\, \Phi_{|_m}(u) = 0\},$$
2. and there is the *Strong Kernel*, i.e. the subspace $\mathrm{Sker}(\Phi, \mathrm{m}) \subset \mathrm{B_m}$ of all elements through which there is a neighborhood $\mathcal{U}$ of m in M and a local section in the kernel of $\Phi \colon \Gamma_{\mathcal{U}}(B) \to \Gamma_{\mathcal{U}}(C)$. In equation:
$$\mathrm{Sker}(\Phi, \mathrm{m}) := \{\mathrm{u} \in \mathrm{B_m} \text{ s.t. } \exists \mathrm{U} \in \Gamma(\mathrm{B}) \text{ with } \Phi(\mathrm{U}) = 0 \text{ and } \mathrm{U}_{|_\mathrm{m}} = \mathrm{u}\}.$$

Of course, there is an inclusion:
$$\mathrm{Sker}(\Phi, \mathrm{m}) \subset \ker(\Phi|_\mathrm{m}).$$

Moreover, the dimensions of the distributions have opposite behaviour:

1. the map $m \mapsto \dim(\ker(\Phi_{|_\mathrm{m}}))$ is upper semi-continuous, i.e. if a sequence (x_n) in M has limit x, then
$$\dim(\ker(\Phi_{|_\mathrm{x}})) \geq \text{ upper limit of } \dim(\ker(\Phi_{|_{\mathrm{x_n}}}))$$
2. the map $m \mapsto \dim(\mathrm{Sker}(\Phi_{|_\mathrm{m}}))$ is lower semi-continuous, i.e. if a sequence (x_n) in M has limit x, then
$$\dim(\mathrm{Sker}(\Phi, \mathrm{x})) \leq \text{ lower limit of } \dim(\mathrm{Sker}(\Phi, \mathrm{x_n})).$$

Proposition 2.2.1. *Let $\Phi \colon B \to A$ be a vector bundle morphism over the identity of M. For any $m \in M$, the following two assertions are equivalent:*

1. *the kernel and the strong kernel coincide at m.*
2. *There is a neighborhood $\mathcal{U}$ of m in M on which the kernel and the strong kernel coincide at all points.*

In this case moreover, these coinciding kernels form a sub-vector bundle of the restriction to $\mathcal{U}$ of B.

2.2.2 The isotropy Lie algebra I: the space

Let $\mathcal{F}$ be a singular foliation on a manifold M. Let $\mathcal{U}$ be an open neighborhood of m on which $\mathcal{F}$ is finitely generated. In view of Proposition 2.1.13, there exists an anchored bundle $(A \to \mathcal{U}, \rho)$ over $\mathcal{F}$. We call *isotropy vector space at m* the quotient space:

$$\mathfrak{g}_m(\mathcal{F}) = \frac{\ker(\rho_{\mathrm{m}})}{\mathrm{Sker}(\rho, \mathrm{m})}.$$

Notice that the notation $\mathfrak{g}_m(\mathcal{F})$ makes no reference to the chosen anchored bundle. This is justified by the following proposition:

Proposition 2.2.2: The holonomy vector space at m makes sense

Let $\mathcal{F}$ be a singular foliation. The isotropy vector spaces associated to any two anchored bundles are canonically isomorphic.

Proof. This is an immediate consequence of Theorem 2.1.8. ☐

Here is an important theorem, due to [AS09].

Theorem 2.2.3: Ranks and dimensions

Let $(M, \mathcal{F})$ be a singular foliation. For every $m \in M$,

1. *the rank $\mathrm{rk}_m(\mathcal{F})$ of $\mathcal{F}$ at m (i.e. the minimal number of local generators),*
2. *the dimension $\dim(L_m)$ of the leaf through m,*
3. *and the dimension $\dim(\mathfrak{g}_m(\mathcal{F}))$ of the holonomy vector space[a] at m,*

are related by the relation

$$\mathrm{rk}_m(\mathcal{F}) = \dim(\mathfrak{g}_m(\mathcal{F})) + \dim(L_m).$$

[a]that will be soon equipped with a Lie bracket making it Androulidakis-Skandalis isotropy Lie algebra.

Proof. Using the local splitting theorem, locally, we obtain that $\mathcal{F}$ is the product of $\mathfrak{X}(\mathbb{R}^a)$ (with a the dimension of $T_m\mathcal{F}$) and a foliation vanishing at the origin, called the transverse foliation. The rank of foliation $\mathfrak{X}(\mathbb{R}^r)$ is r. So we are left with the task of proving that the transverse foliation $\mathcal{T}$ (which is made of vector fields vanishing of the origin) has a rank equal to the dimension of $\mathfrak{g}_m(\mathcal{T})$. We can pick a minimal system of generators $Y_1, \dots, Y_r$. It is clear that the class $[Y_i]$'s of these vectors in $\mathfrak{g}_0(\mathcal{T})$, so the dimension of $\mathfrak{g}_m(\mathcal{T})$ is less or equal the rank r of $\mathcal{T}$. Thus, it only remains to show that these classes $[Y_i] \in g_m(\mathcal{T})$ are linearly independent. Assume that they are not, i.e. (without loss of generality): $[Y_1] = \sum_{i=2}^r \alpha_i [Y_i]$ for some $\alpha_i \in \mathbb{R}$. This means that $Y_1 = \sum_{i=2}^r \alpha_i Y_i + Y$ with Y in $I_0\mathcal{F}$. We can write $Y = \sum g_i Y_i$ with g_i vanishing at the origin and obtain

$$(1 - g_1)Y_1 = \sum_{i=2}^r (\alpha_i + g_i) Y_i$$

Near the origin, we can invert $(1 - g_k)$ and get an expression for Y_1 in term of the other Y's which contradicts the minimality of $Y_1, ..., Y_r$. This cncludes the proof. ☐

2.2.3 The isotropy Lie algebra II: the bracket

Again, $\mathcal{F}$ is a singular foliation, m is a point, $\mathcal{U}$ open and $(A \to \mathcal{U}, \rho)$ an anchored bundle of $\mathcal{F}$.

According to Proposition **??**, $(A \to \mathcal{U}, \rho)$ can be equipped with an almost Lie algebroid bracket[2] that we denote by

$$[\cdot,\cdot]_A \colon \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A).$$

Choose two sections a, b of $A \to \mathcal{U}$. In view of the Leibniz identity, the value at m of the Lie bracket $[a,b]$ depends on the 1-jet at m of the sections of A. However, if $\rho(a) = 0$, then

$$\begin{aligned} [a, fb]|_m &= f(m)[a,b]|_m + \rho(a|_m)[f]b|_m \\ &= f(m)[a,b]|_m \end{aligned}$$

for any function f, this implies that $[a,b]|_m$ depends only on the value of the section b at m. More generally, if $\rho(a) = \rho(b) = 0$, then $[a,b]|_m$ depends only on the value of a and b at m, and the bracket $[\cdot,\cdot]_A$ therefore induces a bilinear map

$$[\cdot,\cdot]_{A,m} \colon \wedge^2 \ker(\rho_m) \longrightarrow A_m$$

by

$$[a,b]_{A,m} = [\tilde{a}, \tilde{b}]_A(m)$$

for any sections $\tilde{a}, \tilde{b}$ through a and b. However,$[a,b]_{A,m}$ is in fact valued in $\ker(\rho_m)$: this follows easily from the anchor condition, since

$$[a,b]_{A,m} = [\tilde{a}, \tilde{b}]_A(m)$$

Lastly, we claim that the strong kernel at m is an "ideal" of that bracket, i.e. $[\mathrm{Sker}(\rho, m), \ker(\rho_m)] \subset \mathrm{Sker}(\rho, m)$ so that the skew-symmetric bilinear map $[\cdot,\cdot]_{A,m}$ goes to the quotient to a bilinar map

$$[\cdot,\cdot]_m \colon \wedge^2 \mathfrak{g}_m(\mathcal{F}) \longrightarrow \mathfrak{g}_m(\mathcal{F}) \tag{2.7}$$

Proposition 2.2.4

The bilinear map (2.7)*:*

1. *is a Lie bracket on the holonomy vector space* $\mathfrak{g}_m(\mathcal{F})$*,*
2. *is canonically defined, i.e. does not depend on the choice of an anchored bundle and of an almost Lie algebroid bracket.*

It is called the isotropy Lie algebra of $\mathcal{F}$ at $m \in M$*.*

Proof. The Jacobi identity follows from (3.1.1): Now, for any two almost Lie algebroid brackets on A, we have, for any two sections $\tilde{a}, \tilde{b}$

$$\rho\left([\tilde{a}, \tilde{b}]'_A - [\tilde{a}, \tilde{b}]_A\right) = 0$$

so that $\tilde{a}, \tilde{b}$ is valued in the Strong kernel of ρ. This implies that the induced bracket (2.7) does not depend on the choice of an almost Lie algebroid bracket on given anchored bundle (A, ρ). More generally, given two anchored bundles (A, ρ) and (A', ρ'), Theorem 3.1.1 implies the existence of anchored bundle morphisms $\Phi : (A, \rho) \longrightarrow (A', \rho')$. For any two almost Lie algebroid structures on A and A', we have

$$\rho\left(\Phi([\tilde{a}, \tilde{b}]_A) - [\Phi(\tilde{a}), \Phi(\tilde{b})]_A\right) = 0$$

so that $(\Phi([\tilde{a}, \tilde{b}]_A) - [\Phi(\tilde{a}), \Phi(\tilde{b})]_A$ is valued in the Strong kernel, so that Φ induces a Lie algebra morphism

$$\Phi_{|_m} \colon \frac{\ker(\rho_\mathrm{m})}{\mathrm{Sker}(\rho, \mathrm{m})} \simeq \frac{\ker(\rho'_\mathrm{m})}{\mathrm{Sker}(\rho', \mathrm{m})}.$$

The same construction, applied to Ψ, gives an inverse map to that Lie morphism. The result follows $\square$

[2]At least in the smooth setting. In complex or real analytic setting, one may have to restrict to a smaller open neighborhood.

2.2.4 Androulidakis-Skandalis construction of the isotropy Lie algebra

Let us now present the original definition of the isotropy Lie algebra of a singular foliation by Androulidakis and Skandalis.

Let $\mathcal{F}$ be a singular foliation. For every $m \in M$, we denote by $\mathcal{F}_m \subset \mathcal{F}$ the sub-Lie algebra of vector fields in $\mathcal{F}$. Let $\mathcal{I}_m$ be the ideal of functions vanishing at m. There is an inclusion $\mathcal{I}_m\mathcal{F} \subset \mathcal{F}_m$, where $\mathcal{I}_m\mathcal{F}$ stands for teh space of vector fields on the form $\sum_{i=1}^s f_i X_i$ with $f_1, \ldots, f_s \in \mathcal{I}_m$ and $X_1, \ldots, X_s \in \mathcal{F}$. Moreover, $\mathcal{I}_m\mathcal{F}$ is a Lie ideal of $\mathcal{F}_m$, since for all $X \in \mathcal{F}, Y \in \mathcal{F}_m$ and $F \in \mathcal{I}_m$:

$$[FX, Y] = \underbrace{F}_{\in \mathcal{I}_m} \underbrace{[X,Y]}_{\in \mathcal{F}} - \underbrace{Y[F]}_{\in \mathcal{I}_m} \underbrace{X}_{\in \mathcal{F}}.$$

Proposition 2.2.5: The isotropy Lie algebra: original construction

Let $(M, \mathcal{F})$ be a singular foliation. For every $m \in M$, the isotropy Lie algebra at m is canonically isomorphic to the quotient Lie algebra:

$$\mathfrak{g}_m(\mathcal{F}) \xrightarrow{\simeq} \frac{\mathcal{F}_m}{\mathcal{I}_m\mathcal{F}}.$$

Proof. Let (A, ρ) be an achored bundle over $\mathcal{F}$. For any $a \in Ker(\rho|_m)$, let $\rho(\tilde{a})$ where $\tilde{a}$ is any section of A through a. By construction, $\rho(\tilde{a})$ is in $\mathcal{F}_m$. Since any two sections $\widetilde{a}_1, \widetilde{a}_2$ through a differ by a section of the form

$$\widetilde{a}_1 - \widetilde{a}_2 = \sum_{i=1}^s F_i b_i$$

with $F_i \in \mathcal{I}_m$. As a consequence:

$$\rho\left(\widetilde{a}_1\right) - \rho\left(\widetilde{a}_2\right) = \sum_{i=1}^s F_i\, \rho(b_i)$$

so that $\rho\left(\widetilde{a}_1\right) - \rho\left(\widetilde{a}_2\right) \in \mathcal{I}_m\mathcal{F}$. The map

$$Ker(\rho_m) \longrightarrow \frac{\mathcal{F}_m}{\mathcal{I}_m\mathcal{F}}$$

is therefore well-defined. It is also surjective by construction. It clearly has the strong kernel in its kernel, and therefore goes down to a morphism:

$$\frac{Ker(\rho_m)}{SKer(\rho, m)} \longrightarrow \frac{\mathcal{F}_m}{\mathcal{I}_m\mathcal{F}}\ .$$

The anchor condition implies that it is a Lie algebra morphism from $\mathfrak{g}_m(\mathcal{F})$ onto $\frac{\mathcal{F}_m}{\mathcal{I}_m\mathcal{F}}$. The kernel of this map is zero: we leave it to the reader. □

2.2.5 The linear isotropy Lie algebra

The linear part of a vector field vanishing at a point

Let m be a point in a manifold M, and $\mathcal{I}_m$ be the ideal of functions vanishing at $m \in M$. Denote by $\mathfrak{X}_m(M)$ the Lie algebra of vector fields vanishing at m. The purpose of this preliminary section is to show that there exists a natural Lie algebra morphism:

$$\mathfrak{X}_m(M) \longrightarrow \mathrm{gl}(T_m M).$$

There are several equivalent manners to see this Lie algebra homomorphism, that we now detail.

1. One manner is simply to take local coordinates $(x_1, \ldots, x_n)$ in which m has coordinates $(0, \ldots, 0)$. The vector fields
$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_1},$$
restricted to $T_m M$ form a basis of that vector space that we shall denote by $\delta_1, \ldots, \delta_n$. We then map a vector field:
$$\sum_{i=1}^{n} X_i(x_1, \ldots, x_n) \frac{\partial}{\partial x_i}$$
to the linear endomorphism of $T_m M$ whose matrix in the basis $\delta_1, \ldots, \delta_n$ is
$$\left(\quad \frac{\partial X_i}{\partial x_j}(0, \ldots, 0) \quad \right)$$
We leave it to the reader to check that this is indeed a Lie algebra morphism.

   Although very explicit, this method has a drawback: we have to check it does not depend on the choice of local coordinates. It is therefore better to use the coming two descriptions then to show that, in local coordinates, they take the previous form.

2. The second manner is to use the flow ϕ_t^X of a vector field $X \in \mathfrak{X}_m(M)$. Since X vanishes at m, for every $\eta > 0$, there is an neighborhood $\mathcal{U}_m$ of m on which ϕ_t^X is well-defined for all $t \in -]\eta, \eta[$. Also, $\phi_t^X(m) = m$, so that the differential of ϕ_t^X at m is a family depending on $t \in -]\eta, \eta[$ of invertible linear endomorphisms
$$T_m \phi_t^X : T_m M \longleftrightarrow T_m M$$
We then define a linear endomorphism of $T_m M$ by
$$X \mapsto \left. \frac{\partial}{\partial t} \right|_{t=0} T_m \phi_t^X$$
It is obvious that the previous map is well-defined, but it is not clear that it is a Lie algebra morphism. Also, defining it required the notion of flow, which does not make sense in algebraic geometry.

3. The third manner is to look, for any vector field X vanishing at m, at the adjoint action:
$$Y \mapsto [X, Y]$$
and to check that $[X, Y]_{|_m}$ only depends on $Y_{|_m}$, so that the adjoint action induces a linear endomorphism of $T_m M$. The Jacobi identity implies that this map is a Lie algebra morphism.

4. The fourth manner is to use the canonical identification
$$T_m^* M \simeq \frac{\mathcal{I}_m}{\mathcal{I}_m^2}$$
with $\mathcal{I}_m$ the[3] ideal of functions vanishing on M (which, in the algebraic geometry setting, is in fact a definition of the cotangent space). Consider vector fields as derivations of the sheaf of functions: a vector field X vanishes at m if and only if $X[\mathcal{I}_m] \subset \mathcal{I}_m$. By derivation properties, this implies $X[\mathcal{I}_m^2] \subset \mathcal{I}_m^2$, so that X induces a linear endomorphism of of $T_m^* M \simeq \frac{\mathcal{I}_m}{\mathcal{I}_m^2}$. Since the bracket of vector fields is their commutator, when seen as a derivation, it is obvious that the map above is a Lie algebra morphism:
$$\mathfrak{X}_m(M) \longrightarrow \mathrm{gl}(T_m^* M).$$
The desired Lie algebra morphism is obtained by composing the latter morphism with the canonical dualization Lie algebra isomorphism $\mathrm{gl}(T_m^* M) \simeq \mathrm{gl}(T_m M)$.

[3]In real analytic or complex geometry setting, "ideal" must be understood as "sheaf of ideals"

Notation 2.2.6: Linear Part of a vector field

We denote by $\mathfrak{Lin}$ *the Lie algebra morphism*

$$\{\textit{Vector fields vanishing at } m\} \longrightarrow \mathrm{gl}(T_m M).$$

described in the lines above.

The linear isotropy Lie algebra of a foliation vanishing at a point

Let us consider a foliation $\mathcal{F}$ on a manifold M made of vector fields vanishing at a point m (equivalently, such that $\{m\}$ is a leaf).

To our knowledge, Dominique Cerveau is the first one to have understood the importance and studied the following Lie algebra.

Definition 2.2.7: Linear isotropy Lie algebra

Let $\mathcal{F}$ *be a singular foliation vanishing at point* m*. We call* linear isotropy Lie algebra of $\mathcal{F}$ *the image of* $\mathcal{F}$ *through the linear part morphism:* $\mathfrak{Lin}$.
We denote it by $\mathfrak{g}_m^{lin}(\mathcal{F})$.

Remark 2.2.8. In equation

$$\mathfrak{g}_m^{lin}(\mathcal{F}) := \mathfrak{Lin}(\mathcal{F})$$

Remark 2.2.9. Upon choosing local coordinates, and therefore a basis of $T_m M$, the linear isotropy Lie algebra of $\mathcal{F}$ at m at the origin is the sub-Lie-algebra of all matrices (a_{ij}) such that there exists $X \in \mathcal{F}$ whose Taylor expansion at the origin reads:

$$X = \sum_{i,j} a_{ij} x_i \frac{\partial}{\partial x_j} + \text{higher order terms}$$

Example 2.2.10. Let $\mathcal{F}$ be the singular foliation induced by a Lie algebra action of $\mathfrak{g} \subset \mathfrak{gl}(\mathbb{R}^d)$ on $\mathbb{R}^d$. Then the linear holonomy of $\mathcal{F}$ at 0 is $\mathfrak{g}$ itself.

Example 2.2.11. Let $\mathcal{F} \subset I_0^2 \mathfrak{X}(\mathbb{R}^d)$, i.e. a foliation made of vector fields vanishing quadratically in the origin. Then the linear isotropy Lie algebra of $\mathcal{F}$ at 0 is $\{0\}$.

By construction,

$$\mathfrak{Lin} \colon \mathcal{F} \longrightarrow \mathfrak{g}_m^{lin}(\mathcal{F}) \tag{2.8}$$

is a surjective morphism of Lie algebras. Here is therefore a natural question, that one could ask for any Lie algebra morphism: Does (2.8) admit a section which is a Lie algebra morphism? If yes, it means, geometrically, that $\mathcal{F}$ contains a sub-singular foliation associated to the Lie algebra action of $\mathrm{g}_m^{lin}(\mathcal{F})$ on M.

Question 2.2.12: Natural question.

Does $\mathfrak{Lin}$ *admits sections? I.e, does* $\mathcal{F}$ *contains, in a neighborhood* $\mathcal{U}$ *of* m*, a sub-singular foliation given by a Lie algebra action of* $\mathfrak{g}_m^{lin}(\mathcal{F})$ *on* $\mathcal{U}$*?*

In general, the answer of this kind of question tends to be ”no, unless the image is semi-simple”. And in the infinite dimensional case, the answer tends to be ”no, unless the image is compact and semi-simple. If the image is semi-simple, then there are only formal sections”. There are several results in that vein, by Conn for Poisson structures and Zung for Lie algebroids.

To our knowledge, the compact case is widely open. Here is an important result by Dominique Cerveau for the semi-simple case. A more recent proof can also be found in [LGR21].

Theorem 2.2.13: A linearisation theorem by Dominique Cerveau

If the linear isotropy Lie algebra of $\mathcal{F}$ at m is a semi-simple Lie algebra, then the map:

$$\mathfrak{Lin}\colon \mathcal{F} \longrightarrow \mathfrak{g}_m^{lin}(\mathcal{F})$$

admits a formal section which is a formal Lie algebra morphism.

We will not prove this theorem, but let us say a word about its meaning. Formal functions[4] at a point $m \in M$ form an algebra that we should denote by $\hat{\mathcal{O}}_m$. Formal functions $\hat{\mathcal{O}}_m$ are a module over over germs of smooth, complex, polynomials, or real analytic functions (that we should denote by $\mathcal{O}$). As a consequence, the tensor product

$$\hat{\mathcal{O}}_m \otimes_{\mathcal{O}} \mathcal{F}$$

is a finitely generated $\hat{\mathcal{O}}_m$ module stable under Lie bracket[5], and the linear map extends easily to a Lie algebra morphism:

$$\mathfrak{Lin}\colon \hat{\mathcal{O}}_m \otimes_{\mathcal{O}} \mathcal{F} \longrightarrow \mathfrak{g}_m^{lin}(\mathcal{F})$$

The result of Dominique Cerveau states that this Lie algebra morphism admits a section which is a Lie algebra morphism.

Exercice 2.2.14. Let $(M, \mathcal{F})$ be a singular foliation vanishing at $m \in M$. Show that the quotient space[6]

$$\frac{\mathcal{F}}{\mathcal{F} \cap \mathcal{I}_m^2 \mathfrak{X}(M)},$$

is a Lie algebra isomorphic to $\mathfrak{g}_m^{lin}(\mathcal{F})$.

We would like to define the linear holonomy at an arbitrary point in a way that does not require the choice of local coordinates. This can be done algebraically as follows:

Definition 2.2.15. Let $(M, \mathcal{F})$ be a foliated manifold and $m \in M$. The *linear isotropy Lie algebra of $\mathcal{F}$ at m* is the following space

$$\mathfrak{g}_m^{lin}(\mathcal{F}) = \frac{\mathcal{F} \cap I_m \mathfrak{X}(M)}{\mathcal{F} \cap I_m^2 \mathfrak{X}(M)},$$

where $I_m \subset \mathcal{C}^\infty(M)$ is the ideal of all functions vanishing in m. Its Lie bracket is induced by the Lie bracket of vector fields.

Roughly speaking, we take all vector fields in $\mathcal{F}$ vanishing in m and consider two such vector fields equivalent, if they differ by a vector field vanishing quadratically. The fact that this is a finite-dimensional Lie algebra can be verified by hand or using the local splitting theorem and the above discussion.

Also, the linear holonomy at m is canonically isomorphic to the one of its transverse singular foliation.

2.2.6 The isotropy Lie algebra and its linear part

The linear isotropy Lie algebra captures the "linear approximation" of the foliation at a given point. As a consequence, for foliations vanishing quadratically at a point, this Lie algebra is trivial.

Exercice 2.2.16. Show that $m \in M$ is a regular point if and only if $\mathfrak{g}_m(\mathcal{F}) = \{0\}$.

Example 2.2.17. Let $\mathcal{F} = I_0^n \mathbb{R}^d$. Then $\mathfrak{g}_0(\mathcal{F})$ will have dimension $d \times \binom{n+d-1}{n}$, while the linear one will be trivial for $n \geq 2$.

The isotropy Lie algebra is a well-behaved object:

Exercice 2.2.18. Let $(M, \mathcal{F})$ be a foliated manifold and $m \in M$ a point. Show the following points:

1. Let U be an open subset of M containing m, then $\mathfrak{g}_m(\mathcal{F}) = \mathfrak{g}_m(\mathcal{F}|_U)$.

[4] In the smooth setting, it is the quotient of $\mathcal{C}^\infty(M)$ by the ideal of functions vanishing with all their derivatives. In the other settings, it is a formal completion, i.e. the ring of formal power series in d variables near m.

[5] It is in fact an algebraic singular foliation in the sense of Definition xxx for the ring $\hat{\mathcal{O}}_m$

[6] where $\mathcal{I}_m \subset \mathcal{C}^\infty(M)$ is the ideal of all functions vanishing in m

2. Let $(\tilde{M}, \tilde{F})$ be another foliated manifold and $\tilde{m} \in M'$. Then $\mathfrak{g}_{(m,\tilde{m})}(\mathcal{F} \times \tilde{\mathcal{F}}) = \mathfrak{g}_m(\mathcal{F}) \oplus \mathfrak{g}_{(\tilde{m})}(\tilde{\mathcal{F}})$. (The direct sum should be understood as a Lie algebra direct sum).

3. For $\varphi\colon N \mapsto M$ a surjective submersion, $\mathfrak{g}_n(\varphi^{-1}\mathcal{F}) = \mathfrak{g}_m(\mathcal{F})$ for every $n \in \varphi^{-1}(m)$.

Proof. This is a easy consequence of the local splitting theorem. □

When $\mathcal{F}$ admits real analytic generators, one can prove that the linear holonomy contains all the semi-simplicity of $\mathfrak{g}_m(\mathcal{F})$, i.e.:

Proposition 2.2.19: The semi-simple part is linear

For a real analytic foliation $\mathcal{F}$, the kernel of the linearization map $\mathfrak{g}_m(\mathcal{F}) \to \mathfrak{g}_m^{lin}(\mathcal{F})$ is a nilpotent Lie algebra.

Proof. We refer to [LGR21] for the complete proof. The main ingredient of the proof is the Artin-Rees Lemma, which is valid for Noetherian rings (i.e. the ring of analytic functions but not the ring of smooth functions). □

Whether the theorem holds also in the smooth category is still an open problem:

Question 2.2.20: Smooth Case

Is it possible to omit the assumption "locally real analytic" in Proposition 2.2.6?

While the algebra of smooth functions is not Noetherian, its infinite jets (formal Taylor power series) form a Noetherian ring. In particular by a reasoning analogous to Proposition 2.2.6, we can obtain the following series of Lie algebras[7]

$$\mathcal{F}(m) \to \mathfrak{j}_m^\infty(\mathcal{F}) \to \ldots \to \mathfrak{j}_m^N(\mathcal{F}) \to \mathfrak{j}_m^{N-1}(\mathcal{F}) \to \ldots \to \mathfrak{j}_m^1(\mathcal{F}) = \mathfrak{g}_m^{lin}(\mathcal{F}),$$

where for each finite N, the kernel of $\mathfrak{j}_m^N(\mathcal{F}) \to g_m^{lin}(\mathcal{F})$ is nilpotent. Here $\mathfrak{j}_m^N(\mathcal{F}) = \frac{\mathcal{F}(m)}{I_m^{N+1}\mathfrak{X}(M)}$ are N-jets of vector fields on $\mathcal{F}(m)$ and $\mathfrak{j}_m^\infty(\mathcal{F})$ their projective limit, i.e. the space of Taylor expansions of elements in $\mathcal{F}(m)$.

The Lie algebra $g_m^{lin}(\mathcal{F})$ might still contain a solvable ideal. By dividing out the maximal solvable ideal $\mathfrak{r}$, we obtain a semisimple Lie algebra $g_m^{lin}(\mathcal{F})^{ss} = \frac{g_m^{lin}(\mathcal{F})}{\mathfrak{r}}$, which could be appended on the right in the above diagram to obtain a series of surjections:

$$\mathcal{F}(m) \to \mathfrak{j}_m^\infty(\mathcal{F}) \to \ldots \to \mathfrak{j}_m^N(\mathcal{F}) \to \mathfrak{j}_m^{N-1}(\mathcal{F}) \to \ldots \to \mathfrak{g}_m^{lin}(\mathcal{F}) \to g_m^{lin}(\mathcal{F})^{ss}.$$

It turns out that on a formal level foliations satisfies a sort of Levi-Malcev-theorem:

Proposition 2.2.21 (Cerveau, 1977)**.** *There is a formal Lie algebra section of $\mathfrak{j}_m^\infty(\mathcal{F}) \to g_m^{lin}(\mathcal{F})^{ss}$., i.e. a Lie algebra homomorphism from the semi-simple part of the linear isotropy Lie algebra at m to the Lie algebra $\mathfrak{j}_m^\infty(\mathcal{F})$.* quote proof, e.g. cer selves?

Here, the question gets much harder, when considering convergence:

Question 2.2.22. *Does a Lie algebra section of $\mathcal{F}(m) \to g_m^{lin}(\mathcal{F})^{ss}$ exist?*

The above question is an open problem, even for foliations admitting real analytic generators. say that cerveau clai proof is missing or fi

2.3 The holonomy Lie algebroid of a leaf

In the above, we have introduced Lie algebras that measure the singular dynamics of the foliation near a point. As a consequence of Theorem , the Lie algebras of two points in the same leaf will be isomorphic. add number Moreover, they all fit together into an object that also captures the dynamics along a leaf, namely the holonomy Lie algebroid. Similarly to the isotropy Lie algebra, the simplest way to define it, is algebraically as a quotient.

Let $m \in M$ be a point and L the leaf through m. We denote by $I_L \subset \mathcal{C}^\infty(M)$ the ideal of all functions vanishing on L. definition like that?

[7] $\mathcal{F}(m)$ stands here for vector fields on M that vanish at m.

Definition 2.3.1: Holonomy Lie algebroid

Let L be a leaf of a singular foliation $\mathcal{F}$. We call holonomy Lie algebroid *of $\mathcal{F}$ along L a Lie algebroid $(A_L, \rho, [\cdot,\cdot])$ such that there exists an isomorphism $I : \Gamma(A_L) \to \frac{\mathcal{F}}{\mathcal{I}_L\mathcal{F}}$ such that*

1. *I is a Lie algebra isomorphism. (The bracket on $\frac{\mathcal{F}}{\mathcal{I}_L\mathcal{F}}$ is induced from the numerator).*
2. *The $\rho_* : \Gamma(A_L) \to \mathfrak{X}(L)$ corresponds to the map $\frac{\mathcal{F}}{\mathcal{I}_L\mathcal{F}} \to \mathfrak{X}(L)$ induced by the restriction.*

The validity of the above definition is far from obvious a priori. In order for such a vector bundle to exist, by the Serre-Swan-theorem, we need $\frac{\mathcal{F}}{I_L\mathcal{F}}$ to be a projective $\mathcal{C}^\infty(L)$-module, which heavily relies on the validity of the splitting theorem. Indeed, we get the following short exact seuqence of Lie algebroids over L:

$$\mathfrak{g}_L(\mathcal{F}) \to A_L \to TL$$

where $\mathfrak{g}_L(\mathcal{F}) = \bigsqcup_{m' \in L} \mathfrak{g}_m(\mathcal{F})$ is a bundle of Lie algebras. Hence, as a vector bundle A_L is isomorphic to $\mathfrak{g}_L(\mathcal{F}) \times TL$, however, the Lie bracket might intertwine the factors in a non-trivial way.

2.3.1 Definition through the almost Lie algebroid of a leaf

rticaft, we already

In the present section, we will use the notion presented in the next chapter "almost Lie algebroid", to give an alternative description of the holonomy Lie algebroid of a leaf. Of course, the next section is independent from the present one, so there is no contradiction.

2.4 Bi-submersions over a singular foliation

2.4.1 Definition

The most crucial and intriguing object, in order to understand singular foliations, is certainly Androulidakis-Skandalis holonomy groupoid. This is constructed out of bisubmersions, which plays in some sense the role of representatives of the differential stack for Lie groupoids. It was also introduced by Androulidakis and Skandalis in [AS09]. We first recall the definition.

Definition 2.4.1: Androulidakis-Skandalis Bi-submersions

Let M be a manifold equipped with a singular foliation $\mathcal{F}$. A bi-submersion over a $(M, \mathcal{F})$ is a triple (M, s, t) where:

1. *X is a manifold,*
2. *$s, t \colon X \to M$ are submersions, respectively called* source *and* target*,*

such that

1. *The pull-back singular foliations $s^{-1}\mathcal{F}$ and $t^{-1}\mathcal{F}$ are equal,*
2. *and $s^{-1}\mathcal{F} = t^{-1}\mathcal{F}$ coincides with the sheaf of sections of the form $X+Y$ with $X \in \Gamma(\ker(Ts))$ and $Y \in \Gamma(\ker(Tt))$.*

Bi-submersions over $(M, \mathcal{F})$ shall be denoted by $M \overset{s}{\leftarrow} X \overset{t}{\rightarrow} M$.

Definition 2.4.2: Some important notions: units and bisections

bisection *of a bisubmersion $M \overset{s}{\leftarrow} X \overset{t}{\rightarrow} M$ is a submanifold $\Sigma \subset X$ to which the restrictions of s, t are diffeomorphism onto open M (or at least open subsets[a] of M)*
A map $\epsilon \colon M \longrightarrow X$ is said to be a unit map *if it is a section of both s and t.*

[a]We then should speak of local bisubmersions, but most of the time we will just say "bisubmersions"

Remark 2.4.3. The image of the unit map is a bisection.

Of course, any bisection Σ induces a diffeomorphism:

$$\underline{\Sigma} \colon s(\Sigma) \longrightarrow t(\Sigma),$$

that makes the following diagram commutative:

$$\begin{tikzcd} & \Sigma \arrow[dl, "s"'] \arrow[dr, "t"] & \\ s(\Sigma) \arrow[rr, "\underline{\Sigma}"] & & t(\Sigma) \end{tikzcd}$$

Examples

Exercice 2.4.4. Any source-connected Lie groupoid is a bi-submersion over its basic singular foliation. Warning: it is not easy!

Exercice 2.4.5. Here are non-examples of bisubmersions. Let $\mathcal{F}$ be a singular foliation on M which is different from 0.

1. Show that $X := M \times M$ equipped with the projections onto the first and second components is not a bisubmersion for $\mathcal{F}$, unless $\mathcal{F} = \mathfrak{X}(M)$.

2. Show that $X := M$ equipped with the the identity maps as source and target is not a bisubmersion for $\mathcal{F}$, unless $\mathcal{F} = 0$.

3. Give an example of a manifold X, equipped with two surjective submersions $s, t \colon X \to M$, that do satisfy $s^*\mathcal{F} = t^*\mathcal{F}$, and is still not a bi-submersion for $\mathcal{F}$.

Exercice 2.4.6. Assume a unit map $\epsilon \colon M \to X$ exists, then:

1. The normal bundle N of $\epsilon(M)$ into X is canonically isomorphic to $\ker(Ts) \subset TX|_{\epsilon(M)}$ and to $\ker(Ts) \subset TX|_{\epsilon(M)}$

2. The map $Ts - Tt : TX|_{\epsilon(M)} \to TM$ goes to the quotient to a vector bundle morphism $\rho_X : N \to TM$ over the identity of M.

Theses exercices mean that $\mathcal{F}$ is hidden inside the data of (X, M, s, t, ϵ). That it to say, assume that you are given a bi-submersion for $\mathcal{F}$ with unit, but you forgot what $\mathcal{F}$ is: then you can reconstruct it. The next exercise shows that ϵ is even not needed.

Exercice 2.4.7. Consider a bisubmersion $M \xleftarrow{s} X \xrightarrow{t} M$ over $\mathcal{F}$.

1. There exists a local bisection through every point $x \in X$.

2. Let Σ be a local bisubmersion.The restriction to Σ of the singular foliation $\ker(Ts) + \ker(Tt)$

   (a) is a a singular foliation on Σ,

   (b) the restriction to Σ of the target t (resp. the source s) is a diffeomorphism of singular foliations from $(\Sigma, \mathcal{F}_\Sigma)$ to $(t(\Sigma), \mathcal{F}_\Sigma)$ (resp. $(\Sigma, \mathcal{F}_\Sigma)$ to $(s(\Sigma), \mathcal{F}_\Sigma)$).

2.4.2 The fundamental example

A bisubmersion for every finitely generated singular foliations

There is a very natural bisubmersion over any finitely generated singular foliation $\mathcal{F}$ on M. Let $X_1, \dots, X_r$ be generators of $\mathcal{F}$. For the sake of simplicity we assume them to be complete. Consider the following triple:

1. The manifold $\mathbb{R}^r \times M$

2. The map $s \colon \mathbb{R}^r \times M \to M$ given by the projection on the second factor.

$$\begin{array}{cccc} s\colon & \mathbb{R}^r \times M & \to & M \\ & ((t_1 \dots, t_r), m) & \to & m \end{array}$$

3. The map $t \colon \mathbb{R}^r \times M \to M$ given by:

$$\begin{array}{cccc} t\colon & \mathbb{R}^r \times M & \to & M \\ & ((t_1 \dots, t_r), m) & \mapsto & \phi_1^{\sum_{i=1}^r t_i X_i}(m) \end{array}$$

When the vector fields $X_1, \dots, X_r$ are not complete, the previous map t still makes sense, but the initial manifold has to be replaced by a neighborhood of $(0, \dots, 0) \times M$ (= the zero section of the trivial bundle $\mathbb{R}^r \times M \to M$).

The map s is always a submersion, and so is the map t, at least in a neighborhood of the zero section. The following proposition is non-trivial:

Proposition 2.4.8: A Crucial Example

Let $\mathcal{F}$ be a finitely generated singular foliation. There is a neighborhood $\mathcal{V}$ of the zero section in $\mathbb{R}^r \times M \to M$ on which s, t make $M \overset{s}{\leftarrow} \mathcal{V} \overset{t}{\to} M$ a bi-submersion for $\mathcal{F}$.

Remark 2.4.9. We could replace the target map by

$$((t_1 \dots, t_r), m) \mapsto \phi_{t_1}^{X_1} \circ \dots \circ \phi_{t_r}^{X_r}(m)$$

and the same statement would still hold. This is numerically interesting, for it means that if $\mathcal{F}$ is generated, as a module, by vector fields integrable by quadrature ($\simeq$ such that the flow can be described "by hand"), then there is a bisubmersion which can be found explicitly.

The same example made more abstract: using anchored bundles

This can extended by using an anchored bundle such that[8] $\rho(\Gamma(A)) = \mathcal{F}$.

Let us choose a connection[9]

$$\begin{array}{cccc} \nabla : & \mathfrak{X}(M) \times \Gamma(A) & \to & \Gamma(A) \\ & (X, a) & \mapsto & \nabla_X a \end{array}$$

on $A \to M$.

[8] We refer to Section xxx for results are the existence of those objects. In short: it the smooth case, it exists if and only if $\mathcal{F}$ is finitely generated, and it always exist locally.

[9] They always exist in the smooth setting, and always exist locally on every setting

Definition 2.4.10: Anchored paths

Let $A \xrightarrow{\pi} M$ be an anchored bundle. We say that a path $a\colon I \to A$ is anchored *if*

$$\frac{d\gamma(t)}{dt} = \rho_{\gamma(t)}(a(t))$$

where $\gamma = \pi \circ a : I \to M$ is the projection of $a(t)$ onto M. We say that it is ∇-parallel if, in addition of being anchored, it satisfies:

$$\nabla_{\dot\gamma(t)} a(t) = 0.$$

Here is a result which is purely a differential geometry result and that we will therefore not prove.

Lemma 2.4.11. *Every element $a \in A$ is the starting point of a parallel anchored anchored path $\Phi_t^{\nabla,\rho}(a)$.*

The path $t \mapsto \Phi_t^{\nabla,\rho}(a)$ may not be defined for all t, but there is a neighborhood $\mathcal{U}_A$ of the zero section where it is defined for $t = 1$.

We consider the triple made of

1. the neighborhood $\mathcal{U}_A$ of the zero section in A
2. the projection $A \xrightarrow{\pi} M$ that we now call s
3. the composition t of $a \mapsto \Phi_1^{\nabla,\rho}(a)$ with the projection $A \xrightarrow{\pi} M$

Proposition 2.4.12: The same as Prop 3.1.1 but more abstract

Let (A, ρ) be any anchored bundle such that $\mathcal{F} = \rho(\Gamma(A))$. There is a neighborhood $\mathcal{V}_A$ of the zero section in A on which $M \xleftarrow{s} \mathcal{U}_A \xrightarrow{t} M$, with s, t as above, is a bisubmersion of $\mathcal{F}$.
We call it a fundamental bisubmersion *associated the anchored bundle (A, ρ).*

Discussion on the notion of a bisubmersion

A word of caution: the word "bisubmersion" alone does not make sense : only the expression "bi-submersion over the singular foliation $\mathcal{F}$" makes sense.

We do not claim that the following notion is interesting by itself, but it is practical to introduce it for pedagogical purposes.

Definition 2.4.13. We call *twin-submersions* the data of

1. two manifolds X, M,
2. two surjective submersions $s, t\colon X \to M$.

A *unit map for twin-submersions* is a smooth map $\epsilon\colon M \to X$ which is a section of both s and t.

The next exercises answer to the natural question:

Question 2.4.14. *Given a twin-submersion (X, M, s, t), when is it a bi-submersion for some singular foliation $\mathcal{F}$?*

Exercice 2.4.15. Show that a twin-submersion can not be a bi-submersion for two different singular foliations on M.

Exercice 2.4.16. Does there exist twin-submersions which are NOT bisubmersions for any singular foliation $\mathcal{F}$?

Exercice 2.4.17. Let (X, M, s, t) be a twin-submersion with connected fibers. There exists a singular foliation $\mathcal{F}$ with respect to which (X, M, s, t) is a bi-submersion if and only if one of the equivalent conditions below hold.

1. $\ker(Ts) + \ker(Tt)$ is stable under Lie bracket.

2. $\ker(Tt)$ is generated by s-projectable vector fields and $\ker(Ts)$ is generated by t-projectable vector fields

3. If for every bisections Σ_1, Σ_2, the induced distributions by $\ker(Ts) + \ker(Tt)$ maps to the same singular foliations on $s(\Sigma_1) \cap s(\Sigma_2)$, $t(\Sigma_1) \cap s(\Sigma_2)$, $s(\Sigma_1) \cap t(\Sigma_2)$, and $t(\Sigma_1) \cap t(\Sigma_2)$.

Here is a first operation on bisubmersion.

Definition 2.4.18: Composing bisubmersions with symmetries

Let $M \leftarrow^s X \rightarrow^t M$ be a bisubmersion of $\mathcal{F}$, and $\phi\colon M \to M$ a symmetry of M. Then

$$M \overset{\phi\circ s}{\leftarrow} X \overset{t}{\rightarrow} M \text{ and } M \overset{s}{\leftarrow} X \overset{\phi^{-1}\circ t}{\rightarrow} M$$

are bisubmersions of $\mathcal{F}$ again. We call them, respectively, the right and left composition by the symmetry *ϕ.*

Exercice 2.4.19. For Σ a bisection of a bisubmersion $M \leftarrow^s X \rightarrow^t M$ of $\mathcal{F}$. Let $\mathcal{N}_\Sigma$ be the normal bundle of Σ in X, and A_Σ be its push forward through t to a vector bundle on the open subset $t(\Sigma) \subset M$.

1. Show that for any $\sigma \in \Sigma$, the following diagram is commutative:

$$\begin{tikzcd} & T_\sigma X \arrow[dl, "Ts"'] \arrow[dr, "Tt"] & \\ T_{s(\sigma)}M \arrow[rr, "T\Sigma"] & & T_{t(\sigma)}M \end{tikzcd}$$

2. Deduce that for any $\sigma \in \Sigma$, the tangent space $T_\sigma\Sigma$ is in the kernel of

$$Ts - T\underline{\Sigma} \circ Tt : T_\sigma X \to T_{t(\sigma)}M$$

3. Show the induced vector bundle morphism

$$\begin{tikzcd} A_\Sigma := \frac{TM|_\Sigma}{T\Sigma} \arrow[r, "\rho"] \arrow[d] & TM \arrow[d] \\ \Sigma \arrow[r, "t"] & M \end{tikzcd},$$

is an anchored bundle such that $\rho(\Gamma(A_\Sigma)) = \mathcal{F}$.

Show that there is a neighborhood of any bisubmersion Σ in X isomorphic, as a bisubmersion, to the left composition by $\underline{\Sigma}$ of a coordinate neighborhood (see Proposition 2.4.12) for (A_Σ, ρ).

2.4.3 Bisubmersion, left and right invariant vector fields, and almost Lie algebroids

Let $\mathcal{F}$ be a singular foliation on M. So far, we have seen two classes of objects "over $\mathcal{F}$".

1. bisubmersions $M \leftarrow^s X \rightarrow^t M$ of $\mathcal{F}$.

2. anchored bundles (A, ρ) of $\mathcal{F}$

Let us assume that we are given both. What is the relation between them? If X is a Lie groupoid with algebroid A, then the kernels of s and t at $x \in X$ are canonically isomorphic to $A_{t(x)}$ and $A_{s(x)}$ respectively: the isomorphism being defined by left and right actions respectively.

There is a very similar phenomena for bi-submersions. There are two maps

ny anchored bun-
nersion, or are we
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$$\begin{array}{ccc} \Gamma(A) & \to & \mathfrak{X}(X) \\ a & \mapsto & \overrightarrow{a} \\ a & \mapsto & \overleftarrow{a} \end{array}$$

satisfying the following conditions:

1. the vector field $\overrightarrow{a}$ (resp. $\overleftarrow{a}$) is s-compatible (resp. t-compatible) with $\rho(a) \in \mathfrak{X}(M)$
2. the vector field $\overrightarrow{a}$ (resp. $\overleftarrow{a}$) is tangent to the fibers of t (resp. s).

It could be stated as being the existence of vector bundle morphisms:

$$s^! A \xrightarrow{L} TX \xleftarrow{R} t^! A$$

such that L takes values in $ker(Tt)$, R takes values in $ker(Ts)$ and such that the following diagrams commute

$$\begin{tikzcd} s^! A \arrow[r, "L"] \arrow[d, "\rho"] & TX \arrow[dl, "Ts"] \arrow[dr, "Tt"] & t^! A \arrow[l, "R"'] \arrow[d, "\rho"'] \\ s^! TM & & t^! TM \end{tikzcd}$$

It is interesting to notice that for every choice of an almost Lie algebroid bracket on (A, ρ), the vector fields that measures the default of the left and right actions to preserve the brackets, i.e. the vector fields

$$\overleftarrow{[a,b]_A} - [\overleftarrow{a}, \overleftarrow{b}] \text{ and } \overrightarrow{[a,b]_A} - [\overrightarrow{a}, \overrightarrow{b}] \text{ with } a, b \in \Gamma(A),$$

are valued in both the s-fiber and the t-fiber. They are therefore valued on $\mathrm{Ker}(Ts) \cap \mathrm{Ker}(Tt)$, which is an involutive family, which is a singular folaition provided it is finitely generated, and is interesting by itself, see Ruben Louis's [Lou22].

2.4.4 Products and inverse of bisubmersions

As we said, bisubmersions for a given singular foliation are like differentiable stacks, i.e. like Lie groupoids. But this analogy may seem strange: we defined an unit map, but there is still no inverse and no product. The following statement presents an analogy of those:

Definition 2.4.20: Product and Inverse

Let $\mathcal{F}$ be a singular foliation on a manifold M.

1. *The* inverse *of a bisubmersion $M \overset{s}{\leftarrow} X \overset{t}{\rightarrow} M$ for $\mathcal{F}$ is simply the bisubmersion $M \overset{t}{\leftarrow} X \overset{s}{\rightarrow} M$.*
2. *The* composition *of two bisubmersions $M \overset{s}{\leftarrow} X \overset{t}{\rightarrow} M$ and $M \overset{s'}{\leftarrow} X' \overset{t'}{\rightarrow} M$ for $\mathcal{F}$ is the fibered product*
$$X \times_{s,M,t'} X' = \{(x, x') \in X \times X' \ s.t.\ t(x) = s'(x')\}$$
equipped with the source $(x, x') \mapsto s(x)$ and target $(x, x') \mapsto t'(x')$.

Exercice 2.4.21. We leave it as an exercise to check that the product of bisubmersions for $\mathcal{F}$, defined as above, is a bisubmersion for $\mathcal{F}$ again.

A *bisection* of a bisubmersion of $\mathcal{F}$ is a submanifold $\Sigma \subset X$ on which both s and t restrict to diffeomorphisms onto their image. Any bisection Σ induces a diffeomorphism:

$$\underline{\Sigma} : s(\Sigma) \to t(\Sigma)$$

mapping $m \in s(\Sigma)$ to the the target of the only point in Σ that s maps to m. Said otherwise, $\underline{\Sigma}$ is the diffeomorphism making this diagram commutative:

$$\begin{tikzcd} & s(\Sigma) \arrow[dd, "\underline{\Sigma}"] \\ \Sigma \arrow[ur, "s"] \arrow[dr, "t"'] & \\ & t(\Sigma) \end{tikzcd}$$

We already stated the next result as an exercise, now we prove it:

Lemma 2.4.22. *Through every $x \in X$, there exists a least one bisubmersion (in fact, infinitely many)*

Proof. There exists vector subspaces $S \subset T_xX$ that intersect neither $\mathbb{K}\diagdown(T_xs)$ nor $\mathbb{K}\diagdown(T_xt)$, so that both T_xs and T_xX are invertible onto their images. Any submanifold through x admitting such a sub-space as its tangent space is a bisubmersion in a neighborhood of x. $\square$

Proposition 2.4.23: Bisubmersions induce Symmetries

Consider a bisubmersion $M \xleftarrow{s} X \xrightarrow{t} M$ of a singular foliation $\mathcal{F}$. For any bisection Σ, the induced diffomorphism

$$\underline{\Sigma}\colon s(\Sigma) \longrightarrow t(\Sigma)$$

in an isomorphism of the singular foliation $\mathcal{F}$.

For any two points in a foliated manifold (M, π) Let us denote by $Iso_{\mathcal{F}}(m, m')$ the set of isomorphisms of $\mathcal{F}$, defined from a neighborhood of m to a neighborhood of m'. And let us define by $Germs_{\mathcal{F}}(m, m')$ the set of germs of isomorphisms of $\mathcal{F}$, defined from a neighborhood of m to a neighborhood of m'.

As a consequence, to any bisubmersion $M \xleftarrow{s} X \xrightarrow{t} M$ of a singular foliation $\mathcal{F}$, and any $x \in X$ one can associate a subset of $Iso_{\mathcal{F}}(s(x), t(x))$, by considering, all diffeomorphisms arising from a bisubmersion through x:

$$\begin{array}{ccl} X & \to & Iso_{\mathcal{F}}^{X}(s(x), t(x)) \\ x & \to & \{\underline{\Sigma},\ \Sigma \text{ a bisubmersion through } x\ \} \end{array}$$

Taking the germs of of the previously defined local isomorphisms for all one defines a subset $Germ_{\mathcal{F}}(X)$ of $\cup_{m,m' \in M} Germ_{\mathcal{F}}(m, m')$: that we call the *symmetry-germ of the bi-submersion $M \xleftarrow{s} X \xrightarrow{t} M$*.

2.4.5 Equivalence of bisubmersions (and their compositions)

There is a "Morita equivalence-like" equivalence relation on the set of all bisubmersions for $\mathcal{F}$.

Its definition is very natural.

Definition 2.4.24: Equivalence of bisubmersions

Consider two bisubmersions $M \xleftarrow{s} X \xrightarrow{t} M$ and $M \xleftarrow{s'} X' \xrightarrow{t'} M$ for $\mathcal{F}$.

1. *A* morphism *from the first one to the second one is a map $X \to X'$ making the following diagram commutative:*

$$\begin{tikzcd} M \arrow[d, equal] & X \arrow[l, "s"'] \arrow[r, "t"] \arrow[d] & M \arrow[d, equal] \\ M & X' \arrow[l, "s'"] \arrow[r, "t'"'] & M \end{tikzcd}$$

2. *An* equivalence *is a third bisubmersion $M \xleftarrow{s''} X'' \xrightarrow{t''} M$ of $\mathcal{F}$ equipped with two surjective*

submersions $X'' \to X$ and $X'' \to X'$ making the following diagram commutative:

$$\begin{tikzcd}
& X \arrow[dl, "s"] \arrow[dr, "t"] & \\
M & X'' \arrow[l, "s''"'] \arrow[r, "t''"] \arrow[u, twoheadrightarrow] \arrow[d, twoheadrightarrow] & M \\
& X' \arrow[ul, "s'"] \arrow[ur, "t'"'] &
\end{tikzcd}$$

Exercice 2.4.25. Show that the equivalence defined above is an equivalence relation on bisubmersions for $\mathcal{F}$.

Here is an important theorem that entirely explains the notion of equivalence of bisubmersions:

Theorem 2.4.26: Equivalence of bisubmersions

Consider two bisubmersions $M \overset{s}{\leftarrow} X \overset{t}{\rightarrow} M$ and $M \overset{s'}{\leftarrow} X' \overset{t'}{\rightarrow} M$ for $\mathcal{F}$. The following statement are equivalent:

(i) Both bisubmersions are equivalent.

(ii) The following two conditions hold:

(a) any $x \in X$ admits a neighborhood $\mathcal{U}$ on which a morphism $\mathcal{U} \to X'$ exists,

(b) and any $x' \in X'$ admits a neighborhood $\mathcal{U}'$ on which a morphism $\mathcal{U}' \to X$ exists,

(iii) Both bisubmersions induce the same symmetry-germs, i.e.

$$Germs_{\mathcal{F}}(X) = Germs_{\mathcal{F}}(X')$$

This theorem is proved through a proposition, which is interesting by itself:

Proposition 2.4.27. *Consider two bisubmersions $M \overset{s}{\leftarrow} X \overset{t}{\rightarrow} M$ and $M \overset{s'}{\leftarrow} X' \overset{t'}{\rightarrow} M$ for $\mathcal{F}$. Fro any two points $x \in X$ and $x' \in X'$, the following statements*[10] *are equivalent:*

(i) A neighborhood $\mathcal{U}$ of x is X is equipped with a morphism of bisubmersion $\phi \colon \mathcal{U} \to X'$ mapping x to x'.

(ii) A neighborhood $\mathcal{U}'$ of x' is X' is equipped with a morphism of bisubmersion $\phi' \colon \mathcal{U}' \to X$ mapping x' to x.

(iii) There exist bisections through x and x' that induce the same germ of isomorphisms of $\mathcal{F}$.

(iv) $Iso_{\mathcal{F}}^{X}(x)$ and $Iso_{\mathcal{F}}^{X'}(x')$ coincide at the level of germs.

2.5 Holonomy groupoid

Let us give the most important steps in the construction of the holonomy groupoid made by its inventors Androulidakis and Skandalis in [AS09]. We shall use a presentation which is quite different from the original one in its form, but quite similar in its structure.

2.5.1 Atlas

Let $(M, \mathcal{F})$ be a smooth singular foliation.

[10] Notice that all these statements imply $s(x) = s(x')$ and $t(x) = t(x')$.

Definition 2.5.1: Groupoid-like bisubmersion

We say that a bisubmersion $M \overset{s}{\leftarrow} X \overset{t}{\rightarrow} M$ *of* $\mathcal{F}$ *is an* atlas of $\mathcal{F}$ *when*

1. X *is equivalent to its inverse,*
2. *the composition* $X \times_M X$ *is equivalent to* X*.*
3. X *admits local units*[a]

[a] I.e. every point n m admits a neighborhood on which a unit exists.

In fact, the third assumption in the above definition is a consequence of the two first ones.

Example 2.5.2. We call *fundamental atlas* the atlas obtained by taking:

1. An anchored bundle (A, ρ) over $\mathcal{F}$ (we assume for the sake of simplicity that $\mathcal{F}$ is globally finitely generated, so that (A, ρ) exists on the whole manifold M).
2. a neighborhood $\mathcal{A}$ of the zero section on which is is a bi-submersion over $\mathcal{F}$ (see Proposition **??**)
3. Then by considering the disjoint union over n of all direct products
$$\mathcal{A}^\star \times_M \cdots \times_M \mathcal{A}^\star \text{ (n times)}$$
where * means that we consider $\mathcal{A}$ or its inverse $\mathcal{A}^{-1}$. It is an atlas by construction.

It deserves to be noticed that every connected component of this atlas is finitely dimensional, although it is not globally finitely dimensional.

2.5.2 Holonomy groupoid

The next proposition is easy to prove. But we invite the reader to remember that "groupoid" does not mean "Lie groupoid".

Proposition 2.5.3: From atlases to groupoids

Let $M \overset{s}{\leftarrow} X \overset{t}{\rightarrow} M$ *of* $\mathcal{F}$ *be an atlas of* $\mathcal{F}$*. Consider the equivalence relation on* X *given by* $x \sim x'$ *if and only if* x *and* x' *have neighborhoods which are equivalent as bisubmersions of* $\mathcal{F}$*.*
The equivalence classes of this relation form a groupoid over M*.*

Proof. The inverse of an equivalence represented by $x \in X$ is represented by any point in X^{-1} which equivalent to x. The same applies to product. Given two compatible x_1, x_2, there exists $x_3 \in X$ such that x_3 and (x_1, x_2) are equivalent. This construction goes to the quotient w.r.t. the equivalence relation on X and defines the groupoid product. □

Remark 2.5.4. It is easily checked that equivalent atlases induce canonically isomorphic groupoids.

Definition 2.5.5: Holonomy groupoid

We call holonomy groupoid of $\mathcal{F}$ *the groupoid associated to a fundamental atlas.*

Proposition 2.5.6: Holonomy groupoid is well-defined

The holonomy groupoid does not depend on the choice of an anchored bundle.

2.5.3 About smoothness of the holonomy groupoid: two theorems by Claire Debord

Recall that a *Lie groupoid* is a groupoid $\Gamma \rightrightarrows M$ such that Γ and M are manifolds, the source and targets are smooth surjective submersions, and all structural maps (unit, product, inverse) are smooth[11].

The holonomy groupoid is certainly not a smooth groupoid in general. It is a topological groupoid, and the topology may be quite horrible -very far away from a manifold topology.

However, the following theorem was proven by Claire Debord [Deb13b].

Theorem 2.5.7: Along a leaf

[Deb13b] The orbits of the holonomy groupoid $Hol(\mathcal{F})$ of a singular foliation $\mathcal{F}$ are the leaves of $\mathcal{F}$. Moreover, its restriction to any leaf L:

1. *is a Lie groupoid (separated),*
2. *whose Lie algebroid is the holonomy Lie algebroid of the L.*

Proof. The proof relies on a theorem that bounds below the periods of a periodic orbit of a vector field in a neighborhood of a point. This lower bound forbids a bisbmersion to have "too many quotients", it makes the quotient discrete-like. We refer to [Deb13b] $\square$

Smoothness of the holonomy groupoid happens in a second situation. This theorem is also due to Claire Debord [Deb01].

Theorem 2.5.8: Projective case

The holonomy groupoid of a Debord foliation is a Lie groupoid (separated).

2.6 Geometric resolutions of a singular foliation

Introduction

We saw in Section **??** that for every finitely generated singular foliation $\mathcal{F}$ (and more generally for any finitely generated module over the algebra of functions), there exists a anchored vector bundle

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[d] & TM \arrow[d] \\ M \arrow[r, equal] & M \end{tikzcd}$$

such that $\rho(\Gamma(A)) = \mathcal{F}$. Now, consider the kernel $\ker(\rho)$ of

$$\rho\colon \Gamma(A) \longrightarrow \mathfrak{X}(M).$$

Obviously $\ker(\rho)$ is again a $\mathcal{C}^\infty(M)$-module. If it is finitely generated, then there exists a second vector bundle $B \to M$ and a vector bundle $\mathrm{d}\colon B \longrightarrow A$ such that

$$\mathrm{d}\left(\Gamma(B)\right) = \ker(\rho)$$

In particular, we have $\rho \circ \mathrm{d} = 0$, and

$$\begin{tikzcd} B \arrow[r, "\mathrm{d}^{(1)}"] \arrow[d] & A \arrow[r, "\rho"] \arrow[d] & TM \arrow[d] \\ M \arrow[r, equal] & M \arrow[r, equal] & M \end{tikzcd}$$

[11]or real analytic, or holomorphic, depending on the setting

Again, the kernel of

$$\mathrm{d}\colon (\Gamma(B)) \longrightarrow (\Gamma(A))$$

is a complex of vector bundles which is exact at the level of section, i.e that the sequence

$$\Gamma(B) \xrightarrow{\mathrm{d}} \Gamma(A) \xrightarrow{\rho} \mathcal{F}$$

is exact. Again, when the kernel of

$$\mathrm{d}^{(1)}\colon \Gamma(B) \longrightarrow \Gamma(A)$$

is finitely generated, there exists a vector bundle $C \to M$ and a vector bundle morphism $\mathrm{d}\colon C \to B$ such that $\mathrm{d}^{(2)}(\Gamma(C)) = Ker(\mathrm{d}^{(1)})$. By construction,

$$\begin{tikzcd}
C \arrow[r, "\mathrm{d}^{(2)}"] \arrow[d] & B \arrow[r, "\mathrm{d}^{(1)}"] \arrow[d] & A \arrow[r, "\rho"] \arrow[d] & TM \arrow[d] \\
M \arrow[r, equal] & M \arrow[r, equal] & M \arrow[r, equal] & M
\end{tikzcd}$$

Again, the kernel of

$$\mathrm{d}^{(1)}\colon \Gamma(B) \longrightarrow (\Gamma(A))$$

is a complex of vector bundles and

$$\Gamma(C) \xrightarrow{\mathrm{d}^{(2)}} \Gamma(B) \xrightarrow{\mathrm{d}^{(1)}} \Gamma(A) \xrightarrow{\rho} \mathcal{F}$$

is an exact complex.

Question 2.6.1: Resolutions of singular foliations

When can the construction of the complex of vector bundles described above be continued "up to infinity" (i.e.: can one be certain that the kernels are finitely generated?).
Does it stop at some point? (i.e. can we manage that the kernel of d *is trivial for k large enough?*
Assume it can be constructed, what kind of geometric information is encoded in that complex?

Let us start with by precise definition and a precise vocabulary.

Definition 2.6.2: Anchored complex of vector bundles

An anchored complex of vector bundles *consists of a triple* $(E_\bullet, \mathrm{d}^\bullet, \rho)$*, where*

1. $E_\bullet = (E_{-i})_{i\geq 1}$ *is a family of vector bundles over* M*, indexed by negative integers.*
2. $\mathrm{d}^{(i+1)} \in \mathrm{Hom}(E_{-i-1}, E_{-i})$ *is a vector bundle morphism over the identity of* M *called the* differential map
3. $\rho\colon E_{-1} \longrightarrow TM$ *is a vector bundle morphism over the identity of* M *called the* anchor map.

i.e. $(E_\bullet, \mathrm{d}^\bullet, \rho) :=$

$$\begin{tikzcd}
\arrow[r] & E_{-i-1} \arrow[r, "\mathrm{d}^{(i+1)}"] \arrow[d] & E_{-i} \arrow[r, "\mathrm{d}^{(i)}"] \arrow[d] & E_{i-1} \arrow[r] \arrow[d] & \cdots\cdots \arrow[r, "\mathrm{d}^{(2)}"] & E_{-1} \arrow[r, "\rho"] \arrow[d] & TM \arrow[d] \\
\arrow[r, equal] & M \arrow[r, equal] & M \arrow[r, equal] & M \arrow[r, equal] & \cdots\cdots \arrow[r, equal] & M \arrow[r, equal] & M
\end{tikzcd}$$

which form a complex, i.e. such that

$$\mathrm{d}^{(i)} \circ \mathrm{d}^{(i+1)} = 0 \text{ and } \rho \circ \mathrm{d}^{(2)} = 0$$

Let us fix some vocabulary about such complexes:

1. The integer $-i$ is called the *degree* of the vector bundle E_{-i}. The choice of negative numbers may seem surprising: it will be justified when introducing Lie ∞-algebroid structures.
2. The anchored bundle $(E_\bullet, \mathrm{d}^\bullet, \rho)$ is said to be of *(finite) length* $n \in \mathbb{N}$ if $E_{-i} = 0$ except to finitely many indices.
3. We shall speak of *anchored complex of trivial bundles* when all the vector bundles $(E_{-i})_{i\geq 1}$ are trivial vector bundles. We do not assume TM to be a trivial bundle.

There are two main cohomologies that one can associate to an anchored bundle.

1. **Cohomology at the level of sections**. The complex of vector bundles (xxx) induces a complex of sheaves of modules over functions. More explicitly, for every open subset $\mathcal{U} \subset M$, there is a complex:

$$\cdots \longrightarrow \Gamma_{\mathcal{U}}(E_{-i-1}) \stackrel{\mathrm{d}^{(i+1)}}{\longrightarrow} \Gamma_{\mathcal{U}}(E_{-i}) \stackrel{\mathrm{d}^{(i)}}{\longrightarrow} \Gamma_{\mathcal{U}}(E_{-i+1}) \longrightarrow \cdots \longrightarrow \Gamma_{\mathcal{U}}(E_{-1}) \stackrel{\rho}{\longrightarrow} \mathcal{F}_{\mathcal{U}} \subset \mathfrak{X}(\mathcal{U}).$$

In particular, $\mathrm{Im}(\mathrm{d}^{(i+1)}) \subseteq \ker \mathrm{d}^{(i)}_{|_m}$ for every $i \in \mathbb{N}$, so that the quotient spaces:

$$H^{-i}(E_\bullet, \mathcal{U}) = \begin{cases} \frac{\ker \rho}{\mathrm{Im}(\mathrm{d}^{(2)})} & \text{for } i = 1 \\ \frac{\ker \mathrm{d}^{(i)}}{\mathrm{Im}(\mathrm{d}^{(i+1)})} & \text{if } i \geq 2 \end{cases}$$

is a module over functions on $\mathcal{U}$ that we call *i-th cohomology of $(E_\bullet, \mathrm{d}^\bullet, \rho)$ at the level of sections.*

2. **Cohomology at an arbitrary point** $m \in M$. The complex of vector bundles (xxx), at an arbitrary point $m \in M$, restricts to a complex of vector spaces

$$\cdots \longrightarrow E_{-i-1|_m} \stackrel{\mathrm{d}^{(i+1)}_{|_m}}{\longrightarrow} E_{-i|_m} \stackrel{\mathrm{d}^{(i)}_{|_m}}{\longrightarrow} E_{-i+1|_m} \longrightarrow \cdots \longrightarrow E_{-1} \stackrel{\rho_m}{\longrightarrow} T_m M.$$

In particular, $\mathrm{Im}(\mathrm{d}^{(i+1)}) \subseteq \ker \mathrm{d}^{(i)}_{|_m}$ for every $i \in \mathbb{N}$, ans we call the quotient vector spaces:

$$H^{-i}(E_\bullet, m) = \begin{cases} \frac{\ker \rho_m}{\mathrm{Im}(\mathrm{d}^{(2)}_{|_m})} & \text{for } i = 1 \\ \frac{\ker \mathrm{d}^{(i)}_{|_m}}{\mathrm{Im}(\mathrm{d}^{(i+1)}_{|_m})} & \text{if } i \geq 2 \end{cases}$$

the *i-th cohomology of $(E_\bullet, \mathrm{d}^\bullet, \rho)$ at the point m.*

It is important to notice that $H^{-i}(E_\bullet, m)$ may be non-zero at a point m even if $H^{-i}(E_\bullet, \mathcal{U})$ is zero in every neighborhood $\mathcal{U}$ of m. The converse, however, is not possible.

Proposition 2.6.3. *Let $i \in \mathbb{N}$. Every $m \in M$ such that $H^{-i}(E_\bullet, m) = 0$ has an open neighborhood $\mathcal{U}$ for which $H^{-i}(E_\bullet, \mathcal{U}) = 0$.*

Definition 2.6.4. Let (E, d) and (E', d) be complexes of vector bundles.

1. A *chain map* or *complex of vector bundle morphisms* between the complexes of vector bundles (E, d) and (E', d) is collection of vector bundle morphisms of degree zero, $\varphi_\bullet \colon E_{-\bullet} \longrightarrow E'_{-\bullet}$, such that the following diagram commutes

$$\begin{tikzcd} \cdots \arrow[r] & E_{-i} \arrow[r, "\mathrm{d}^{(i)}"] \arrow[d, "\varphi_{-i}"'] & E_{-i+1} \arrow[r] \arrow[d, "\varphi_{-i+1}"] & \cdots \\ \cdots \arrow[r] & E'_{-i} \arrow[r, "\mathrm{d}'^{(i)}"] & E'_{-i+1} \arrow[r] & \cdots \end{tikzcd} \tag{2.9}$$

i.e., $\mathrm{d}'^{(i)} \circ \varphi_{-i} = \varphi_{-i+1} \circ \mathrm{d}^{(i)}$ for every $i \in \mathbb{N}$.

2. A *homotopy* between two complexes of vector bundle morphisms $\varphi_\bullet, \psi_\bullet \colon E_{-\bullet} \longrightarrow E'_{-\bullet}$ is the datum $\{h_i \colon E_{-i} \longrightarrow E'_{-i-1}\}_{i\geq 1}$ of vector bundles morphisms that satisfies $\psi_1 - \varphi_1 = \mathrm{d}'^{(2)} \circ h_1$ and for each $i \geq 2$, $\psi_i - \varphi_i = \mathrm{d}'^{(i+1)} \circ h_i + h_{i-1} \circ \mathrm{d}^{(i)}$

$$
\begin{tikzcd}
\cdots \arrow[r] & E_{-i-1} \arrow[r, "\mathrm{d}^{(i-1)}"] \arrow[d, "\psi_{-i-1}-\varphi_{-i-1}"'] & E_{-i} \arrow[r, "\mathrm{d}^{(i)}"] \arrow[d, "\psi_i-\varphi_i"'] \arrow[dl, dotted, "h_i"'] & E_{-i+1} \arrow[r] \arrow[d, "\psi_{-i+1}-\varphi_{-i+1}"] \arrow[dl, dotted, "h_{-i+1}"'] & \cdots \\
\cdots \arrow[r] & E'_{-i-1} \arrow[r, "\mathrm{d}'^{(i-1)}"] & E'_{-i} \arrow[r, "\mathrm{d}'^{(i)}"] & E'_{-i+1} \arrow[r] & \cdots
\end{tikzcd}
\tag{2.10}
$$

(a) When there is a homotopy between two complexes of vector bundle morphisms $\varphi_\bullet, \psi_\bullet \colon E_{-\bullet} \longrightarrow E'_{-\bullet}$ we write $\varphi \sim \psi$.

(b) Two complexes of vector bundles (E, d) and (E', d') are said to be *homotopy equivalent*, if there exist chain maps $\varphi_\bullet \colon E_{-\bullet} \longrightarrow E'_{-\bullet}$ and $\psi_\bullet \colon E'_{-\bullet} \longrightarrow E_{-\bullet}$ such that $\varphi \circ \psi \sim \mathrm{id}_{E'_\bullet}$ and $\psi \circ \varphi \sim \mathrm{id}_{E_\bullet}$.

Check that $\sim$ is an equivalence relation on the class of complexes of vector bundle morphisms.

Lemma 2.6.5. *Let (E, d) and (E', d) two homotopy equivalent complexes of vector bundles of finite length n and n' respectively. The alternating sum of the ranks of the vector bundles $(E_{-i})_{i\in\mathbb{N}}$ and $(E'_{-i})_{i\in\mathbb{N}}$ respectively, are equal, i.e.*

$$\sum_{i=0}^{n}(-1)^i \mathrm{rk}(E_{-i}) = \sum_{i=0}^{n'}(-1)^i \mathrm{rk}(E'_{-i}).$$

Here rk *stands for the rank of a vector bundle.*

Proof. Note first that the restriction of both complexes to a point $m \in M$ give two finite length complexes of vector spaces of finite dimension. The result is an immediate consequence of the fact that in every degree the cohomology group of two equivalent complexes of vector spaces are isomorphic. It follows by taking the alternating sum of their dimensions and using the Rank–nullity theorem. □

Definition 2.6.6: Geometric resolution of a singular foliation

Let $\mathcal{F} \subseteq \mathfrak{X}(M)$ be a singular foliation on a manifold M. A complex of vector bundles $(E_\bullet, \mathrm{d}^\bullet, \rho) :=$

$$
\begin{tikzcd}
{} \arrow[r] & E_{-i-1} \arrow[r, "\mathrm{d}^{(i+1)}"] \arrow[d] & E_{-i} \arrow[r, "\mathrm{d}^{(i)}"] \arrow[d] & E_{i-1} \arrow[r] \arrow[d] & \cdots\cdots \arrow[r, "\mathrm{d}^{(2)}"] & E_{-1} \arrow[r, "\rho"] \arrow[d] & TM \arrow[d] \\
{} \arrow[r, equal] & M \arrow[r, equal] & M \arrow[r, equal] & M \arrow[r, equal] & \cdots\cdots \arrow[r, equal] & M \arrow[r, equal] & M
\end{tikzcd}
$$

is said

1. *to* terminate in $\mathcal{F}$ *if* $\rho\Gamma(E_{-1}) \subset \mathcal{F}$
2. *to be* over $\mathcal{F}$ *if* $\rho\Gamma(E_{-1}) = \mathcal{F}$
3. *to be a* geometric resolution of $\mathcal{F}$ *if the following complex of sheaves is exact:*

$$\cdots \longrightarrow \Gamma(E_{-i-1}) \overset{\mathrm{d}^{(i+1)}}{\longrightarrow} \Gamma(E_{-i}) \overset{\mathrm{d}^{(i)}}{\longrightarrow} \Gamma(E_{-i+1}) \longrightarrow \cdots \longrightarrow \Gamma(E_{-1}) \overset{\rho}{\longrightarrow} \mathcal{F}.$$

A geometric resolution $(E_\bullet, \mathrm{d}^\bullet, \rho)$ is said to be minimal *at a point $m \in M$ if for each $i \geq 2$ the linear map $\mathrm{d}^{(i)}_{|_m} \colon E_{-i} \longrightarrow E_{-i+1|_m}$ vanishes.*

Let us recall what we mean precisely by the sheaf condition above, and explain how this condition simplifies in the smooth case. "Exact as sheaves" means that for any $i \in \mathbb{N}$ and any $m \in M$, there is a neighborhood $\mathcal{V}$ of m such that for any $\mathcal{U}$ in $\mathcal{V}$ the short complex:

$$\cdots \longrightarrow \Gamma_{\mathcal{U}}(E_{-i-1}) \overset{\mathrm{d}^{(i+1)}}{\longrightarrow} \Gamma_{\mathcal{U}}(E_{-i}) \overset{\mathrm{d}^{(i)}}{\longrightarrow} \Gamma_{\mathcal{U}}(E_{-i+1}) \longrightarrow \cdots .$$

is exact.

Remark 2.6.7. When a geometric resolution $(E_\bullet, \mathrm{d}^\bullet, \rho)$ is minimal at a point $m \in M$ then one has, $H^{-i}(E_\bullet, m) = E_{-i_{|m}}$ for all $i \geq 2$.

In the smooth setting, it is equivalent to require that condition or to requite that it is exact at the level of global sections.

In the smooth setting therefore, the notion of geometric resolution is much easier.

Definition 2.6.8: Geometric resolution of a singular foliation (smooth case)

Let $\mathcal{F} \subseteq \mathfrak{X}_c(M)$ be a singular foliation on a smooth manifold M. A geometric resolution *of the singular foliation $\mathcal{F}$ is a complex of vector bundles $(E_\bullet, \mathrm{d}^\bullet, \rho) :=$*

$$\begin{tikzcd}
{} \arrow[r] & E_{-i-1} \arrow[r, "\mathrm{d}^{(i+1)}"] \arrow[d] & E_{-i} \arrow[r, "\mathrm{d}^{(i)}"] \arrow[d] & E_{i-1} \arrow[r] \arrow[d] & \cdots \arrow[r, "\mathrm{d}^{(2)}"] & E_{-1} \arrow[r, "\rho"] \arrow[d] & TM \arrow[d] \\
{} \arrow[r, equal] & M \arrow[r, equal] & M \arrow[r, equal] & M \arrow[r, equal] & \cdots \arrow[r, equal] & M \arrow[r, equal] & M
\end{tikzcd}$$

such that the following complex is exact:

$$\cdots \longrightarrow \Gamma_c(E_{-i-1}) \overset{\mathrm{d}^{(i+1)}}{\longrightarrow} \Gamma_c(E_{-i}) \overset{\mathrm{d}^{(i)}}{\longrightarrow} \Gamma_c(E_{-i+1}) \longrightarrow \cdots \longrightarrow \Gamma_c(E_{-1}) \overset{\rho}{\longrightarrow} \mathcal{F}.$$

Remark 2.6.9. A smooth singular foliation $\mathcal{F}$ is projective if and only if there exists a geometric resolution of length 1.

Remark 2.6.10. The same simplification occurs for singular foliations on affine varieties, if the geometric resolution is by trivial vector bundles.

Theorem 2.6.11

Let $\mathcal{F} \subseteq \mathfrak{X}(M)$ be a singular foliation on a smooth manifold M that admits a geometric resolution $(E_\bullet, \mathrm{d}^\bullet, \rho)$.

1. *For any anchored complex of vector bundles $(E'_\bullet, (\mathrm{d}^\bullet)', \rho')$ that terminates inside $\mathcal{F}$, there exists a chain map of anchored vector bundles*

$$(E'_\bullet, \mathrm{d}^\bullet, \rho') \longrightarrow (E_\bullet, \mathrm{d}^\bullet, \rho)$$

   and any two such chain maps are homotopy equivalent.

2. *In particular, two geometric resolutions of the same singular foliations are homotopy equivalent.*

The same results hold in the complex and real analytic setting, but in a neighborhood of a point and for anchored bundles of finite length. The same result also holds for foliations over algebraic varieties, and trivial anchored bundles.

Proof. These are reinterpretations of classical results of algebraic topology, see [LGLS20]. □

The fact that two geometric resolutions of $\mathcal{F}$, when they exist, are homotopy equivalent, has many consequences around the topic "whatever is canonically invariant under homotopy equivalence is canonically attached to the singular foliation - provided it admits geometric resolutions." It is the case, for instance, of the alternating sums of the ranks. We can give a precise meaning to the later:

Corollary 2.6.12. *[LGLS20] Let $\mathcal{F} \subseteq \mathfrak{X}(M)$ be a singular foliation on a smooth manifold M that admits a geometric resolution $(E_\bullet, \mathrm{d}^\bullet, \rho)$ of finite length. Then:*

1. *the regular leaves all have the same dimension ℓ,*
2. *the alternating sum of the ranks of $E_\bullet$ is equal to the dimension of the regular leaves, i.e.*

$$\ell = \sum_{i \geq 1} (-1)^{i+1} \mathrm{rk}(E_{-i}).$$

If two geometric resolutions are homotopy equivalent, their restrictions to a point $m \in M$ are also homotopy equivalent. In consequence, the complexes have the same cohomologies. This proves the first part of the following corollary:

Corollary 2.6.13. *Let $\mathcal{F} \subseteq \mathfrak{X}(M)$ be a singular foliation on a smooth manifold M that admits a geometric resolution. Then for every $m \in M$, the cohomologies $H^{-i}(E_\bullet, m) \simeq H^{-i}(E'_\bullet, m)$ are canonically isomorphic. In particular, the dimensions $d_1, \ldots, d_i, \ldots$ of these spaces are canonically attached to $\mathcal{F}$. Also, the following items are equivalent:*

1. *m is a regular point,*
2. *$H^1(\mathcal{F}, m) = 0$*
3. *$H^i(\mathcal{F}, n) = 0$ for every $i \geq 1$ and every n in a neighborhood of m.*

Remark 2.6.14. The integers $d_1, \ldots, d_i, \ldots$ were constructed without making any use of the Lie bracket of vector fields, so that they are, as a matter of fact, attached to $\mathcal{F}$ seen as a module over functions, and not to $\mathcal{F}$ seen as a singular foliation. We suggest to interpret them as follows:

1. d_1 is the minimal number of generators of $\mathcal{F}$ near m minus the dimension of the leaf through m.
2. d_2 is the minimal number of generators of relations between the previous generators.
3. d_3 is the minimal number of generators of relations between relations between generators.
4. ... and so on

Existence of geometric resolutions: Noether, Malgrange and Syzygies

Here are some cases where geometric resolutions of a singular foliation always exist at least locally, and are of finite length.

Proposition 2.6.15: A particular case of Syzygy theorem

Any algebraic singular foliation on $\mathbb{K}^d$ admits geometric resolutions by trivial vector bundles and of length $\leq d+1$.
The same holds for a real analytic of holomorphic singular foliation but only in a neighborhood of a point.

Proof. See [LGLS20] □

Smooth functions are a flat ring over polynomial functions on $\mathbb{R}^n$.

Proposition 2.6.16: Malgrange flatness theorems

A geometric resolution of finite length by trivial bundles of an algebraic singular foliation on $\mathbb{R}^n$ is also a real analytic geometric resolution.
A geometric resolution of a real analytic singular foliation is also a smooth geometric resolution.
A geometric resolution of finite length by trivial bundles of an algebraic singular foliation on $\mathbb{C}^n$ is also a real holomorphic geometric resolution.

Proposition 2.6.17: Glueing of resolutions in the smooth setting [LGLS20]

In the smooth setting, there is a natural manner to glue two finite length geometric resolutions defined on two open sets into a geometric resolution on their unions.

Altogether, these results imply:

Theorem 2.6.18: Existence of geometric resolutions [LGLS20]

A locally real analytic singular foliation on a manifold of dimension d admits a geometric resolutions of length $\leq d+1$ on any relatively compact open subset of M.

Here we have some examples of geometric resolutions of singular foliations.

Example 2.6.19. Let $\mathcal{F}_0 = \{X \in \mathfrak{X}(V) \mid X(0) = 0\}$ be the singular foliation made of all vector fields vanishing at the origin of a vector space V (e.g think of $\mathbb{C}^N$ or $\mathbb{R}^N$). The contraction by the Euler vector field $\overrightarrow{E} = \displaystyle\sum_{i=1}^{N} x_i \frac{\partial}{\partial x_i}$ gives rise to a complex of trivial vector bundles

$$\cdots \longrightarrow \wedge^3 T^*V \xrightarrow{\iota_{\overrightarrow{E}}} \wedge^2 T^*V \xrightarrow{\iota_{\overrightarrow{E}}} T^*V \xrightarrow{\iota_{\overrightarrow{E}}} \mathbb{C} \times V =: \underline{\mathbb{C}}, \tag{2.11}$$

whose complex on the sections level is $(\Omega^\bullet(V), \iota_{\overrightarrow{E}})$. Here $(x_1, \ldots, x_N)$ are the canonical coordinates on V. The latter is the Kozul complex associated to the coordinate functions $x_1, \ldots, x_N$ of V. Since the x_i's form a regular sequence, it is well known that $(\Omega^\bullet(V), \iota_{\overrightarrow{E}})$ is exact .

The following complex of vector bundles over V

$$\cdots \longrightarrow \wedge^3 T^*V \otimes TV \xrightarrow{\iota_{\overrightarrow{E}} \otimes \mathrm{id}} \wedge^2 T^*V \otimes TV \xrightarrow{\iota_{\overrightarrow{E}} \otimes \mathrm{id}} T^*V \otimes TV \xrightarrow{\iota_{\overrightarrow{E}} \otimes \mathrm{id}} \underline{\mathbb{C}} \otimes TV. \tag{2.12}$$

is a geometric resolution of $\mathcal{F}_0$ since $\left(\Omega^\bullet(V) \otimes \mathfrak{X}(V), \iota_{\overrightarrow{E}} \otimes \mathrm{id}\right)$ is also exact (here $\Omega^i(V) := \Gamma(\wedge^i T^*V)$ stands for the sheaf of i-forms on V).

More generally, the construction we have made in (2.12) is still possible by contracting with any vector field $X = \displaystyle\sum_{i=1}^{N} X^i \frac{\partial}{\partial x_i} \in \mathfrak{X}(V)$. The latter will yield a complex of vector bundles that covers the singular foliation $\mathcal{F}_X$ generated by the $X^i \frac{\partial}{\partial x_j}$'s. For instance, if X is a polynomial vector field and $(X^1, \ldots, X^N)$ form a regular sequence, we will get a geometric resolution of $\mathcal{F}_X$.

Example 2.6.20. Let $\mathcal{F}_2 = \mathcal{I}_0^2 \mathfrak{X}(\mathbb{K}^2) \subset \mathcal{F}_0$ be the sub-singular singular foliation made of vector fields vanishing at order 2 at the origin of $\mathbb{K}^2$, where $\mathcal{I}_0^2 \subset \mathcal{O}(\mathbb{K}^2)$ is the ideal generated by the monomials x^2, xy, y^2. Note that the ideal $\mathcal{I}_0^2$ admits a free resolution of the form

$$0 \longrightarrow \mathcal{O}(\mathbb{K}^2) \oplus \mathcal{O}(\mathbb{K}^2) \xrightarrow{\delta_1} \mathcal{O}(\mathbb{K}^2) \oplus \mathcal{O}(\mathbb{K}^2) \oplus \mathcal{O}(\mathbb{K}^2) \xrightarrow{\delta_0} \mathcal{I}_0^2 \longrightarrow 0, \tag{2.13}$$

where for all $f, g, h \in \mathcal{O}(\mathbb{K}^2)$,

$$\delta_0(f, g, h) = x^2 f + xyg + y^2 h \text{ and } \delta_1(f, g) = (xf, xf - yg, xg).$$

The free resolution (2.13) has to take the form

$$0 \longrightarrow \Gamma(\mathcal{I}_{-2}) \xrightarrow{\delta_1} \Gamma(\mathcal{I}_{-1}) \xrightarrow{\delta_0} \mathcal{I}_0^2 \longrightarrow 0, \tag{2.14}$$

for sum trivial vector bundles $\mathcal{I}_{-1}, \mathcal{I}_{-2}$ on $\mathbb{K}^2$. Thus, the following complex

$$0 \longrightarrow \mathcal{I}_{-2} \otimes T\mathbb{K}^2 \xrightarrow{\delta_1 \otimes \mathrm{id}} \mathcal{I}_{-1} \otimes T\mathbb{K}^2 \xrightarrow{\delta_0 \otimes \mathrm{id}} \mathcal{I}_0^2 \otimes T\mathbb{K}^2 \longrightarrow 0 \tag{2.15}$$

is a geometric resolution of $\mathcal{F}_2$. Note that $\mathcal{I}_{-1}$ can be identified with the tivial vector bundle $S^2((\mathbb{K}^2)^*)$.

More generally, let $\mathcal{F}_k$ be the singular foliation made of vector fields vanishing at order k at the origin of a vector space V of dimension N over $\mathbb{R}$ or $\mathbb{C}$. The Hilbert's syzygy theorem assures the existence of a free resolution of length $N+1$ of the ideal $\mathcal{I}_0^k$ made of functions on V vanishing to order k at the origin. This resolution is of the form

$$\cdots \longrightarrow \Gamma(\mathcal{I}_{-2}) \xrightarrow{\delta_1} \Gamma(\mathcal{I}_{-1}) \xrightarrow{\delta_0} \mathcal{I}_0^2 \longrightarrow 0, \tag{2.16}$$

for some family of trivial vector bundles $(\mathcal{I}_{-i})_{i\geq 1}$ over V. We obtain a geometric resolution of $\mathcal{F}^k$ of the form

$$\cdots \longrightarrow \Gamma(\mathcal{I}_{-2} \otimes TV) \xrightarrow{\delta_1 \otimes \mathrm{id}} \Gamma(\mathcal{I}_{-1} \otimes TV) \xrightarrow{\delta_0 \otimes \mathrm{id}} \mathcal{I}_0^2 \otimes \mathfrak{X}(V) = \mathcal{F}^k \tag{2.17}$$

Example 2.6.21. Let φ be a polynomial function on $V := \mathbb{C}^N$. Consider the singular foliation

$$\mathcal{F}_\varphi := \{X \in \mathfrak{X}(V) \mid X[\varphi] = 0\}.$$

The contraction by $\mathrm{d}\varphi$ gives a complex of vector bundles

$$\cdots \xrightarrow{\iota_{\mathrm{d}\varphi}} \wedge^3 TV \xrightarrow{\iota_{\mathrm{d}\varphi}} \wedge^2 TV \xrightarrow{\iota_{\mathrm{d}\varphi}} TV \xrightarrow{\iota_{\mathrm{d}\varphi}} \mathbb{C} \times V. \tag{2.18}$$

The induced complex on the section level

$$\ldots \xrightarrow{\iota_{\mathrm{d}\varphi}} \mathfrak{X}^3(V) \xrightarrow{\iota_{\mathrm{d}\varphi}} \mathfrak{X}^2(V) \xrightarrow{\iota_{\mathrm{d}\varphi}} \mathfrak{X}(V) \xrightarrow{\iota_{\mathrm{d}\varphi}} \mathcal{O}(V) \tag{2.19}$$

(where $\mathfrak{X}^i(V) := \Gamma(\wedge^i TV)$ stands for the sheaf of i-multivector fields on V) is exact in all degree, except in degree 0 if $\left(\frac{\partial\varphi}{\partial x_1}, \cdots, \frac{\partial\varphi}{\partial x_N}\right)$ is a regular sequence. For instance, if φ is weight-homogeneous and admits an isolated singularity at the origin. In this case,

$$\cdots \xrightarrow{\iota_{\mathrm{d}\varphi}} \mathfrak{X}^3(V) \xrightarrow{\iota_{\mathrm{d}\varphi}} \mathfrak{X}^2(V) \xrightarrow{\iota_{\mathrm{d}\varphi}} \ker(\iota_{\mathrm{d}\varphi}) = \mathcal{F}_\varphi \tag{2.20}$$

is a resolution of $\mathcal{F}_\varphi$.

2.6.1 Geometric resolutions of length ≤ 2 and singular foliations

In this section we discuss the case when a singular foliation $\mathcal{F}$ admits a geometric resolution of length 1 and 2. In those cases, we claim that there are Lie algebra-like structures on them.

For any geometric resolution $(E_\bullet, \mathrm{d}^\bullet, \rho)$, the pair $(E_{-1} \to M, \rho)$ is an anchored bundle such that $\rho(\Gamma((E_{-1})) = \mathcal{F}$. Therefore, by Proposition 2.1.2(1), $(E_{-1} \to M, \rho)$ can be endowed with an almost Lie algebroid structure $(E_{-1}, [\cdot\,,\cdot]_{E_{-1}}, \rho)$.

In general, the Jacobiator of $[\cdot,\cdot]_{E_{-1}}$ is non-zero, but is valued kernel of ρ, see Remark 2.1.11.

Geometric resolutions of length 1

As we saw in Proposition 2.6.12, a singular foliation $\mathcal{F}$ admits a geometric resolution of length 1 if and only if it is projective. It that case, the almost Lie algebroid bracket $[\cdot,\cdot]_{E_{-1}}$ is a Lie algebroid bracket. In conclusion: geometric resolutions of length 1 admit a Lie algebroid structure.

Geometric resolutions of length 2

Let $(M,\mathcal{F})$ be a singular foliation that admits a geometric resolution of length 2, namely

$$(E_\bullet, \mathrm{d}^\bullet, \rho): \quad 0 \longrightarrow E_{-2} \stackrel{\mathrm{d}^{(2)}}{\longrightarrow} E_{-1} \stackrel{\rho}{\longrightarrow} TM. \tag{2.21}$$

Since Equation (2.21) is a geometric resolution of $\mathcal{F}$, $(E_{-1} \to M, \rho)$ is an anchored bundle such that $\Gamma(E_{-1}) = \mathcal{F}$. It is quite judicious to ask whether we can extend this bracket to sections of degree -2. If yes, what structures will we have?

Since the complex (2.21) is a geometric resolution of $\mathcal{F}$, the complex

$$0 \longrightarrow \Gamma(E_{-2})_U \stackrel{\mathrm{d}^{(2)}}{\longrightarrow} \Gamma(E_{-1})_U \stackrel{\rho}{\longrightarrow} \mathcal{F}_U \longrightarrow 0$$

is exact for all open subsets $U \subset M$.

Let $U \subset M$ be a an open subset in M. Let $(e'_1, \ldots, e'_{r_2})$ and $(e_1, \ldots, e_{r_1})$ be local trivializations of the vector bundles E_{-2} and E_{-1} on U, respectively. For all $i,j,k \in \{1, \ldots, r_1, r_1+1, \ldots, r_2\}$ we have

1. $$\rho([\mathrm{d}^{(2)}e'_i, e_j]_{E_{-1}}) = [\rho \circ \mathrm{d}^{(2)} e'_i, \rho(e_j)] = 0, \text{ (by Equation (2.4) and since } \rho \circ \mathrm{d}^{(2)} \equiv 0).$$

   In other words, $[\mathrm{d}^{(2)}e'_i, e_j]_{E_{-1}} \in \ker\rho$. By exactness of the complex (2.6.1) there exists a local section denoted by $\nabla_{e'_i} e_j \in \Gamma(E_{-2})_U$ such that

   $$\mathrm{d}^{(2)} \nabla_{e'_i} e_j = [\mathrm{d}^{(2)} e'_i, e_j]_{E_{-1}}. \tag{2.22}$$

   Equation (2.22) allows to define a bilinear map:

   $$\begin{array}{ccc} \Gamma(E_{-1})_U \otimes \Gamma(E_{-2})_U & \to & \Gamma(E_{-2})_U \\ (x,y) & \mapsto & \nabla_x y \end{array}$$

   by extending the $\nabla_{e'_i} e_j$'s by linearity and Leibniz identity with the understanding that the anchor map ρ vanishes on sections of E_{-2} in order to have

   (a) $\mathrm{d}^{(2)} \nabla_x y = [\mathrm{d}^{(2)} x, y]_{E_{-1}}$, $\forall x\ \Gamma(E_{-2})_U$, $y \in \Gamma(E_{-1})_U$,

   (b) for all function $f \in \mathcal{O}(U)$: $\nabla_x f y = f \nabla_x y + \rho(x)[f]\, y$ and $\nabla_{fx} y = f\nabla_x y$, for all $x \in \Gamma(E_{-1}), y \in \Gamma(E_{-2})$,

2. Remember that

   $$\mathrm{Jac}(e_i, e_j, e_k) := [e_i, [e_j, e_k]_2]_2 + [e_j, [e_k, e_i]_2]_2 + [e_k, [e_i, e_j]_2]_2 \in \ker\rho.$$

   By using again exactness of the complex (2.6.1) there is a local section that denote by $[e_i, e_j, e_k]_{E_{-1}} \in \Gamma(E_{-2})_U$ that satisfiies

   $$\mathrm{d}^{(2)}[e_i, e_j, e_k]_{E_{-1}} = \mathrm{Jac}(e_i, e_j, e_k). \tag{2.23}$$

   Thus, we can define a skew-symmetric trilinear map:

   $$[\cdot\,,\cdot\,,\cdot]_{E_{-1}} : \Gamma(E_{-1})_U \wedge \Gamma(E_{-1})_U \wedge \Gamma(E_{-1})_U \longrightarrow \Gamma(E_{-2})_U$$

   such that

   $$\mathrm{d}^{(2)}[x,y,z]_{E_{-1}} = [x,[y,z]_2]_2 + [y,[z,x]_2]_2 + [z,[x,y]_2]_2, \ \forall x,y,z \in \Gamma(E_{-1})_U.$$

In the smooth case these operators can be glued to global ones by taking a partition of unity.

The following Proposition recapitulates the discussion above.

Proposition 2.6.22. *Let $(M,\mathcal{F})$ be a singular foliation that admits a geometric resolution of length* 2 *as in* (2.21)*, and $(E_{-1}, [\cdot\,,\cdot]_{E_{-1}}, \rho)$ an almost Lie algebroid.*

1. *There is a bilinear map:*

$$\begin{array}{ccc} \Gamma(E_{-1}) \otimes \Gamma(E_{-2}) & \to & \Gamma(E_{-2}) \\ (x,y) & \mapsto & \nabla_x y \end{array}$$

and a skew-symmetric trilinear map:

$$[\cdot\,,\cdot\,,\cdot]_{E_{-1}} \colon \Gamma(E_{-1}) \wedge \Gamma(E_{-1}) \wedge \Gamma(E_{-1}) \longrightarrow \Gamma(E_{-2})$$

2. *such that for all function f:*

   (a) $\nabla_x fy = f\nabla_x y + \rho(x)[f]\, y$ *and* $\nabla_{fx} y = f\nabla_x y$, *for all* $x \in \Gamma(E_{-1}), y \in \Gamma(E_{-2})$,

   (b) $[fx, y, z]_{E_{-1}} = f[x, y, z]_{E_{-1}}$ *for all* $x, y, z \in \Gamma(E_{-1})$,

   such that the 2-ary bracket *on* $\Gamma(E_{-1} \oplus E_{-2})$ *defined by:*

$$[x,y]_2 = \begin{cases} [x,y]_{E_{-1}} & \text{for} \quad x, y \in \Gamma(E_{-1}) \\ \nabla_x y & \text{for} \quad x \in \Gamma(E_{-1}),\, y \in \Gamma(E_{-2}) \\ -\nabla_y x & \text{for} \quad x \in \Gamma(E_{-1}),\, y \in \Gamma(E_{-2}) \\ 0 & \text{for} \quad x,\, y \in \Gamma(E_{-2}) \end{cases}$$

   together with the 3-ary bracket *on* $\Gamma(E_{-1} \oplus E_{-2})$ *defined by* $[x, y, z]_3 = [x, y, z]_{E_{-1}}$ *if* $x, y, z \in \Gamma(E_{-1})$ *and zero otherwise, satisfies*

   (a) for all $x \in \Gamma(E_{-2}), y \in \Gamma(E_{-1})$,

$$\mathrm{d}^{(2)}[x,y]_2 + [\mathrm{d}^{(2)}x, y]_2 = 0, \tag{2.24}$$

   (b) for all $x, y, z \in \Gamma(E_{-1})$

$$\mathrm{d}^{(2)}[x,y,z]_3 + [x,[y,z]_2]_2 + [y,[z,x]_2]_2 + [z,[x,y]_2]_2 = 0$$

   (c) for all $x, y \in \Gamma(E_{-1})$ *and* $z \in \Gamma(E_{-2})$

$$[x, y, \mathrm{d}^{(2)}z]_3 + [x,[y,z]_2]_2 + [y,[z,x]_2]_2 + [z,[x,y]_2]_2 = 0.$$

Definition 2.6.23. The structure $(E_\bullet, \mathrm{d}^\bullet, \rho, [\cdot\,,\cdot]_2, [\cdot\,,\cdot,\cdot]_3)$ described in Proposition 2.6.22 is called 2-*Lie algebroid.*

A generalization will be discussed later.

2.7 Universal Q-manifold

Beyond the world of manifolds is the universe of "manifolds up to homotopy", which are known under names: some or more or less equivalent, and some are mostly dual notions:

♣ Lie ∞-algebroids.

♢ Q-manifolds, also called dg-manifolds (dg= differential graded).

The two notions are in fact "equivalent" in the sense that they are dual one to the other.

2.7.1 Two dual point of views on Lie algebras

To explain where the notion of dg-manifold comes from, let us look at the case of Lie algebras:

Definition 2.7.1: A weird definition of Lie algebra

A co-Lie algebra is a vector space V equipped with a degree $+1$ derivation

$$\delta \colon \wedge^\bullet V \longrightarrow \wedge^{\bullet+1} V$$

such that $\delta^2 = 0$

Before explaining this definition, let us start with a few comments:

1. we write $\delta\colon \wedge^\bullet V \mapsto \wedge^{\bullet+1} V$ to mean that for every $k \geq 0$, δ maps $\wedge^k V$ to $\wedge^{k+1} V$
2. By a degree $+1$ derivation, we mean that
$$\delta(\alpha \wedge \beta) = \delta(\alpha) \wedge \beta + (-1)^k \alpha \wedge \delta(\beta)$$
for all $\alpha \in \wedge^k V$ and $\beta \in \wedge^\bullet V$. The signs are exactly those of the de Rham differential (which is also a degree $+1$ derivation).
3. For any degree $+1$ derivation, δ^2 is easily seen to be a degree $+2$ derivation[12].
4. A derivation of $\wedge^\bullet V$ is entirely determined by its restriction to V, which is a map $\mu\colon V \longrightarrow \wedge^2 V$ that we call the *co-Lie-bracket*. This comes from the very derivation property:
$$\delta(v_1 \wedge \cdots \wedge v_k) = \sum_{i=1}^{k} (-1)^{i+1} v_1 \wedge \cdots \wedge v_{i-1} \ \wedge \mu(v_i) \wedge v_{i+1} \wedge \cdots \wedge v_k \tag{2.25}$$
5. Conversely, any linear map $\mu\colon V \longrightarrow \wedge^2 V$ extends to a degree $+1$ derivation by using (2.25).

Proposition 2.7.2: Lie algebras are dual to co-Lie algebra

There is a one-to-one correspondence between finite dimensional Lie algebras and finite dimensional Lie coalgebras.

The correspondence goes as follows.

1. The dual of the Lie algebra bracket $[\cdot,\cdot]\colon \wedge^2 \mathfrak{g} \to \mathfrak{g}$ is a map $\mu\colon \mathfrak{g}^* \to (\wedge^2\mathfrak{g})^*$. Since the dimensions are finite, there is a canonical isomorphism $(\wedge^2\mathfrak{g})^* \simeq \wedge^2\mathfrak{g}^*$, and we still denote by μ the map $\mu\colon \mathfrak{g}^* \to \wedge^2\mathfrak{g}^*$. Using (2.25), one extends μ to a degree $+1$ derivation δ of $\wedge^\bullet\mathfrak{g}^*$. It is routine to check that the Jacobi identity holds for $[\cdot,\cdot]$ implies $\delta^2 = 0$.
2. Conversely, given a co-Lie algebra, the dual of the co-Lie bracket $\mu\colon V \to \wedge^2 V$ is a linear map $\wedge^2 V^* \to V^*$. It is routine to check that $\delta^2 = 0$ implies the Jacobi identity for $[\cdot,\cdot]$.

Remark 2.7.3. The degree $+1$ derivation corresponding to a finite-dimensional Lie algebra is its Chevalley-Eilenberg differential computing the Lie algebra cohomology in trivial coefficients.

To put it all in a nutshell:

<table>
<tr><td colspan="2" align="center">What is a finite dimensional Lie algebra?
Two dual answers</td></tr>
<tr><td>Direct notion:

A vector space $\mathfrak{g}$
+ linear map: $[\cdot,\cdot]\colon \wedge^2 \mathfrak{g} \longrightarrow \mathfrak{g}$</td><td>Dual notion:

A vector space V
+ degree $+1$ derivation $\delta\colon \wedge^\bullet V \longrightarrow \wedge^{\bullet+1} V$</td></tr>
<tr><td colspan="2" align="center">such that</td></tr>
<tr><td>The Jacobi identity holds</td><td>$\delta^2 = 0$</td></tr>
</table>

[12] Of course, this is not true for degree 0 derivation, otherwise the formula $(fg)'' = f''g + fg''$ would not be classical confusion of undergraduate students.

2.7.2 Graded symmetric algebras

Throughout of this section we are working on a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Co-Lie algebras have been described in the previous sections. They are nothing else than dual of Lie algebras, but it is conceptually important to understand how Lie ∞-algebras are defined.

Consider now a positively $\mathbb{Z}$-graded vector space:

$$V_\bullet := \oplus_{i\in\mathbb{Z}} V_i$$

Definition 2.7.4: Graded Symmetric algebras

We call graded symmetric algebra of $V_\bullet$ and denote by $S(V_\bullet)$ *the quotient of the tensor algebra*

$$\oplus_{i\geq 0} V^{\otimes n}$$

by the ideal generated by

$$x \otimes y - (-1)^{ij} y \otimes x$$

for all $x \in V_i, y \in V_j$. We denote by $\odot$ the induced product on $S(V_\bullet)$.

Let us state a few basic facts about this quotient space. To start with, the tensor algebra comes with two different "degrees" that we have to distinguish, and that go to the quotient: Elements of

$$V_{i_1} \otimes \cdots \otimes V_{i_n}$$

shall be said of

1. *polynomial degree n*,

2. *degree $i_1 + \cdots + i_n$*.

With both the polynomial degree and the degree, the quotient $S(V_\bullet)$ is "graded" in the sense that the degree of the product is the sum of the degrees. But with respect to the degree, it is also graded commutative, i.e.

$$P \odot Q = (-1)^{ij} Q \odot P$$

For homogenous elements $x_1, \ldots, x_n \in V$, the *Koszul sign* denoted by, $\epsilon(\sigma, x_1, \ldots, x_n)$ or simply by $\epsilon(\sigma)$ when there no ambiguity, is the sign induced by the permutation of the x_i's which is defined by:

$$x_{\sigma(1)} \odot \cdots \odot x_{\sigma(n)} = \epsilon(\sigma, x_1, \ldots, x_n) x_1 \odot \cdots \odot x_n. \tag{2.26}$$

For $V = \oplus_{i=1}^{\infty} V_{-i}$ a graded vector space we denote by $S(V^*)$ the graded commutative symmetric algebra generated by $V^* = \oplus_{i=1}^{\infty} V_{-i}^*$, with the understanding that elements in E_{-i}^* are of degree $+i$. Elements in $V_{-i}^* \odot \cdots \odot V_{-i_k}^*$ are therefore of polynomial degree k and degree $i_1 + \cdots + i_k$. We shall define elements of arity zero to be elements in $\mathbb{K}$.

In the present section, we will be interested in two kinds of symmetric algebras:

♣ those of the form $S(\oplus_{i\geq 1} V_{-i})$ whose non-trivial components are of negative degrees.

♢ those of the form $S(\oplus_{i\geq 1} V_i)$ whose non-trivial components are of positive degrees.

By our convention, both symmetric algebras are in duality, i.e. if all spaces are of finite dimension, and if E_{-i} is the dual of V_i for all $i \geq 1$, then there is a duality between:

♣ elements of polynomial degree k and degree $-i$ in $S(\oplus_{i\geq 1} E_{-i})$

♢ elements of polynomial degree k and degree $+i$ in $S(\oplus_{i\geq 1} V_i)$

Question 2.7.5: Towards Lie ∞-algebras

Let $V_\bullet = \oplus_{i\geq 1} V_i$ be a positively graded vector space. Assume $S(V_\bullet)$ comes equiped with a degree $+1$ derivation δ such that $\delta^2 = 0$. What kind of structures do me obtain on the dual spaces $\oplus_{i\geq 1} E_{-i}$?

Here is an answer. The derivation δ is entirely determined by its restriction to V. Decomposing according to polynomial degree, we see that $\delta = \sum_{k\geq 1} \delta^{(k)}$, with $\delta^{(k)} \colon V \mapsto S^k V$ a degree $+1$ map.

By duality, there is a one-to-one correspondence between:

- ♣ the datum $(V, (\ell_k)_{k\geq 1})$ made of a collection of vector spaces $V = (V_{-i})_{i\geq 1}$ together with a family of degree $+1$ linear maps $(\ell_k \colon S^\bullet(V) \longrightarrow V)_{k\geq 1}$ called k-ary brackets,

- ♢ a sequence $\delta^{(k)}$ of linear maps $V \longrightarrow S^k(V)$.

for all $k \in \mathbb{N}$, denote by $\ell_k \colon S^k E_\bullet \to E_\bullet$ the dual of the differential $\delta^{(k)} \colon$.

It is then a complicated by direct computation to check that $\delta^2 = 0$ holds if and only if the ℓ_k equip $E_\bullet$ with a Lie ∞-algebra structure, the latter being defined as follows:

Definition 2.7.6: Lie ∞-algebras

A negatively graded Lie ∞-algebra *is the datum $(V, (\ell_k)_{k\geq 1})$ made of a collection of vector spaces $V = (V_{-i})_{i\geq 1}$ together with a family of degree $+1$ linear maps $(\ell_k \colon S^\bullet(V) \longrightarrow V)_{k\geq 1}$ called k-ary brackets, which fulfill the compatibility conditions the so-called* higher Jacobi identities*: for all homogeneous elements $v_1, \ldots, v_n \in V$*

$$\sum_{i=1}^{n} \sum_{\sigma \in \mathfrak{S}_{i,n-i+1}} \epsilon(\sigma) \ell_{n-i+1} \left(\ell_i(v_{\sigma(1)}, \ldots, v_{\sigma(i)}), v_{\sigma(i+1)}, \ldots, v_{\sigma(n)} \right) = 0. \tag{2.27}$$

Here $\epsilon(\sigma)$ is the Kozsul sign associated to $v_1, \ldots, v_n$.

In conclusion:

Proposition 2.7.7

Let $E_\bullet = \oplus_{i\geq 1} E_{-i}$ and $V_\bullet = \oplus_{i\geq 1} V_{+i}$ be finite dimensional graded vector spaces in duality. There is a one-to-one correspondence between

- ♣ *Lie ∞-algebras brackets $(\ell_k)_{k\geq 1}$ on $E_\bullet$,*

- ♢ *degree $+1$ derivations squaring to zero of $S(V_\bullet)$.*

2.7.3 NQ-manifolds

We will now extend the previous discussion from Lie ∞-algebras to Lie ∞-algebroids. And from $S(V)$ equipped with a degree $+1$ differential to the so-called dg-manifolds.

Graded manifolds

Let us first define $\mathbb{N}$-graded manifolds.

Definition 2.7.8: Graded manifolds: the objects

Let M be a smooth, real analytic or complex manifold. A (positively) graded manifold *over a manifold*[a] *M is a sheaf*

$$\mathcal{E} \colon \mathcal{U} \mapsto \mathcal{E}(\mathcal{U})$$

of graded commutative algebras over $\mathbb{K}$ such that every $m \in M$ admits an open neighborhood $\mathcal{U} \subset M$ on which the sheaf structure takes the form

$$\mathcal{E}(\mathcal{U}) = \mathcal{O}_{\mathcal{U}} \otimes_{\mathbb{K}} S(V_\bullet)$$

for some some graded vector space $V = \oplus_{i=1}^{\infty} V_{+i}$. If $E_\bullet = \oplus_{i \geq 1} E_{-i}$ and $V_\bullet = \oplus_{i \geq 1} V_{+i}$ are in duality, sections of the sheaf $\mathcal{E}$ are called functions on E. *It is convenient to denote a graded manifold as a pair $(M, \mathcal{E})$.*

[a] called the *base* of the sheaf

Remark 2.7.9. In the smooth setting, it can be proven that there exists a globally and canonically defined graded vector bundle $V_\bullet \to M$ such that $\mathcal{E}$ is isomorphic to the graded commutative algebra bundle $S(V_\bullet) \to M$. Although $V_\bullet$ is canonical, the isomorphism of sheaves

$$\mathcal{E} \xleftrightarrow{\simeq} \Gamma(S(V))$$

is not canonical. For a statement adapted to the present situation, see [**?**]. Upon choosing such an isomorphism, a function $\xi \in \mathcal{E}_j$ is a formal sum

$$\xi = \sum_{i \geq 0} \xi^{(i)} \tag{2.28}$$

with $\xi^{(i)} \in \mathcal{E}$ an element of polynomial degree i and degree j. For degree reasons, the sum must be finite.

Local coordinates of a graded manifold

Recall that for $\mathcal{U} \subset M$ an open set, one has $(V_i)_{\mathcal{U}} \xrightarrow{\sim} \mathcal{U} \times \mathbb{K}^{\mathrm{rk}(V_i)}$ for every $i \geq 1$. Hence, the *graded coordinates* on the graded manifold $(M, \mathcal{E})$ is the data made of:

In degree 0: a system of coordinates $(x_1, \ldots, x_n)$ of M on $\mathcal{U}$

In degree $i \geq 1$: a local trivialization $(\xi_i^1, \ldots, \xi_i^{\mathrm{rk}(V_i)})$ of V_i on $\mathcal{U}$.

That is, a system of graded coordinates of $(M, \mathcal{E})$ on $\mathcal{U}$ is

$$(x_1, \ldots, x_n, \xi_1^1, \ldots, \xi_1^{\mathrm{rk}(V_1)}, \ldots, \xi_i^1, \ldots, \xi_i^{\mathrm{rk}(V_i)}, \ldots).$$

Therefore, the elements of $\mathcal{E}(\mathcal{U})$ are "polynomials" in $\{(\xi_i^j)_{j=1,\ldots,\mathrm{rk}(V_{-i})}, \ i \geq 1\}$ with coefficients in $\mathcal{O}(\mathcal{U})$.

Example 2.7.10. The sheaf of differential forms $(M, \mathcal{E} = \Omega(M))$ on a manifold M is a graded manifold since for every point $m \in M$, it takes the form $\mathcal{O}_{\mathcal{U}} \otimes_{\mathbb{K}} \wedge^\bullet T_m^* M$ where $\mathcal{U}$ is an open neighborhood of m. Exterior forms can be seen as sections on the graded vector bundle $E_{-1} = TM$.

Example 2.7.11. Let k be a positive integer. A finite dimensional vector space E and its dual V can be seen as graded vector bundles of respective degree $-k$ and k over a point. E is a graded manifold over $M = \{pt\}$, with functions isomorphic to $\wedge V$ for k odd and $S(V)$ for k even.

Definition 2.7.12: Graded manifolds: the Morphisms

A morphism of graded manifolds *between the two graded manifolds* $(M, \mathcal{E})$ *and* $(M', \mathcal{E}')$ *with respective base manifolds* M *and* M' *is a pair made of a smooth or real analytic or holomorphic map* $\phi\colon M \longrightarrow M'$ *called the* base map *and a sheaf morphism over it, i.e. a family of graded algebra morphisms:*

$$\mathcal{E}'(\mathcal{U}') \to \mathcal{E}(\phi^{-1}(\mathcal{U}')),$$

compatible with the restriction maps, such that

$$\Phi(f\alpha) = \phi^*(f)\Phi(\alpha). \tag{2.29}$$

for all $f \in \mathcal{O}'_{\mathcal{U}'}$ *and* $\alpha \in \mathcal{E}'(\mathcal{U}')$.
A homotopy *between two morphisms of graded manifolds* $\Phi, \Psi\colon (M, \mathcal{E}) \longrightarrow (N, \mathcal{E}')$ *is a morphism of graded manifold*

$$(M, \mathcal{E}) \times ([0,1], \Omega([0,1])) \longrightarrow (M', \mathcal{E}')$$

whose restrictions to the extremities of the interval $[0,1]$ *coincide with* Φ *and* Ψ *respectively.*

Vector fields on graded manifolds

Vector fields on manifolds are derivations of its algebra of functions. For a graded manifold, the equivalent of functions are the sections of the sheaf $\mathcal{E}$. Since it is not commutative but graded commutative, one has to consider graded derivations. A *graded derivation of degree* k of $\mathcal{E}$ is the data, for every $\mathcal{U} \subset M$ of a linear map

$$Q\colon \mathcal{E}_\bullet(\mathcal{U}) \longrightarrow \mathcal{E}_{\bullet+k}(\mathcal{U}),$$

compatible with all restriction maps, that increases the degree by $+k$ and satisfies:

$$Q[FG] = Q[F]G + F(-1)^{ki}Q[G]$$

for every $F \in \mathcal{E}_i(\mathcal{U}), G \in \mathcal{E}(\mathcal{U})$. Since we think geometrically, we will simply say "vector fields of degree k" instead of graded derivations.

Definition 2.7.13. Let $(M, \mathcal{E})$ be a graded manifold. For $\mathcal{U} \subset M$ and $k \in \mathbb{Z}$ let

$$\mathfrak{X}_k(E)(\mathcal{U}) := \mathrm{Der}_k(\mathcal{E}(\mathcal{U}))$$

be the $\mathcal{E}(\mathcal{U})$-module of derivation of degree k on $\mathcal{E}(\mathcal{U})$. The correspondence $\mathcal{U} \longmapsto \mathfrak{X}_\bullet(E)(\mathcal{U})$ is a sheaf of $\mathcal{E}$-modules. Its sections are called *vector fields on* E.

Let us list some important facts on vector fields on E:

1. the $\mathcal{E}$-module $\mathfrak{X}_\bullet(E) := \oplus_{k\mathbb{Z}}\mathfrak{X}_k(E)$ of vector fields on E is naturally graded. The $\mathcal{E}$-module $\mathfrak{X}_\bullet(E)$ of vector fields on E is a graded Lie sub-algebra of the graded Lie algebra $\mathrm{Hom}_{\mathbb{K}}(\mathcal{E}, \mathcal{E})$ whose graded Lie bracket is the graded commutator. Precisely, the graded Lie bracket

$$[P, Q] = P \circ Q - (-1)^{kl} Q \circ P \tag{2.30}$$

of two vector fields P, Q of degree k, l respectively is a vector field of degree $k + l$. It is easily checked that the bracket (2.30) fulfills

   (a) $[P, Q] = -(-1)^{jk}[Q, P]$ (graded skew-symmetry)
   (b) $(-1)^{jl}[P, [Q, R]] + (-1)^{jk}[Q, [R, P]] + (-1)^{kl}[R, [P, Q]] = 0,$ (graded Jacobi identity)

   for vector fields P, Q, R of degree j, k and l respectively.

2. Their description in local coordinates: note that any homogeneous element $e \in E_{-k}$ corresponds to a *vertical* vector field $\iota_e \in \mathfrak{X}_{-k}(E)$ (i.e it is $\mathcal{O}$-linear) of degree $-i$ defined by contraction with e

$$\iota_e(\xi) := \langle \xi, e \rangle, \quad \xi \in \Gamma(V) \tag{2.31}$$

and we extend by $\mathcal{O}$-linear derivation, where $\langle\cdot\,,\cdot\rangle$ is the dual pairing between V and E. Let $(\mathcal{U}, x_1, \ldots, x_n)$ a coordinate chart of M and $(\xi_i^j)_{j=1,\ldots,\mathrm{rk}(V_i)}$ with $i \geq 1$ be a homogeneous local trivialization of V, it should be understood that ξ_i^j is the j-th elements of the local frame in $\Gamma(E^*_{-i})$. If $(e_i^j)_{j=1,\ldots,\mathrm{rk}(E_{-i})}, i \geq 1$ is the dual basis of $(\xi_i^j)_{j=1,\ldots,\mathrm{rk}(V_i)}, i \geq 1$, then for every pair i, j, $\iota_{e_i^j} = \frac{\partial}{\partial \xi_i^j}$ is the partial derivative with respect to $\xi_i^j \in \Gamma(V_i)$. By choosing a TM-connection on E, it is easy to check that for any $k \in \mathbb{Z}$ the family

$$\left(\xi_{i_1}^{j_1} \odot \cdots \odot \xi_{i_l}^{j_l} \frac{\partial}{\partial x_j}\right)_{\substack{l \geq 0 \\ i_1 \cdots i_l = k \\ j_1, \ldots, j_l \\ j = 1, \ldots, n}} \cup \left(\xi_{i_1}^{j_1} \odot \cdots \odot \xi_{i_l}^{j_l} \frac{\partial}{\partial \xi_i^j}\right)_{\substack{l \geq 0 \\ i_1 \cdots i_l - i = k \\ j_1, \ldots, j_l \\ i \geq 1,\, j = 1, \ldots, \mathrm{rk}(E_{-i})}}$$

form a basis for $\mathfrak{X}_k(E)(\mathcal{U})$ up to permutations of the $\xi_{i_1}^{j_1} \odot \cdots \odot \xi_{i_l}^{j_l}$'s. Here we adopt the convention $i_0 = j_0 = 0$ and $\xi^0 = 1 \in \Gamma(S^0(V)) \simeq \mathcal{O}$. Whence, any vector field $Q \in \mathfrak{X}_k(E)(U)$ admits coordinates decomposition as follow

$$Q = \sum_{\substack{l \geq 0 \\ i_1 \cdots i_l = k \\ j_1, \ldots, j_l \\ j = 1, \ldots, n}} \frac{1}{l!}\, {}^{j}Q_{i_1 \cdots i_l}^{j_1 \cdots j_l}\, \xi_{i_1}^{j_1} \odot \cdots \odot \xi_{i_l}^{j_l} \frac{\partial}{\partial x_j} + \sum_{\substack{i \geq 1,\, l \geq 0 \\ i_1 \cdots i_l - i = k \\ j_1, \ldots, j_l \\ j = 1, \ldots, \mathrm{rk}(E_{-i})}} \frac{1}{l!}\, {}^{ij}Q_{i_1 \cdots i_l}^{j_1 \cdots j_l}\, \xi_{i_1}^{j_1} \odot \cdots \odot \xi_{i_l}^{j_l} \frac{\partial}{\partial \xi_i^j}.$$

for some functions $Q_{i_1 \cdots i_l}^{j_1 \cdots j_l} \in \mathcal{O}$. These functions can be chosen in a unique manner to satisfy e.g ${}^{ij}Q_{i_{\sigma(1)} \cdots i_{\sigma(l)}}^{j_{\sigma(1)} \cdots j_{\sigma(l)}} = \epsilon(\sigma) Q_{i_1 \cdots i_l}^{j_1 \cdots j_l}$ for any permutation σ of $\{1, \ldots, l\}$.

For example, if Q is of degree $+1$ it can be written in these notations as

$$Q = \sum_{\substack{1 \leq u \leq \mathrm{rk}(E_{-1}) \\ j = 1, \ldots, n}} {}^{j}Q_1^u\, \xi_1^u \frac{\partial}{\partial x_j} + \sum_{\substack{i \geq 1,\, = l \geq 0 \\ i_1 \cdots i_l - i = 1 \\ j_1, \ldots, j_l \\ j = 1, \ldots, n}} \frac{1}{l!}\, {}^{ij}Q_{i_1 \cdots i_l}^{j_1 \cdots j_l}\, \xi_{i_1}^{j_1} \odot \cdots \odot \xi_{i_l}^{j_l} \frac{\partial}{\partial \xi_i^j}.$$

Definition 2.7.14: dg-manifolds = Q-manifolds

A dg-manifold *or* NQ-manifold *is a positively graded manifold $(M, \mathcal{E})$ endowed with a degree $+1$ homological vector field Q on E, i.e., $Q \in \mathfrak{X}_1(E)$ is such that $Q^2 = 0$.*

They shall be denoted as a triple $(M, \mathcal{E}, Q)$.

Example 2.7.15. Given a finite dimension Lie algebra $(\mathfrak{g}, [\cdot\,,\cdot])$ of dimension d. We assume that $\mathfrak{g}$ is concentrated in degree -1. It is clear that $(M = \{pt\}, \mathcal{E} = \wedge^\bullet \mathfrak{g}^*)$ is a graded manifold over $M = \{pt\}$. This graded manifold carries a dg-manifold structure. Precisely, we define the corresponding homological vector field as follow: fix a basis $(e_i)_{i=1,\ldots,n}$ of $\mathfrak{g}$ and let these global coordinate functions $(\xi^i)_{i=1,\ldots,n}$ on $\mathfrak{g}$ be its dual. Write

$$[e_i, e_j] = \sum_{l=1}^{n} \lambda_{ij}^l e_l$$

for some scalars $\lambda_{ij}^l \in \mathbb{K}$. One can check that the degree $+1$ vector field

$$Q = \frac{1}{2} \sum_{i,j,l=1}^{n} \lambda_{ij}^l \xi^i \wedge \xi^j \frac{\partial}{\partial \xi^l}$$

corresponds to the Chevalley-Eilenberg differential $(\mathrm{d}^{\mathrm{CE}}, \wedge^\bullet \mathfrak{g}^*)$. Therefore, $Q^2 = 0$ and is equivalent to Jacobi identity.

Example 2.7.16. Given a differential graded vector bundle $((E_{-i})_{i\geq 1}, \mathrm{d})$ over M. There is a natural *dg* manifold given by its sheaf of sections $(M, \mathcal{E} = \Gamma(S(E^*))$. In particular, the deferential map $\mathrm{d}\colon E \longrightarrow E$ is dualized as a degree $+1$ map $S^1(E^*) \longrightarrow S^1(E^*)$ that we extend to a $\mathcal{C}^\infty(M)$-linear derivation on $\mathcal{E}$ squared to zero.

Example 2.7.17. Let $E = T[1]M$ be the shifted bundle of M. It induces a graded manifold structure $(M, \mathcal{E} = \Omega(M))$ over M. This graded manifold carries a *dg*-manifold structure Q that corresponds to the de Rham differential on $\Omega(M)$. In term of coordinates, the homological vector field Q reads

$$\sum_{i=1}^{n} \mathrm{d}x_i \frac{\partial}{\partial x_i}.$$

Let us introduce some vocabulary that will need to use.

Definition 2.7.18. Let $(M, \mathcal{E}', Q')$ and $(M, \mathcal{E}, Q)$ be two NQ-manifolds.

1. A linear map $\Phi\colon \mathcal{E} \longrightarrow \mathcal{E}'$ is said to be of *polynomial degree/degree* $j \in \mathbb{Z}$ provided that for all functions $\alpha \in \mathcal{E}$ of polynomial degree/degree i, $\Phi(\alpha)$ is of polynomial degree/degree $i + j$. Any map $\Phi\colon \mathcal{E} \longrightarrow \mathcal{E}'$ of degree i decomposes w.r.t the polynomial degree as follows:

$$\Phi = \sum_{r\in\mathbb{Z}} \Phi^{(r)}$$

with $\Phi^{(r)}\colon \mathcal{E} \longrightarrow \mathcal{E}'$ a map of arity r.

Remark 2.7.19. When $\Phi\colon \mathcal{E} \longrightarrow \mathcal{E}'$ is a graded morphism of algebras necessarily one has $\Phi^{(r)} = 0$ for all $r < 0$. Furthermore, for all $n, r \in \mathbb{N}$ and all $\xi_1, \ldots, \xi_k \in \Gamma(V)$ one has:

$$\Phi^{(r)}(\xi_1 \odot \cdots \odot \xi_n) = \sum_{i_1+\cdots+i_n=r} \Phi^{(i_1)}(\xi_1) \odot \cdots \odot \Phi^{(i_n)}(\xi_n). \tag{2.32}$$

Obviously, in this case Φ is determined uniquely by the image of $\Gamma(V)$.

Definition 2.7.20: Morphisms

Let $(M, \mathcal{E}, Q)$ and $(M, \mathcal{E}', Q')$ be two NQ-manifolds over M with sheaves of functions $\mathcal{E}$ and $\mathcal{E}'$ respectively. A morphism of NQ-manifold over M *from $(M, \mathcal{E}', Q')$ to $(M, \mathcal{E}, Q)$ is a morphism of graded manifolds $\Phi\colon \mathcal{E} \longrightarrow \mathcal{E}'$ (of degree 0) over the identity map which intertwines Q and Q', i.e.,*

$$\Phi \circ Q = Q' \circ \Phi. \tag{2.33}$$

Remark 2.7.21. Note that morphisms of NQ-manifolds over M are by definition $\mathcal{O}$-linear since they are defined over the identity map. The component $\Phi^{(r)}$ of arity $r \geq 0$ of any $\mathcal{O}$-linear map $\Phi\colon \mathcal{E} \longrightarrow \mathcal{E}'$ maps $\Gamma(V)$ to $\Gamma(S^{r+1}(V'))$. By $\mathcal{O}$-linearity, it gives rise to a section $\phi_r \in \Gamma(S^{r+1}(V') \otimes E)$. Therefore one has,

$$\Phi^{(r)}(\xi) = \langle \phi_r, \xi \rangle \tag{2.34}$$

for all $\xi \in \Gamma(V)$. It follows that Φ is entirely determined by the collection $\left(\phi_r \in \Gamma(S^{k+1}(V') \otimes E)\right)_{r\geq 0}$ when Φ is a algebra morphism or a Ξ-derivation for some map $\Xi\colon \mathcal{E} \longrightarrow \mathcal{E}'$. In such case, for $r \geq 0$, $\phi_r \in \Gamma(S^{r+1}(V') \otimes E)$ is then called the *r-th Taylor coefficient* of Φ. We also call the 0-th Taylor coefficient $\phi_0\colon E' \to E$ the *linear part* of Φ. The latter is a chain map

$$\begin{tikzcd}
\cdots \arrow[r] & E'_{-3} \arrow[r, "\mathrm{d}'^{(3)}"] \arrow[d, "\phi_0"] & E'_{-2} \arrow[r, "\mathrm{d}'^{(2)}"] \arrow[d, "\phi_0"] & E'_{-1} \arrow[r, "\rho'"] \arrow[d, "\phi_0"] & TM \arrow[d, "\mathrm{id}"] \\
\cdots \arrow[r] & E_{-3} \arrow[r, "\mathrm{d}^{(3)}"] & E_{-2} \arrow[r, "\mathrm{d}^{(2)}"] & E_{-1} \arrow[r, "\rho"] & TM
\end{tikzcd} \tag{2.35}$$

2.7.4 Negatively graded Lie ∞-algebroids and their morphisms

Definition 2.7.22

A negatively graded Lie ∞-algebroid $(E, (\ell_k)_{k\geq 1}, \rho)$ *is a collection of vector bundles $E = (E_{-i})_{i\geq 1}$ over M endowed with a sheaf of Lie ∞-algebra structures $(\ell_k)_{k\geq 1}$ over the sheaf of sections of E together with a vector bundle morphism $\rho\colon E_{-1} \longrightarrow TM$, called the* anchor*, such that the k-ary-brackets are all $\mathcal{O}$-multilinear except when $k = 2$ and at least one of the arguments is of degree -1. The 2-ary bracket satisfies the Leibniz identity*

$$\ell_2(x, fy) = \rho(x)[f]y + f\ell_2(x, y), x \in \Gamma(E_{-1}), y \in \Gamma(E). \tag{2.36}$$

Remark 2.7.23. Definition 2.7.4 implies the following facts

1. $\rho(\ell_2(x, y)) = [\rho(x), \rho(y)]$ for all $x, y \in \Gamma(E_{-1})$ and that $\rho \circ \ell_1 = 0$.
2. The sequence of morphisms of vector bundles

$$\cdots \xrightarrow{\ell_1} E_{-2} \xrightarrow{\ell_1} E_{-1} \xrightarrow{\rho} TM$$

   is a complex of vector bundles that we call the *linear part*. A Lie ∞-algebroid is said to be *acyclic* if its linear part has no cohomology in degree ≤ -1.
3. The 2-ary bracket restricts to an almost-Lie algebroid structure on E_{-1}. Hence, by Lemma 2.1.12, $\mathcal{F} := \rho(\Gamma(E_{-1}))$, a singular foliation called the *basic singular foliation* of $(E, (\ell_k)_{k\geq 1}, \rho)$. We say then, that the Lie ∞-algebroid $(E, (\ell_k)_{k\geq 1}, \rho)$ is *over* $\mathcal{F}$.

Theorem 2.7.24

There is a one-to-one correspondence between

1. *negatively graded Lie ∞-algebroids $(E, (\ell_k)_{k\geq 1}, \rho)$,*
2. *NQ-manifolds (E, Q).*

Moreover, both structures correspond one to the other through the following relations:

1. *for all $f \in \mathcal{O}$, $x \in \Gamma(E_{-1})$*
$$\langle Q(f), x\rangle = \rho(x)[f], \tag{2.37}$$
2. *for all $\xi \in \Gamma(E^*)$ and $x \in \Gamma(E)$:*
$$\langle Q^{(0)}(\xi), x\rangle = (-1)^{|\xi|}\langle \xi, \ell_1(x)\rangle \tag{2.38}$$
   .
3. *for all homogeneous elements $x, y \in \Gamma(E)$ and $\alpha \in \Gamma(E^*)$*
$$\langle Q^{(1)}(\xi), x \odot y\rangle = \rho(x)[\langle \xi, y\rangle] - \rho(y)[\langle \xi, x\rangle] - \langle \xi, \ell_2(x, y)\rangle, \tag{2.39}$$
   with the understanding that the anchor ρ vanishes on E_{-i} when $i \geq 1$.
4. *for every $n \geq 3$, the k-ary brackets $\ell_n\colon \Gamma(S^k_{\mathbb{K}}(E)) \longrightarrow \Gamma(E)$ and the arity $k-1$ component $Q^{(n-1)}\colon \Gamma(E^*) \longrightarrow \Gamma(S^k_{\mathbb{K}}(E^*))$ of Q are dual to each other.*

Where $\langle \cdot\,, \cdot\rangle$ stands for the duality pairing between sections of a vector bundle and sections of its dual.

Example 2.7.25. Let $(A, [\cdot\,,\cdot\,]_A, \rho)$ be Lie algebroid concentrated in degree -1. The graded manifold $(M, \mathcal{E} = \Gamma(\wedge A^*))$ carries a *dg*-manifold structure Q which is given by

$$\langle Q[f], a\rangle = \rho(a)[f]$$
$$\langle Q[\xi], a \wedge b\rangle = \rho(a)[\langle \xi, b\rangle] - \rho(b)[\langle \xi, a\rangle] - \langle \xi, [a, b]_A\rangle$$

for $f \in \mathcal{O}, \xi \in \Gamma(A^*)$ and $a, b \in \Gamma(A)$. This is sufficient to extend Q by derivation on $\mathcal{E}$. One can check that $Q^2 = 0$ because of Jacobi identity.

Definition 2.7.26

Let $(E', (\ell'_k)_{k\geq 1}, \rho')$ resp. $(E, (\ell_k)_{k\geq 1}, \rho)$ be negatively graded Lie ∞-algebroids and (E', Q') resp. (E, Q) their corresponding NQ-manifolds. A Lie ∞-algebroids morphism *or* Lie ∞-morphism *between $(E', (\ell'_k)_{k\geq 1}, \rho')$ and $(E, (\ell_k)_{k\geq 1}, \rho)$ is a morphism of NQ-manifolds between (E, Q) and (E', Q').*

From now on, we will denote by (E, Q) a Lie ∞-algebroid $(E, (\ell_k)_{k\geq 1}, \rho)$. This notation is justified by Theorem 2.7.4.

Homotopic Lie ∞-algebroids

Definition 2.7.27. Let (E', Q') and (E, Q) be Lie ∞-algebroids over M. A path $t \mapsto \Phi_t$ in the space made of Lie ∞-morphisms from E' to E is said to be *piecewise-smooth* if for every $k \in \mathbb{N}_0$ the map $t \mapsto \phi_k(t)$ which is induced by the Taylor coefficients of arity k is a piecewise-$\mathcal{C}^\infty$ path in $\Gamma(S^{k+1}(E'^*)\otimes E)$ (which is also continuous even at the junction points).

Definition 2.7.28: Homotopies

Let (E, Q) and (E', Q') be Lie ∞-algebroids over M with sheaf of functions $\mathcal{E}$ and $\mathcal{E}'$.

1. *Two Lie ∞-morphisms $\Phi, \Psi\colon (E, Q) \longrightarrow (E', Q')$ are said to be* homotopic *and denoted $\Phi \sim \Psi$ if there is a morphism of graded differential algebras:*

$$(\mathcal{E}, Q) \longrightarrow (\mathcal{E}' \otimes \Omega^\bullet([0, 1]), Q' \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{d}_{dR}) \tag{2.40}$$

   which coincides with Φ and Ψ at $\{0\}$ and $\{1\}$, respectively.

   Equivalently, say $\Phi \sim \Psi$, consists of:

   (a) *a (continuous) piecewise-smooth path $t \mapsto \Phi_t$ valued in Lie ∞-algebroid morphisms between E and E' such that:*

$$\Phi_0 = \Phi \quad \text{and} \quad \Phi_1 = \Psi,$$

   (b) *a piecewise-smooth path $t \mapsto H_t$ valued in Φ_t-derivations of degree -1, such that the following equation:*

$$\frac{d\Phi_t}{dt} = Q' \circ H_t + H_t \circ Q \tag{2.41}$$

   holds for every $t \in [0, 1]$ where it is defined.

   Therefore, we say that the pair (Φ_t, H_t) is a homotopy *between Φ and Ψ.*

2. *(E, Q) and (E', Q') are said to be* homotopic *if there is a pair of Lie ∞-morphisms*

$$(E, Q) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} (E', Q')$$

   whose compositions $\Phi \circ \Psi\colon \mathcal{E} \longrightarrow \mathcal{E}$ and $\Psi \circ \Phi\colon \mathcal{E}' \longrightarrow \mathcal{E}'$ are homotopy equivalent to the identity map $\mathrm{id}_{\mathcal{E}}$ and $\mathrm{id}_{\mathcal{E}'}$, respectively.

Exercice 2.7.29. Let (E, Q) and (E', Q') be Lie ∞-algebroids over M. Show that Definition 2.7.4 implies that for every pair of homotopic Lie ∞-morphisms $\Phi, \Psi \colon (E', Q') \longrightarrow (E, Q)$, there exists an $\mathcal{O}$-linear map $H \colon \mathcal{E} \longrightarrow \mathcal{E}'$ of degree -1 such that:

$$\Psi - \Phi = Q' \circ H + H \circ Q. \tag{2.42}$$

Exercice 2.7.30. Check that the two formulations of homotopies between Lie ∞-morphisms given in Definition 2.7.4 are indeed equivalent.

Exercice 2.7.31. Show that homotopy of Lie ∞-algebroid morphisms is an equivalence relation and also this equivalence relation is compatible with composition.

Exercice 2.7.32. Let (E, Q) and (E, Q) be Lie ∞-algebroids over M and let $w \in \Gamma(S^{r+1}(E'^*) \otimes E)$ be a section of degree -1 for some $r \geq 0$. For a Lie Φ-algebroid morphism $\Phi \colon \mathcal{E} \longrightarrow \mathcal{E}'$ from (E', Q) to (E, Q), let $H^\Phi \colon \mathcal{E} \longrightarrow \mathcal{E}'$ be the $\mathcal{O}$-linear Φ-derivation where the only non zero Taylor coefficient is w.

1. Check that the following differential equation:

$$\begin{cases} \frac{d\Phi_t}{dt} & = Q' \circ H^{\Phi_t} + H^{\Phi_t} \circ Q \\ \Phi_0 & = \Phi. \end{cases} \tag{2.43}$$

   has a unique solution for all $t \in \mathbb{R}$.

2. Show that $(\Phi_t, H^{\Phi_t})_{t \in [0,1]}$ is a homotopy between the Lie ∞-algebroid morphism Φ and the Lie ∞-algebroid morphism Φ_1.

2.7.5 NQ-manifolds and singular foliations

We have seen in the previous section that any Lie ∞-algebroid over M has an induced singular foliation which is its basic foliation. Now we will analyse the opposite direction, i.e., given a singular foliation $\mathcal{F} \subseteq \mathfrak{X}(M)$, can we find a Lie ∞-algebroid over M whose basic foliation is $\mathcal{F}$? in case where it exists do we have uniqueness?

All results below also hold for free resolutions by $\mathcal{O}$-modules and the following statements and their proofs can be deduced from general results on Lie-Rinehart algebras [**?**].

Lemma 2.7.33. *Let (E, d, ρ) a geometric resolution of $\mathcal{F}$. Every almost Lie algebroid structure $(E_{-1}, [\cdot\,, \cdot]_{E_{-1}}, \rho)$ on $E_{-1} \subset E$ can be extended to an almost Lie algebroid structure on E.*

The next theorem is obtained by proving that every graded almost Lie algebroid over a geometric resolution can be extended to a (unique up to homotopy) Lie ∞-algebroid structure. It appeared first in an explicit form in the PhD of Sylvain Lavau [**?**], followed by a referred version by CLG, S. Lavau and T. Strobl in [LGLS20], but the authors acknowledge it was discussed several year earlier by Ralph Mayer and Chenchang Zhu - and also Teodor Voronov and his collaborators in a slightly different context. This result is generalized later for arbitrary Lie-Rinehart algebras over a commutative unital algebra in **??**.

Theorem 2.7.34: Beyond almost Lie algebroids

Let $\mathcal{F}$ be a singular foliation on a manifold M.

1. *Any resolution of $\mathcal{F}$ by free $\mathcal{O}$-modules (which is not necessarily a geometric resolution)*

$$\cdots \xrightarrow{\mathrm{d}} P_{-3} \xrightarrow{\mathrm{d}} P_{-2} \xrightarrow{\mathrm{d}} P_{-1} \xrightarrow{\rho} \mathcal{F} \longrightarrow 0 \tag{2.44}$$

   carries a Lie ∞-algebroid structure over $\mathcal{F}$ whose unary bracket is $\ell_1 := \mathrm{d}$.

2. *In particular, when $\mathcal{F}$ admits a geometric resolution (E, d, ρ), there exists a Lie ∞-algebroid (E, Q) over $\mathcal{F}$ whose linear part is (E, d, ρ).*

Proof. Apply Theorem 2.1 in [**?**] to $\mathcal{F}$ seen as a Lie-Rinehart algebra over $\mathcal{O}$. $\square$

Definition 2.7.35: Universal dg-manifolds: definition

We call universal Lie ∞-algebroid of a singular foliation *any Lie ∞-algebroid whose linear part is a geometric resolution of $\mathcal{F}$.*

The name "universal" is justified: it is indeed a universal object in the category of Lie ∞-algebroids whose anchor is valued inside $\mathcal{F}$. Arrows of that category are defined to be homotopy classes of morphisms.

Theorem 2.7.36: Universal dg-manifold deserve the name "universal"

Let $\mathcal{F}$ be a singular foliation over M. Given,

a) a Lie ∞-algebroid $(M, \mathcal{E}', Q')$ that terminates in $\mathcal{F}$, i.e, $\rho'(\Gamma(E_{-1})) \subseteq \mathcal{F}$,

b) a universal Lie ∞-algebroid $(M, \mathcal{E}, Q)$ of $\mathcal{F}$,

then

1. *there exists a Lie ∞-morphism from $(M, \mathcal{E}', Q')$ to $(M, \mathcal{E}, Q)$.*
2. *and any two such morphisms are homotopic.*

Here is an immediate corollary of this result, which is valid for any universal object in any category.

Corollary 2.7.37: The universal Lie ∞-algebroid is as unique as can be

Two universal Lie ∞-algebroid of a singular foliation are homotopy equivalent. Moreover, the homotopy equivalence between them is unique up to homotopy.

Here are some examples of universal Lie ∞-algebroids of singular foliations

Example 2.7.38. For a regular foliation $\mathcal{F}$ on a manifold M, the Lie algebroid $T\mathcal{F} \subset TM$, whose sections form $\mathcal{F}$, is a universal Lie ∞-algebroid of $\mathcal{F}$.

Exercice 2.7.39. We go back to example 2.6.21. Check that a universal Lie ∞-algebroid of $\mathcal{F}_\varphi \subset \mathfrak{X}(V)$ is given on the free resolution $\left(E_{-\bullet} = \wedge^{\bullet+1}V, \mathrm{d} = \iota_{\mathrm{d}\varphi}, \rho = -\iota_{\mathrm{d}\varphi}\right)$ by defining the following n-ary brackets:

$$\{\partial_{I_1}, \cdots, \partial_{I_n}\}_n := \sum_{i_1 \in I_1, \ldots, i_n \in I_n} \epsilon(i_1, \ldots, i_n)\varphi_{i_1 \cdots i_n}\partial_{I_1^{i_1}\bullet \cdots \bullet I_n^{i_n}}; \tag{2.45}$$

and the anchor map given for all $i, j \in \{1, \ldots, n\}$ by

$$\rho\left(\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}\right) := \frac{\partial \varphi}{\partial x_j}\frac{\partial}{\partial x_i} - \frac{\partial \varphi}{\partial x_i}\frac{\partial}{\partial x_j}. \tag{2.46}$$

Above, for every multi-index $J = \{j_1, \ldots, j_n\} \subseteq \{1, \ldots, d\}$ of length n, ∂_J stands for the n-vector field $\frac{\partial}{\partial x_{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{j_n}}$ and $\varphi_{j_1 \cdots j_n} := \frac{\partial^n \varphi}{\partial x_{j_1} \cdots \partial x_{j_n}}$. Also, $I_1 \bullet \cdots \bullet I_n$ is a multi-index obtained by concatenation of n multi-indices $I_1, \ldots, I_n$. For every $i_1 \in I_1, \ldots, i_n \in I_n$, $\epsilon(i_1, \ldots, i_n)$ is the signature of the permutation which brings $i_1, \ldots, i_n$ to the first n slots of $I_1 \bullet \cdots \bullet I_n$. Last, for $i_s \in I_s$, we define $I_s^{i_s} := I_s \backslash i_s$.

2.7.6 The isotropy Lie ∞-algebra of a singular foliation at a point

So far, to any singular foliation we have associated a homotopy equivalence class of Lie ∞-algebroids. Now, given a point m, there is a functor:

$$Istropy_m \colon \{ \text{ Lie } \infty\text{-algebroids } \} \longrightarrow \{ \text{ Lie } \infty\text{-algebras } \}$$

that we describe in the next lines. Then, we apply this functor to the universal Lie ∞-algebroids at an arbitrary point m, and explain why the henceforth obtained Lie ∞-algebras deserve to be called isotropy Lie ∞-algebras by relating then to AS-isotropy Lie algebras.

Specialization of a Lie ∞-algebroid at a point

- Let $(M, \mathcal{E}, Q) = ((E_\bullet, \ell_\bullet, \rho))$ be a Lie ∞-algebroid with anchor ρ. For every point $m \in M$, the k-ary brackets restrict to the graded vector space

$$ev(E, m) := \left(\bigoplus_{i \geq 2} E_{-i|_m} \right) \oplus \ker(\rho_m)$$

and equipped the latter with a Lie ∞-algebra structure that we denote by $Istropy_m(\mathcal{E}, Q)$ .For every $k \geq 1$, the restriction goes as follows:

$$\{x_1, \ldots, x_k\}_k := \ell_k(s_1, \ldots, s_k)_{|m} \tag{2.47}$$

for all $x_1, \ldots, x_k \in ev(E, m)$ and $s_1, \ldots, s_k \in \Gamma(E)$ sections of E such that $s_i(m) = x_i$ with $i = 1, \ldots, k$. These brackets are well-defined. It is clear that for $k \neq 2$, since ℓ_k is linear over functions. But it is not immediate that the 2-ary bracket is well-defined as well. Let us check that.

On one hand, the new brackets $\{\cdots\}_k$ have values in $ev(E, m)$ for degree reasons, except maybe for the 2-ary bracket when applied to elements of degree -1 (i.e. elements of the kernel of ρ_m) but in that case it is in the kernel of ρ_m since

$$\begin{aligned}\rho_m(\{x_1, x_2\}_2) &= \rho_m(\ell_2(s_1, s_2)_{|m}) \\ &= \rho(\ell_2(s_1, s_2))_{|m} \\ &= [\rho(s_1), \rho(s_2)]_{|m} = 0\end{aligned}$$

In the last line we have used the fact that the Lie bracket of two vector fields that vanish at m is a vector field that vahishes again at m.

On the other hand, the 2-ary bracket $\{\cdot\,,\cdot\}_2$ is also well-defined when applied to elements of degree less or equal to -2, we need to verify when we take the bracket with at least an element of degree -1. Let $(e_1^i, \ldots e^i_{\mathrm{rk}(E_{-i})})$ be a local trivialization of E_{-i} on a neighbourhood $\mathcal{U}$ of the point $m \in M$. For $x_1 \in \ker(\rho_m)$ and $x_2 \in E_{-i|_m}$ write

$$x_1 = \sum_{k=1}^{\mathrm{rk}(E_{-1})} \lambda_k e_k^1(m), \quad x_2 = \sum_{k=1}^{\mathrm{rk}(E_{-i})} \mu_k e_k^i(m)$$

for some scalars (λ_i) in $\mathbb{K}$. The scalars (λ_k), (μ_k) extend to functions (f_k), (g_k) on $\mathcal{U}$. Therefore, we have

$$\{x_1, x_2\}_2 = \ell_2(s_1, s_2)_{|m}$$

with

$$s_1 = \sum_{k=1}^{\mathrm{rk}(E_{-1})} f_k e_k^1, \quad s_2 = \sum_{k=1}^{\mathrm{rk}(E_{-i})} g_k e_k^i.$$

If $\widetilde{s}_2$ is another extension of x_2, then $(s_2 - \widetilde{s}_2)(m) = 0$ and this is equivalent to $(g_k - \widetilde{g}_k)(m) = 0$ for $k = 1, \ldots, \mathrm{rk}(E_{-i})$. It follows that

$$\begin{aligned}\ell_2(s_1, s_2 - \widetilde{s}_2)_{|m} &= \sum_{k=1}^{\mathrm{rk}(E_{-i})} \ell_2\left(s_1, (f_k - \widetilde{g}_k) e_k^i\right)_{|m} \\ &= \sum_{k=1}^{\mathrm{rk}(E_{-i})} \cancel{(f_k - \widetilde{g}_k)(m)} \ell_2\left(s_1, e_k^i\right)_{|m} + \cancel{\rho(s_1)_{|m}[f_k - \widetilde{g}_k]} e_k^i \\ &= 0.\end{aligned}$$

- Any Lie ∞-morphism of algebroids $\Phi\colon (M,\mathcal{E}',Q')\to(M,\mathcal{E},Q)$ induces a graded Lie algebra morphism $\Phi_{|_m}\colon S^\bullet(V'_{|m})\to S^\bullet(V_{|m})$ since it is $\mathcal{O}$-linear. The 0-th Taylor coefficient $\phi_0\colon E_\bullet\to E'_\bullet$ of Φ restrits to Lie ∞-morphism of algebras

$$Istropy_m(\Phi)\colon S^\bullet(ev(E',m)^*)\to S^\bullet(ev(E,m)^*).$$

- Let $\mathcal{F}$ be the basic singular foliation associated to $(M,\mathcal{E},Q)$ i.e. $\mathcal{F}=\rho(E_{-1})$. We define the graded vector space

$$H(\mathcal{F},m):=\bigoplus_{i\geq 1}H^{-i}(E_\bullet,m) \tag{2.48}$$

which is actually the cohomology group of the complex

$$\cdots\xrightarrow{\ell_{1|m}}E_{-3|_m}\xrightarrow{\ell_{1|m}}E_{-2|_m}\xrightarrow{\ell_{1|m}}\ker(\rho_m)\longrightarrow 0.$$

One can check when $(M,\mathcal{E},Q)$ is universal for $\mathcal{F}$, the graded space (2.48) does not depend on the underlying geometric resolution of $\mathcal{F}$.

The isotropy Lie ∞-algebra of a singular foliation at a point

We assume that $(M,\mathcal{E},Q)$ is universal for $\mathcal{F}$. Note that the Lie ∞-algberoid obtained by specialising at some point $m\in M$ does not induce directly a Lie ∞-algberoid on the graded space $H(\mathcal{F},m)$ but the 2-ary bracket $\{\cdot\,,\cdot\}_2$ goes to quotient directly on elements of degree -1 i.e. to $H^{-1}(\mathcal{F},m)$, because

$$\{\mathrm{d}^{(2)}_m(x_1),x_2\}_2=\mathrm{d}^{(2)}_m(\{x_1,x_2\}_2)$$

for all $x_1\in E_{-2|_m}$ and $x_2\in\ker(\rho_m)$. That endows $H^{-1}(\mathcal{F},m)$ with Lie algebra structure.

Proposition 2.7.40. *The Androulidakis and Skandalis isotropy Lie algebra $\mathfrak{g}_m=\frac{\mathcal{F}(m)}{I_m\mathcal{F}}$ of the singular foliation $\mathcal{F}$ at a point $m\in M$, is isomorphic to $H^{-1}(\mathcal{F},m)$ w.r.t the induced Lie algebra structure.*

Proof. For $m\in M$, we construct a Lie algebra is isomorphism $\zeta\colon\frac{\ker(\rho_m)}{\mathrm{im}(\mathrm{d}^{(2)}_m)}\to\mathfrak{g}_m$ as follows: For an element $u\in\ker(\rho_m)$, let $\widetilde{u}$ be an extension of u to a local section on E_{-1}. By construction one has $\rho(\widetilde{u})\in\mathcal{F}(m)$. Let $\widetilde{\rho}_m$ be the surjective linear map defined by

$$\widetilde{\rho}_m\colon\ \ker(\rho_m)\longrightarrow\mathfrak{g}_m,\ u\longmapsto[\rho(\widetilde{u})].$$

Since any other extension $\widetilde{u}$ for u differs from the first one by a section in $\mathcal{I}_m\Gamma(E_{-1})$, the map $\widetilde{\rho}_m$ is well-defined. Surjectivity is due to the fact that every vector field of $\mathcal{F}$ vanishing at $m\in M$ is of the form $\rho(e)$ with e a (local) section of E_{-1} whose value at m belongs to $\ker(\rho_m)$. In addition, it is not hard to see that $\widetilde{\rho}_m$ is a morphism of brackets.

It remains to show that $\ker(\widetilde{\rho}_m)=\mathrm{im}(\mathrm{d}^{(2)}_m)$: let $u\in\ker(\widetilde{\rho}_m)\subset\ker(\rho_m)$ and $\widetilde{u}$ be a local section of E_{-1} that extends u. By definition of u, the class of $\rho(\widetilde{u})$ is zero in $\mathfrak{g}_m$, therefore, there exists some functions $f_i\in\mathcal{I}_m$ and $X_i\in\mathcal{F}, i=1,\ldots,k$, local generators such that $\rho(\widetilde{u})=\sum_{i=1}^k f_iX_i$. This implies that

$$\rho(\widetilde{u}-\sum_{i=1}^k f_ie_i)=0.$$

where for $i=1,\ldots,k$, e_i is a (local) section of E_{-1} whose image through ρ is X_i. Since $(E_\bullet,\mathrm{d}^\bullet,\rho)$ is a geometric resolution, there exists a (local) section $q\in\Gamma(E_{-2})$ such that

$$\widetilde{u}=\sum_{i=1}^k f_ie_i+\mathrm{d}^{(2)}q \tag{2.49}$$

By evaluating Equation (2.49) at m, we find out that $u\in\mathrm{im}(\mathrm{d}^{(2)}_m)$. Conversely, for $v\in E_{-2|_m}$, choose a (local) section q of E_{-2} through v. Therefore, $\mathrm{d}^{(2)}q\in\ker\rho$, is a (local) extension of $\mathrm{d}^{(2)}_m v\in\mathrm{im}(\mathrm{d}^{(2)}_m)$. The image of $\mathrm{d}^{(2)}_m v$ through $\widetilde{\rho}_m$ is obviously zero. This proves that $\ker(\widetilde{\rho}_m)=\mathrm{im}(\mathrm{d}^{(2)}_m)$. $\square$

However, if the underlying complex of $(M, \mathcal{E}, Q)$ is minimal at m then, for every $i \geq 2$, the vector space $H^{-i}(\mathcal{F}, m)$ is canonically isomorphic to $E_{-i|_m}$. Also, $H^{-1}(\mathcal{F}, m)$ is canonically isomorphic to $\ker(\rho_m)$.

Definition 2.7.41. Let $(M, \mathcal{E}, Q)$ be a universal Lie ∞-algebroid of a singular foliation $\mathcal{F}$ whose underlying complex is minimal at m. Then, $H(\mathcal{F}, m)$ carries a Lie ∞-algebra structure given by $Istropy_m(\mathcal{E}, Q)$ called the *isotropy Lie ∞-algebra of the singular foliation $\mathcal{F}$ at m.*

One can show that this definition is independent of any choices made in the construction.

Remark 2.7.42. By Proposition 2.7.40, the isotropy Lie algebra of the singular foliation $\mathcal{F}$ at a point $m \in M$ in the sense of Androulidakis and Skandalis, is isomorphic to the degree minus one component $H^{-1}(\mathcal{F}, m) \simeq \ker(\rho_m)$ of the isotropy Lie ∞-algebra of $\mathcal{F}$ at m.

Chapter 3

State of the Art and open questions

3.1 Open questions

3.1.1 Existence of Lie algebroids generating a singular foliations

Let us start with the most intriguing open question about singular foliations.

Let us start by making the vocabulary precise. So far, it was part of the definition of a "Lie algebroid $(A \to M, \rho, [\cdot,\cdot])$" that $A \to M$ had to be a finite rank vector bundle over M, i.e. that $A \to M$ is a vector bundle modeled over a finite dimensional vector space. In this section however, let us distinguish:

1. *finite rank Lie algebroids*, i.e. Lie algebroids as defined so far, with $A \to M$ a finite rank vector bundle,
2. *infinite rank Lie algebroids*, which have precisely the same definition, except that $A \to M$ is now a vector bundle of infinite rank.

As we saw in Section 1.3.2, for any finite rank Lie algebroid $(A \to M, \rho, [\cdot,\cdot])$, the image of the anchor map $\mathcal{F} = \rho(\Gamma(A))$ is a singular foliation on M.

Exercice 3.1.1. Let $(A \to \mathcal{U}, [\cdot,\cdot], \rho)$ be an infinite rank Lie algebroid. Check that $\mathcal{F} = \rho(\Gamma(A))$ is

1. a $\mathcal{C}^\infty(M)$-module,
2. involutive, i.e. $[\mathcal{F},\mathcal{F}] \subset \mathcal{F}$.

Example 3.1.2. Here is an example for which $\mathcal{F}$ is not locally finitely generated as a $\mathcal{C}^\infty(M)$-module but still comes from an infinite dimensional Lie algebroid:

a) $M = \mathbb{R}$,

b) A is the trivial vector bundles with generators indexed $(e_i)_{i\ \in \mathbb{N}}$ indexed by $\mathbb{N}$,

c) the anchor map is $\rho(e_i) = \frac{1}{x^i} e^{-\frac{1}{x^2}} \frac{\partial}{\partial x}$,

d) the Lie bracket is defined by $[e_i, e_j] = (j-i)e^{-\frac{1}{x^2}} e_{i+j+1}$.

Here is simple open question, that - as far as we know - first appeared in Androulidakis and Zambon's [AZ13b].

Question 3.1.3: [AZ13b]Lie algebroid?

Let $\mathcal{F}$ be a singular foliation on a manifold M. Does every point m admit a neighborhood $\mathcal{U}$ on which there exists a Lie algebroid structure $(A \to \mathcal{U}, [\cdot,\cdot], \rho)$ such that $\mathcal{F} = \rho(\Gamma(A))$?

Here is a slightly more general formulation of the question:

Question 3.1.4: [?] Lie algebroid (version II)?

Is any finitely generated singular foliation the image through the anchor of a finite rank Lie algebroid?

In addition to the local problem, there is also a global "gluing" problem.

Question 3.1.5: Lie algebroid?

Is a smooth singular foliation is the image of the Lie algebroid on open subsets $\mathcal{U}_1$,$\mathcal{U}_2$, is it the image of a Lie algebroid on $\mathcal{U}_1 \cup \mathcal{U}_2$?

Even if we assume both Lie algebroid structures to be defined on the restrictions to $\mathcal{U}_1$ and $\mathcal{U}_2$ of the same vector bundle, Question **??** remains non-trivial.

Example 3.1.6. Singular foliations whose number of generators are not globally bounded can clearly not be, globally, the image through the anchor map of a finite rank Lie algebroid. Hence the singular foliation of Example is not the image through the anchor map of a finite rank Lie algebroid on the whole $\mathbb{R}^2$.

Exercice 3.1.7. The purpose of this exercise is to show that any finitely generated singular foliation is the image through the anchor map of a infinite rank Lie algebroid.

1. Let $X_1, \dots, X_d$ be vector fields on a manifold M, and let $\mathfrak{g}^d_{free}$ be the free Lie algebra with d-generators $e_1, \dots, e_d$. Show that exists a unique Lie algebra morphism $\rho\colon \mathfrak{g}^d_{free} \to \mathfrak{X}(M)$ such that $\rho(e_i) = X_i$.

2. Assume now that $X_1, \dots, X_d$ are generators of a singular foliation $\mathcal{F}$. Use the previous Lie algebra morphism to construct a Lie algebroid structure on the trivial bundle $\mathfrak{g}^d_{free} \times M \to M$ such that the image of its anchor map is $\mathfrak{F}$.

Quite a few singular foliations are the image through the anchor map of a Lie algebroid: symplectic foliations of Poisson structures for instance, or orbits of a Lie algebra action. Here is an example of a singular foliation of rank 6 for which no Lie algebroid is known.

Question 3.1.8: A frustrating example

Is the singular foliation of vector fields on $\mathbb{R}^2$ vanishing quadratically[a] *at the origin* 0 *the image through the anchor map of a finite rank Lie algebroid ?*

[a]See section 1.3.4

Here are other examples of singular foliations for which no finite rank Lie algebroid is known, except in some particular cases:

1. vector fields on $\mathbb{C}^n$ tangent to a given affine variety $W \subset \mathbb{C}^n$,

2. vector fields on $\mathbb{C}^n$ vanishing at every point an affine variety $W \subset \mathbb{C}^n$,

3. vector fields $X \in \mathfrak{X}(\mathbb{C}^n)$ such that $X[\varphi] = 0$ for some polynomial function $\varphi \in \mathbb{C}[X_1, \dots, X_n]$ (see Example).

Exercice 3.1.9. Let $\varphi \in \mathbb{C}[x, y, z]$ be a polynomial function on $\mathbb{C}^3$. Check that the following bivector field :

$$\{x, y\} = \frac{\partial \varphi}{\partial z}\,,\ \{y, z\} = \frac{\partial \varphi}{\partial x}\,,\ \{z, x\} = \frac{\partial \varphi}{\partial y}$$

is a Poisson bivector field, and that φ is a Casimir. Consider the corresponding Lie algebroid on $A = T^*\mathbb{C}^3$. Show that the image of its anchor map is a sub-singular foliation of the singular foliation $\mathcal{F}_\varphi$ of all vector fields $X \in \mathfrak{X}(\mathbb{C}^n)$ such that $X[\varphi] = 0$. Show that if φ is weight homogeneous with an isolated singularity at zero, then $\rho(\Gamma(A)) = \mathcal{F}_\varphi$.

Exercice 3.1.10. Show that any singular foliation $\mathcal{F}$ of rank $k = 1, 2$ comes from a Lie algebroid. (Hint: construct an almost Lie algebroid of rank k over $\mathcal{F}$ and show that its Jacobiator is trivial).

Discussion Question **??** may be misleading, in the sense that that "behind" a singular foliation is a Lie ∞-algebroid[1]. The Lie algebroid, even if there is one, is certainly not unique (one could take the direct product with any Lie algebra for instance). But the universal Lie ∞-algebroid is unique (up to homotopy, see Corollary **??**), so that any homotopy invariant information obtained out of a universal Lie ∞-algebroid is canonically attached to the singular foliation.

Moreover, the universal Lie ∞-algebroid itself gives some hints about a possible Lie algebroid that whose image through the anchor map would be the singular foliation.

Not of rank r It is shown in [LGLS20] that some singular foliations of rank r are not the image through the anchor map of a Lie algebroid of rank r. In fact, the following result is shown in [LGLS20]:

Proposition 3.1.11: No minimal rank Lie algebroid

The singular foliations of all vector fields X on $\mathfrak{C}^4$ such that $X[\phi] = 0$ with $\phi(z_1, z_2, z_3, z_4) = z_1^3 + z_2^3 + z_3^3 + z_4^3$:

1. *has rank* 6,
2. *is not the image through the anchor map of a Lie algebroid of rank* 6.

This relatively elementary result uses the universal Lie ∞-algebroid. In fact, what is shown in [LGLS20] is that if a Lie algebroid of rank r exists in a neighborhood of a leaf reduced to a point, say m, then the holonomy Lie ∞-algebra at m admits a minimal model whose 3-ary bracket vanishes. Now, there are cohomological obstructions to such a cancellation. Here is the exact statement:

Proposition 3.1.12. *[?] A singular foliation, defined in a neighborhood of $0 \in \mathbb{R}^n$ and of rank r at this point, which admits a geometric resolution, and for which the 3-ary bracket of any minimal model of the Lie ∞-isotropy Lie algebra at 0 is not exact as a Chevalley-Eilenberg cocycle for the isotropy Lie algebra at 0 can not be the image through the anchor map of a Lie algebroid of rank r.*

Let us state a striking corollary of this statement. Let $X_1, \dots, X_r$ be generators of a singular foliation $\mathcal{F}$. There exists Christoffel coefficients, i.e. functions c_{ij}^k (with $i, j, k = 1, \dots, r$ satisfying

$$[X_i, X_j] = \sum_{k=1}^{r} c_{ij}^k X_k$$

but those are not unique, since there are relations between the generators. Without any loss of generality, we can assume

$$c_{ij}^k = -c_{ij}^k, \tag{3.1}$$

and, since the Jacobi identity holds, we have:

$$\begin{aligned} 0 &= [X_i, [X_j, X_k]] + c.p. \\ &= \sum_{a=1}^{r} \left(X_i[c_{jk}^a] + \sum_{b=1}^{r} c_{ij}^b c_{bk}^a + c.p. \right) X_a \end{aligned}$$

If for every $a \in \{1, \dots, r\}$,

$$X_i[c_{jk}^a] + \sum_{b=1}^{r} c_{ij}^b c_{bk}^a \; + c.p. = 0 \tag{3.2}$$

[1](= Q-manifold = dg-manifold)

then there exists a Lie algebroid of rank r whose image through the anchor map is $\mathcal{F}$: the Lie algebroid on a trivial bundle of rank r whose bracket is given by

$$[e_i, e_j] = \sum_{k=1}^{r} c_{ij}^k e_k$$

and whose anchor is $\rho(e_k) = X_k$ for all k. Proposition 3.1.1 explains that, if the isotropy Lie ∞-algebra at a point satisfies cohomological condition linked to its 3-ary bracket, then there is no way that coefficients c_{ij}^k could be found that satisfy (3.1) and (3.2).

An other relation between the universal Lie ∞-alegbroid and a Lie algebroid over $\mathcal{F}$. In [**?**], the following result is proven:

Proposition 3.1.13. *[**?**] If a Lie algebroid A over $\mathcal{F}$ exists, then there exists a universal Lie ∞-algebroid $(E_\bullet, Q)$ (with $E_{-1} = A$) for which the restriction to E_{-1} all the n-ary brackets are 0 for $n \geq 3$.*

This proposition makes the next question a natural one:

Question 3.1.14: Univ. Lie ∞-algebroids of SF coming from Lie algebroids

If a singular foliation (i) *admits a geometric resolution and* (ii) *is the image through the anchor map of a Lie algebroid, does it admit a universal Lie ∞-algebroids for which all n-ary brackets are zero for $n \geq 3$?*

So, true or false? Now, is the answer to Question **??** yes or no? Our guess is that the answer is "no", but it seems very hard to prove. For instance, we have the following conjecture, which rather goes in the direction "the Lie algebroid seem to exist" but certainly does not prove it, and leaves room to counter-examples:

Conjecture 3.1.15: Finding counter-examples is tricky.

At every point, the isotropy Lie ∞-algebra of a locally real analytic singular foliation is homotopy equivalent to a finite dimensional differential graded Lie algebra.

The conjecture implies, for instance, that it is not possible to have a homotopy Lie ∞-algebra of the form

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$$

with $\mathfrak{g}_{-1}$ a semi-simple Lie algebra, $\mathfrak{g}_{-2} = \mathbb{R}$, and the 3-ary bracket:

$$\wedge^3 \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$$

to be given by the Cartan 3-form, since the latter is not homotopy equivalent to a finite dimensional differential graded Lie algebra.

3.1.2 Neighborhood of non-simply connected leaves

Let us consider a leaf L of a smooth singular foliation. For simplicity, let us assume L admits a tubular neighborhood $([L, M], p)$ on which there exists an $\mathcal{F}$-connection which is complete, i.e. such that the horizontal curves stay inside $[L, M]$. It is not absolutely necessary to make this last assumption, upon replacing diffeomorphisms or symmetries by germs of diffeomorphisms.

It is not very hard to check that if L is a circle with parameter θ, then

1. parallel lift along the fundamental vector field $\frac{\partial}{\partial \theta}$ of the circle the $\mathcal{F}$-connection defines a symmetry ψ of the transverse foliation $\mathcal{T}$ on $p^{-1}(pt)$, with $pt \in S^1$ a point.

2. the suspension of ψ is isomorphic to the restriction of $\mathcal{F}$ to $[L, M]$.

Question 3.1.16: Neighborhoods

What happens when L is a torus? An arbitrary surface? And, of course, what happens for an arbitrary leaf?

For regular foliations, it is known that a there is a neighborhood of any saturated leaf L is entirely given by a group morphism from $\pi_1(L)$ to diffeomorphisms of an open ball. For singular foliation, the problem is much more involved, as shown in [**?**, LGR21].

3.1.3 Molino-Atiyah classes

To start the discussion, let us recall the construction of the Molino class of a regular foliation $\mathcal{F}$.

As its name indicates, the Molino class is a class in some cohomology: we first describe the cohomology to which it belongs.

Let $\mathcal{F}$ be a regular foliation on M, with tangent bundle $T\mathcal{F} \subset TM$. Notice that $\mathcal{F} = \Gamma(T\mathcal{F})$.

1. The tangent bundle $T\mathcal{F}$ is a Lie sub-algebroid of the tangent Lie algebroid TM, whose anchor map is the inclusion $T\mathcal{F} \hookrightarrow TM$.
2. Consider the *normal bundle* $N_\mathcal{F} := TM/T\mathcal{F}$. Denote by $u \mapsto \overline{u}$ the natural projection $TM \longrightarrow N_\mathcal{F} = TM/T\mathcal{F}$. The normal bundle comes equipped with a $T\mathcal{F}$-connection, called the *Bott connection*, and defined by:
$$\nabla^{Bott}_X \overline{u} = \overline{[X,u]}$$
for all $X \in \mathcal{F}$ and $u \in \Gamma(TM)$.
3. It follows from the Jacobi identity for vector fields on M that the Bott connection is a flat connection. As a consequence $X \mapsto \nabla^{Bott}_X$ turns $N_\mathcal{F}$ into a Lie algebroid representation of $T\mathcal{F}$.
4. The dual of a Lie algebroid representation of $T\mathcal{F}$, and the tensor or symmetric products of two Lie algebroid representations of $T\mathcal{F}$ being Lie algebroid representations of $T\mathcal{F}$ again, the vector bundle $S^2N^*_\mathcal{F} \otimes N_\mathcal{F}$ (i.e. the vector bundle of symmetric bilinear maps from the normal bundle to itself is a Lie algebroid representation of $T\mathcal{F}$.

The Molino class is a cohomology class of degree 1 for the Chevalley-Eilenberg cohomology of $T\mathcal{F}$ valued in the module $S^2N^*_\mathcal{F} \otimes N_\mathcal{F}$. By construction, it is therefore represented by a vector morphism:
$$\alpha \colon T\mathcal{F} \otimes S^2 N_\mathcal{F} \longrightarrow N_\mathcal{F},$$
which has to satisfy (in-order to be a closed-cocycle):
$$\alpha([X,Y],u,v) = \nabla^{Bott}_X \alpha(Y,u,v) - \alpha(Y, \nabla^{Bott}_X u, v) - \alpha(Y,u,\nabla^{Bott}_X v) - \Big(X \longleftrightarrow Y \Big).$$

Let us now construct the Molino class for a regular foliation.

1. Consider a TM-connection[2] ∇ on $N_\mathcal{F}$:
$$(X, \overline{u}) \mapsto \nabla_X \overline{u}$$
whose restriction to $\mathcal{F} \times \Gamma(N_\mathcal{F})$ is the Bott connection, i.e. such that for all $X \in \mathcal{F}$:
$$\nabla_X \overline{u} = \nabla^{Bott}_X \overline{u}.$$
   (a) Such connections always exists.
   (b) Without any loss of generality, we can assume that its torsion is zero. The *torsion* is the vector bundle morphism defined by
$$T^\nabla \colon \begin{array}{ccc} \wedge^2 TM & \to & TM \\ (X,Y) & \mapsto & \overline{\nabla_X \overline{Y} - \nabla_Y \overline{X} - [X,Y]} \end{array}.$$

[2] i.e. a linear connection is teh usual sense

From now on, we shall on, we will assume the torsion to be zero.

2. Consider the curvature κ^∇ of such a connection ∇. By construction, κ^∇ is a vector bundle morphism

$$\kappa^\nabla : TM \wedge TM \otimes N_{\mathcal{F}} \longrightarrow N_{\mathcal{F}}.$$

3. Since the Bott connection is flat, for any $X \in \mathcal{F}$, $\mathfrak{i}_X \kappa^\nabla : Y \longrightarrow \kappa^\nabla(X, Y)$ vanishes as soon as $Y \in \mathcal{F}$. It therefore can be seen as a vector bundle morphism $\overline{\mathfrak{i}_X \kappa^\nabla} : N_{\mathcal{F}} \otimes N_{\mathcal{F}} \longrightarrow N_{\mathcal{F}}$.

4. The map $X \mapsto \overline{\mathfrak{i}_X \kappa^\nabla}$ is therefore a vector bundle morphism from $\mathcal{F}$ to $T\mathcal{F} \otimes N_{\mathcal{F}} \otimes N_{\mathcal{F}} \longrightarrow N_{\mathcal{F}}$.

5. We leave it to the reader to check that the vanishing of the torsion implies that $X \mapsto \overline{\mathfrak{i}_X \kappa^\nabla}$ is symmetric in the two last variables, and is indeed a vector bundle morphism

$$\alpha^\nabla : T\mathcal{F} \otimes S^2 N_{\mathcal{F}} \longrightarrow N_{\mathcal{F}}.$$

The Bianchi identity implies that α satisfies (3.1.1) above, and is therefore a cocycle of the Chevalley-Eilberg cohomology of $T\mathcal{F}$ is the module $S^2 N_{\mathcal{F}}^* \otimes N_{\mathcal{F}}$, called the *Molino cocycle of the torsion-free connection* ∇. It can be shown that different choices of connections ∇ would give the same class in cohomology.

Proposition 3.1.17. *The Molino class is the obstruction to the existence of an extension of the Bott connection whose curvature* 2*-form is zero as soon as one element tangent to the foliation is applied to it.*

Question 3.1.18: Molino class and meaning?

What is the equivalent of the Molino (or Atiyah) class for a singular foliation? And what is its geometrical meaning?

Let us state a few points.

1. The Bott connection has a natural extension to the singular case:
   (a) The formula $(X, \overline{u}) \mapsto \overline{[X, u]}$ defines a flat Lie-Rinehart connection of $\mathcal{F}$ on the $\mathcal{C}^\infty(M)$-module $\mathfrak{X}(M)/\mathcal{F}$.
   (b) The adjoint representation "up to homotopy" of any any universal Lie ∞-algebroid of $\mathcal{F}$ is a flat Lie ∞-algebroid connection on a geometric resolution of the $\mathcal{C}^\infty(M)$-module $\mathfrak{X}(M)/\mathcal{F}$ that can also be understood as a generalization of the Bott connection, see [**?**, **?**].
   (c) The Molino class is an instance of Atiyah classes of Lie algebroid pairs.

2. Geometrically, the vanishing of the Molino class of a regular foliation has several consequences.
   (a) For any leaf L, and any $x \in L$, the holonomy:

   $$Hol_{\mathcal{F},L,l} : \pi_1(x, L) \longrightarrow Diff_0(N_{\mathcal{F}}|_x)$$

   valued in germs at 0 of diffeomorphisms of the normal bundle. If the Molino class vanishes, the holonomy is linearizable, i.e. the group morphism $Hol_{\mathcal{F},L,l}$ can be assumed to be valued in linear invertible endomorphisms of $N_{\mathcal{F}}|_x$. See, e.g., Theorem 8.5 in [**?**].
   (b) We say that two paths $\gamma_1, \gamma_2 : [0,1] \to M, \gamma_2$ are $\mathcal{F}$-related if there exists $F : [0,1]^2 \to M$ such that $F(t, 0) = \gamma_0(t), F(t, 1) = \gamma_1(t)$ and such that for every $t \in [0,1]$, the map $s \mapsto F(t, s)$ is in a fixed leaf. Notice that parallel transportation for ∇ along curves of the form $s \mapsto F(t, s)$ is simply parallel transportation with respect to the Bott-connection. If ∇ is a connection such that the Molino cocycle vanishes, the curvature of ∇ vanishes as soon as a vector tangent to

$\mathcal{F}$ is applied to it, and in particular on the image of F. As the consequence, the following diagram is commutative:

$$\begin{tikzcd} N_{\mathcal{F}}|_{\gamma_0(0)} \arrow[r, "\Phi^{bott}(F)"] \arrow[d] & N_{\mathcal{F}}|_{\gamma_1(0)} \arrow[d] \\ N_{\mathcal{F}}|_{\gamma_0(1)} \arrow[r, "\Phi^{bott}(F)"'] & N_{\mathcal{F}}|_{\gamma_1(1)} \end{tikzcd}$$

where horizontal lines are parallel transportation for the Bott connection, and vertical lines are parallel transportation with respect to ∇ along γ_0 and γ_1.

Question 3.1.19. *Assuming is has been defined, what is the geometrical meaning of the vanishing of the (to be constructed) Molino class for a singular foliation?*

3.1.4 Miscellaneous

Here is a "pot pourri" of several questions, mostly anecdotal at first sight, but to which we have no immediate answer.

Yahya Turki [Tur15] suggested the following notion: we say that a bivector field $\pi \in \Gamma(\wedge^2 TM)$ is *foliated* if $\pi^\sharp(\Omega^1(M))$ is closed under the Lie bracket, i.e. if is a singular foliation.

Example 3.1.20. Poisson bivector fields, but also twisted-Poisson bi-vector fields, are examples. Yahya Turki [?] gave examples of foliated bivector fields that are not of this type, but proved that they are twisted Poisson near any one of their regular points (= points in a neighborhood which $\pi^\#$ has constant rank).

Question 3.1.21. *Foliated bivector fields Let π be a foliated bivector field. Can a Lie algebroid structure with anchor map $\pi^\#$ be constructed on T^*M?*

It is known that T^*M comes equipped with a Lie algebroid structure with anchor $\pi^\sharp\colon T^*M \longrightarrow TM$ when π is twisted Poisson [], so the question makes sense.

Sébastien Michéa asked if for any smooth Poisson structure π on $\mathbb{R}^n$, there is an other structure π' on $\mathbb{R}^n$ which coincides with π in a neighborhood of 0 and vanishes outside a compact subset. The corresponding question for singular foliations is much easier:

Exercice 3.1.22. Given a smooth singular foliation $\mathcal{F}$ on $\mathbb{R}^n$, show that there exist an other singular foliation $\mathcal{F}'$ on $\mathbb{R}^n$ which coincides with $\mathcal{F}$ in a neighborhood of 0 and vanishes outside a compact subset.

Here is however, a more delicate question:

Question 3.1.23: Blocked by spheres?

Given a smooth singular foliation $\mathcal{F}$ on $\mathbb{R}^n$ such that all regular point have rank r, does there exist an other singular foliation $\mathcal{F}'$ on $\mathbb{R}^n$ such that all regular point have rank r, which coincides with $\mathcal{F}$ on the open ball $\sum_{i=1}^n x_i^2 < 1$, but which is made of vector fields all tangent to the sphere $\sum_{i=1}^n x_i^2 = 1$?

Existence of such "deformations" would make simpler to deal with neighborhoods of leaves.

3.1.5 Linearisation

Can we enlarge the classical theorems (by Conn [] or Zung []) about linearizations of Lie algebroid actions or Lie groupoid actions to the context of singular foliations or its holonomy groupoid?

These linearization theorems have the same logic. There are first relatively easy results whose patterns are:

$$\text{Fixed point } + \text{ Semi-simple } \Longrightarrow \text{ Formally Linearizable}$$

For instance, it is not so complicated to show that if a Lie algebroid $(A, \rho, [\cdot,\cdot])$ admits a point m where $\rho_m = 0$ and the isotropy Lie algebra $\mathfrak{g}_m = A_m$ is semi-simple, then the Lie algebroid is formally equivalent to the transformation Lie algebroid $\mathfrak{g}_m \times T_mM \to T_mM$ for some action of $\mathfrak{g}_m$ by linear endomorphisms T_mM. The theorem by Dominique Cerveau 3.1.1 is a result of this type for singular foliations.

Beyond these relatively easy results, there are then much more difficult results whose patterns are:

$$\text{Fixed point } + \text{ Compact } \Longrightarrow \text{ Locally Linear}$$

For instance, Conn's linearization theorem for Lie algebrois Here is natural question in that direction:

Question 3.1.24: Extend Conn's linearization for Lie algebroids

Let $\mathcal{F}$ be a singular foliation on a smooth manifold M made of vector fields that all vanish at a point m. If the isotropy Lie algebra $\mathfrak{g}$ of $\mathcal{F}$ at m is semi-simple of compact type, is there a neighborhood of m on which $\mathcal{F}$ is isomorphic to the singular foliation associated to some representation of the isotropy Lie algebra $\mathfrak{g}_m(\mathcal{F})$ on T_mM?

It suffices to prove that the short exact sequence

$$\begin{tikzcd} \mathcal{I}_m\mathcal{F} \arrow[r, hook] & \mathcal{F} \arrow[r, two heads] & \mathfrak{g}_m(\mathcal{F}) \end{tikzcd}$$

splits with a section σ:

$$\begin{tikzcd} \mathcal{I}_m\mathcal{F} \arrow[r, hook] & \mathcal{F} \arrow[r, two heads] & \mathfrak{g}_m(\mathcal{F}) \arrow[l, bend right, dotted, "\sigma"'] \end{tikzcd}$$

which is a Lie algebra morphism, at least in a neighborhood of m. Since any action of a semi-simple Lie algebra of compact type is linearizable near a fixed point, this section σ may be seen as being an action of $\mathfrak{g}_m(\mathcal{F})$ on a neighborhood of T_mM.

We could of course enlarge these questions to neighborhood of leaves. Again, the formal case is relatively easy: for instance it has been proven that [**?**]:

xxxxxxx

For Lie algebroids or Poisson structures, several authors [**?**, **?**, **?**] have proven recently several linearizations theorems in neighborhood of leaves of Lie algebroids or singular foliations: Pretty much any one of these theorems admit an equivalent for singular foliations.

There are similar questions about the holonomy groupoid. Recall that it is a topological groupoid, although it is not a Lie groupoid. The topology is the push-forward topology of any atlas of bisections that define it. It makes sense, therefore, to speak of a singular foliation $\mathcal{F}$ whose holonomy groupoid $Hol(\mathcal{F})$ is proper: it is a singular foliation for which

$$(s,t)\colon Hol(\mathcal{F}) \longrightarrow M \times M$$

is a proper map.

Definition 3.1.25. We say that a singular foliation $\mathcal{F}$ is *proper* if $Hol(\mathcal{F})$ is a proper topological groupoid.

Example 3.1.26. Consider a proper groupoid $\Gamma \rightrightarrows M$, e.g. the action groupoid associated to an action of a compact group on a manifold.

Then the basic singular foliation[3] is a proper singular foliation. This is easy proven from **??**

Proper groupoids have very strong linearization properties Here is a theorem by Nguyen Tien Zung[4]:

Theorem 3.1.27. Consider a proper Lie groupoid $\Gamma \rightrightarrows M$. Every fixed point $m \in M$ admits a saturated neighborhood $\mathcal{U}$ on which the restriction of Γ is isomorphic, as a Lie groupoid, to a transformation groupoid of the action of the compact isotropy group G_m on the tangent space $V = T_mM$.

It is therefore very natural to guess that the following result should be true:

[3] i.e. $\mathcal{F} = \rho(\Gamma(A))$ with $(A, \rho, [\cdot,\cdot])$ the Lie algebroid of $G \rightrightarrows M$

[4] Recall that for every fixed point $m \in M$ of a Lie groupoid (i.e. any point for which $t(s^{-1}(m)) = \{m\}$), the isotropy group at m acts naturally by linear automorphisms of the tangent space T_mM

Question 3.1.28: Extend Zung's linearization to SF

Consider a singular foliation $\mathcal{F}$ on M whose holonomy groupoid $Hol(\mathcal{F}) \rightrightarrows M$ is proper. Every fixed point $m \in M$ admits a saturated neighborhood $\mathcal{U}$ on which the restriction of $\mathcal{F}$ is isomorphic, as a singular foliation, to a transformation groupoid of some action of the compact isotropy group $Hol(\mathcal{F})_m$ on the tangent space $T_m M$.

For a regular foliations, properness of the holonomy Lie groupoid implies, for instance, that every leaf has a saturated neighborhoods on which the holonomy map is by linear automorphisms of a finite group. Also, proper Lie groupoids have several very strong properties: Nguyen Tien Zung [] has for instance proven that near a fixed point, it has to be isomorphic to a linear action of a compact Lie group by isometries of finite dimensional Euclidian space. Last, Poisson manifolds of compact type [CFMT19] have a rich and complicated geometry. Here is a natural question:

Question 3.1.29: Proper holonomy groupoid

What does properness (or compactness) of the Androulidakis-Skandalis holonomy Lie groupoid implies ? For instance, does it imply that, for any point, the transverse singular foliation is given by a linear action of a compact Lie group by isometries of a finite dimensional Euclidian space ?

It has been proven in [PTW21] that singular foliations arising form a compact Lie groupoid can be made a regular foliation by finitely many blow-up operations of its most singular leaves. It would be interesting to generalize this result to any singular foliation whose holonomy groupoid is compact: a positive answer to the previous question should do it.

3.2 Cohomologies of a singular foliation

We already saw that the derived cohomology $Tor_{\mathcal{C}^\infty(M)}(\mathcal{F}, \mathbb{K})$ comes equipped with a Lie ∞-algebra structure, whose cohomology permits to solve some elementary problems. But these are cohomologies associated to points or to leaves. Our next question is rather vague:

Question 3.2.1: Relevant cohomologies?

What are the interesting global cohomology theories for singular foliations?

Here are several candidates[5]. We denote by $\mathcal{O}$ the algebra of smooth functions on M. Also, for any $\mathcal{O}$-module $\mathcal{E}$, the notation $\mathcal{E} \wedge_{\mathcal{O}} \mathcal{E}$ stands for the wedge product over $\mathcal{O}$, i.e. we allow

$$X \wedge FY = FX \wedge Y \text{ for all } X, Y \in \mathcal{E},\, F \in \mathcal{O}$$

1. *Longitudinal cohomology of a singular foliation* was introduced in [LGLS20]. Let us describe it:

[5]We describe them in the smooth context: for the real-analytic or holomorphic settings, one has to add a Čech-type differential for a good covering - as always in sheaf theory

Chains in degree k	Differential on chains of degree k
Skew symmetric and $\mathcal{O}$-multilinear maps $\underbrace{\mathcal{F}\wedge_{\mathcal{O}}\cdots\wedge_{\mathcal{O}}\mathcal{F}}_{k-\text{times}} \longrightarrow \mathcal{O}$ i.e. $\mathrm{Hom}_{\mathcal{O}}(\wedge_{\mathcal{O}}\mathcal{F},\mathcal{O})$	$\forall \omega \in \mathrm{Hom}^k_{\mathcal{F}}(\mathcal{F},\mathcal{O})$ and all $X_0,\ldots,X_k \in \mathcal{F}$ $\delta\omega\,(X_0,\ldots,X_k) =$ $\sum_{i=0}^{k}(-1)^i Xi \left[\omega(X_0,\ldots,\widehat{X_i},\ldots,X_k)\right]$ $+\sum_{i<j}(-1)^{i+j+1}\omega([X_i,X_j],X_0,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_k)$
Chains in degree 0	Differential on chains of degree 0
In degree 0, chains are simply elements of $\mathcal{O}$	$\forall F \in \mathcal{O}$, $\delta(F)\colon \begin{array}{ccc} \mathcal{F} & \to & \mathcal{O} \\ X & \mapsto & X[F] \end{array}$

For a regular foliation, this cohomology is simply the De Rham cohomology along the leaves, *i.e.* it is the complex $\Gamma(\wedge^\bullet T^*_x\mathcal{F}), d^{\mathcal{F}}_{dR})$ with $d^{\mathcal{F}}_{dR}$ being the De Rham differential, but computed leaf by leaf.

2. The *basic cohomology* is the sub-complex of $(\Omega(M), d_{DR})$ made, in degree k, of all k-forms that vanish on k-elements in $\mathcal{F}$. Equivalently, these are k-forms ω such that $\mathfrak{i}_L\omega = 0$ for every leaf L.

3. The *universal cohomology of* $\mathcal{F}$ is the cohomology of the commutative differential graded algebra of functions[6] on any universal Q-manifold[7] of $\mathcal{F}$. This is more precisely defined as the cohomology of $(\Gamma(S(\oplus_{i\geq 1}E_i^*))), Q)$. The definition makes sense: it can be proven that since any two universal Lie ∞-algebroid of $\mathcal{F}$, say (E,Q) and (E',Q') are homotopy equivalent, the differential graded commutative algebras $(\Gamma(S(\oplus_{i\geq 1}E_i^*))), Q)$ and $(\Gamma(S(\oplus_{i\geq 1}(E_i')^*))), Q')$ are homotopy equivalent in a unique uo to homotopy manner. In particular, their cohomologies are canonically isomorphic.

   Univeral cohomology should be seen as a refinement of the longitudinal cohomology, since there is a map of differential graded commutative algebras:

$$\text{Longitudinal cohomology of } \mathcal{F} \longrightarrow \text{ Universal cohomology of } \mathcal{F}.$$

   See the section on ”longitutinal cohomology” in [LGLS20].

4. The Chevalley-Eilenberg cohomology for the adjoint representation [**?**, **?**] of any universal Lie ∞-algebroid of $\mathcal{F}$. This is a refinement of the basic cohomology of $\mathcal{F}$.

[6]One can also choose compactly supported functions
[7]I.e, the dual of any universal Lie ∞-algebroid of $\mathcal{F}$

Bibliography

[AS09] Iakovos Androulidakis and Georges Skandalis. The holonomy groupoid of a singular foliation. *J. Reine Angew. Math.*, 626:1–37, 2009.

[AS11] Iakovos Androulidakis and Georges Skandalis. The analytic index of elliptic pseudodifferential operators on a singular foliation. *J. K-Theory*, 8(3):363–385, 2011.

[AS19] Iakovos Androulidakis and Georges Skandalis. A Baum-Connes conjecture for singular foliations. *Ann. K-Theory*, 4(4):561–620, 2019.

[AZ13a] Iakovos Androulidakis and Marco Zambon. Smoothness of holonomy covers for singular foliations and essential isotropy. *Math. Z.*, 275(3-4):921–951, 2013.

[AZ13b] Iakovos Androulidakis and Marco Zambon. Smoothness of holonomy covers for singular foliations and essential isotropy. *Math. Z.*, 275(3-4):921–951, 2013.

[AZ16] Iakovos Androulidakis and Marco Zambon. Stefan-Sussmann singular foliations, singular subalgebroids and their associated sheaves. *Int. J. Geom. Methods Mod. Phys.*, 13:17, 2016. Id/No 1641001.

[CDW87] A. Coste, P. Dazord, and A. Weinstein. Groupoïdes symplectiques. Publ. Dép. Math., Nouv. Sér., Univ. Claude Bernard, Lyon 1987, Fasc. 2A, 1-62 (1987)., 1987.

[Cer79] Dominique Cerveau. Distributions involutives singulières. *Ann. Inst. Fourier (Grenoble)*, 29(3):xii, 261–294, 1979.

[CFM21] Marius Crainic, Rui Loja Fernandes, and Ioan Mărcuţ. *Lectures on Poisson geometry*, volume 217 of *Grad. Stud. Math.* Providence, RI: American Mathematical Society (AMS), 2021.

[CFMT19] Marius Crainic, Rui Loja Fernandes, and David Martínez Torres. Poisson manifolds of compact types (PMCT 1). *J. Reine Angew. Math.*, 756:101–149, 2019.

[Daz85] Pierre Dazord. Feuilletages à singularités. *Indag. Math.*, 47:21–39, 1985.

[Deb01] Claire Debord. Holonomy groupoids of singular foliations. *J. Differential Geom.*, 58(3):467–500, 2001.

[Deb13a] Claire Debord. Longitudinal smoothness of the holonomy groupoid. *C. R., Math., Acad. Sci. Paris*, 351(15-16):613–616, 2013.

[Deb13b] Claire Debord. Longitudinal smoothness of the holonomy groupoid. *C. R. Math. Acad. Sci. Paris*, 351(15-16):613–616, 2013.

[DLPR12] Lance D. Drager, Jeffrey M. Lee, Efton Park, and Ken Richardson. Smooth distributions are finitely generated. *Ann. Global Anal. Geom.*, 41(3):357–369, 2012.

[DS21] Claire Debord and Georges Skandalis. Blow-up constructions for Lie groupoids and a Boutet de Monvel type calculus. *Münster J. Math.*, 14(1):1–40, 2021.

[GG20] Katarzyna Grabowska and Janusz Grabowski. Solvable Lie algebras of vector fields and a Lie's conjecture. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 16:Paper No. 065, 14, 2020.

[GV21] Alfonso Garmendia and Joel Villatoro. Integration of Singular Foliations via Paths. *International Mathematics Research Notices*, 09 2021. rnab177.

[Her62] R. Hermann. The differential geometry of foliations. II. *J. Math. Mech.*, 11:303–315, 1962.

[KS02] C. Klimčík and T. Strobl. WZW-Poisson manifolds. *J. Geom. Phys.*, 43(4):341–344, 2002.

[KS05] Yvette Kosmann-Schwarzbach. Quasi, twisted, and all that... in Poisson geometry and Lie algebroid theory. In *The breadth of symplectic and Poisson geometry. Festschrift in honor of Alan Weinstein*, pages 363–389. Boston, MA: Birkhäuser, 2005.

[Lav18] Sylvain Lavau. A short guide through integration theorems of generalized distributions. *Differ. Geom. Appl.*, 61:42–58, 2018.

[LGL22] Camille Laurent-Gengoux and Ruben Louis. Lie-Rinehart algebras $\simeq$ acyclic lie ∞-algebroids. *J. Algebra*, 594:1–53, 2022.

[LGLS20] Camille Laurent-Gengoux, Sylvain Lavau, and Thomas Strobl. The universal Lie ∞-algebroid of a singular foliation. *Doc. Math.*, 25:1571–1652, 2020.

[LGR19] Camille Laurent-Gengoux and Leonid Ryvkin. The holonomy of a singular leaf, arxiv:1912.05286, 2019.

[LGR21] Camille Laurent-Gengoux and Leonid Ryvkin. The neighborhood of a singular leaf. *J. Éc. Polytech., Math.*, 8:1037–1064, 2021.

[Lic77] Andre Lichnerowicz. Les variétés de Poisson et leurs algèbres de Lie associées. *J. Differ. Geom.*, 12:253–300, 1977.

[Lou22] Ruben Louis. On symmetries of singular foliations, arXiv2203.01585, 2022.

[Mac05] Kirill C. H. Mackenzie. *General theory of Lie groupoids and Lie algebroids*, volume 213 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2005.

[Miy21] David Miyamoto. The basic de rham complex of a singular foliation, arxiv:2102.10091, 2021.

[MM03] Ieke Moerdijk and J. Mrčun. *Introduction to foliations and Lie groupoids*, volume 91 of *Camb. Stud. Adv. Math.* Cambridge: Cambridge University Press, 2003.

[Nag66] T. Nagano. Linear differential systems with singularities and an application to transitive Lie algebras. *J. Math. Soc. Japan*, 18:398–404, 1966.

[Nes20] Jet Nestruev. *Smooth manifolds and observables*, volume 220 of *Grad. Texts Math.* Cham: Springer, 2nd revised and expanded edition edition, 2020.

[PTW21] Hessel Posthuma, Xiang Tang, and Kirsten Wang. Resolutions of proper Riemannian Lie groupoids. *Int. Math. Res. Not.*, 2021(2):1249–1287, 2021.

[Ste74] P. Stefan. Accessibility and foliations with singularities. *Bull. Am. Math. Soc.*, 80:1142–1145, 1974.

[Ste80] P. Stefan. Integrability of systems of vectorfields. *J. Lond. Math. Soc., II. Ser.*, 21:544–556, 1980.

[Sus73a] Hector J. Sussmann. Orbits of families of vector fields and integrability of distributions. *Trans. Am. Math. Soc.*, 180:171–188, 1973.

[Sus73b] Hector J. Sussmann. Orbits of families of vector fields and integrability of systems with singularities. *Bull. Am. Math. Soc.*, 79:197–199, 1973.

[Tou68] Jean-Claude Tougeron. Idéaux de fonctions différentiables. I. *Ann. Inst. Fourier (Grenoble)*, 18(fasc., fasc. 1):177–240, 1968.

[Tur15] Yahya Turki. A Lagrangian for Hamiltonian vector fields on singular Poisson manifolds. *J. Geom. Phys.*, 90:71–87, 2015.

[Wei83] Alan Weinstein. The local structure of Poisson manifolds. *J. Differ. Geom.*, 18:523–557, 1983.