Poisson 2022 Summer School
Wonderful Compactifications
Problem Sheet

Problems for Lecture 1:

Problem 1. Let G be a semisimple simply-connected complex algebraic group, let $T \subset B$ be a pair of maximal torus and Borel, and let W be the Weyl group assocaited to T. Let λ be a dominant weight of T corresponding to the irreducible representation V, and let $v_\lambda \in V$ be a highest weight vector. Show that the following are equivalent:

(a) λ is regular.

(b) The stabilizer of $\mathbb{C}v_\lambda \subset V$ in G is B.

(c) The stabilizer of λ in W is trivial.

Problem 2. Let $\widetilde{G}$ be a semisimple simply-connected complex algebraic group, V a regular irreducible representation of $\widetilde{G}$, and G the quotient of $\widetilde{G}$ by its center. Show that the representation map

$$\widetilde{G} \longrightarrow \text{End } V$$

descends to an embedding

$$G \hookrightarrow \mathbb{P}(\text{End } V).$$

Problem 3. Let $V = \mathbb{C}^3$ be the standard representation of SL_3. Show that the compactification of PGL_3 given by embedding it into the projectivized space of 3×3-matrices

$$\mathbb{P}(\text{End } V) = \mathbb{P}(M_{3\times 3})$$

is *not* the wonderful compactification of PGL_3, by showing that it doesn't satisfy all of the defining properties of wonderful varieties.

Problems for Lecture 2:

Problem 4. Let H be a connected complex algebraic group with Lie algebra $\mathfrak{g}$ and let X be a smooth H-variety. Show that X is homogeneous for the action of H if and only if the vector bundle map

$$X \times \mathfrak{h} \longrightarrow T_X$$

is surjective.

Problem 5. Let $\overline{G}$ be the wonderful compactification of a semisimple group G with trivial center, and write $\mathfrak{g}$ for the Lie algebra of G. Consider the logarithmic infinitesimal action map

$$\rho_D : \overline{G} \times \mathfrak{g} \times \mathfrak{g} \longrightarrow T_{\overline{G},d},$$

where $T_{\overline{G},d}$ is the logarithmic tangent bundle of $\overline{G}$.

(a) Show that the kernel of ρ_D at $1 \in G$ is

$$\{(x, x) \in \mathfrak{g} \times \mathfrak{g} \mid x \in \mathfrak{g}\}.$$

(b) Show that the kernel of ρ_D at any $g \in G$ is

$$\{(\mathrm{Ad}_g\, x, x) \in \mathfrak{g} \times \mathfrak{g} \mid x \in \mathfrak{g}\}.$$

(c) Show that the subalgebras in parts (a) and (b) are Lagrangian with respect to the nondegenerate symmetric form on $\mathfrak{g} \times \mathfrak{g}$ given by

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \kappa(x_1, x_2) - \kappa(y_1, y_2),$$

where κ denotes the Killing form on $\mathfrak{g}$.

Problem 6. Let X be a smooth complex variety with a simple normal crossings divisor $D \subset X$, and write $\mathring{X} = X \backslash D$. Show that the canonical symplectic structure on the cotangent bundle $T^*_{\mathring{X}}$ extends to a log-symplectic structure on the logarithmic cotangent bundle $T^*_{X,D}$.

Problems for Lecture 3:

Problem 7. Let $(\mathfrak{d}, \mathfrak{u}, \mathfrak{u}^*)$ be a Manin triple, let (U, π_U) be a Poisson–Lie group integrating $\mathfrak{u}$, and let (M, π) be a (U, π_U)-homogeneous space. For each $m \in M$, view $\pi(m)$ as an element of

$$\wedge^2(\mathfrak{u}/\mathfrak{u}_m) \cong T_{M,m},$$

and consider the subspace

$$\mathfrak{l}_m = \{(x, \xi) \in \mathfrak{u} \bowtie \mathfrak{u}^* \mid \xi_{|\mathfrak{u}_m} = 0 \text{ and } \pi(m)^{\#}(\xi) = x + \mathfrak{u}_m\}.$$

Show that $\mathfrak{l}_m$ is a Lagrangian subalgebra of $\mathfrak{d}$.

Problem 8. Let $(\mathfrak{d}, \mathfrak{u}, \mathfrak{u}^*)$ be a Manin triple, and define $R \in \wedge^2\mathfrak{d}$ by

$$R((\xi_1, x_1), (\xi_2, x_2)) = \xi_2(x_1) - \xi_1(x_2) \quad \text{for all } x_1, x_2 \in \mathfrak{u} \text{ and } \xi_1, \xi_2 \in \mathfrak{u}^*.$$

Let Π be the corresponding bivector induced on the Grassmannian $\mathrm{Gr} = \mathrm{Gr}(\dim \mathfrak{u}, \mathfrak{d})$ by the natural action of $\mathfrak{d}$. If $\mathfrak{l} \in \mathrm{Gr}$ is a Lagrangian subalgebra, show that the Schouten bracket

$$[\Pi, \Pi]$$

vanishes at $\mathfrak{l}$.