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Singular Foliations : Problems sessions

Solutions

We intend to study vector fields that are tangent to a subvariety $W \subset M$. We shall deal with the smooth, complex, algebraic settings altogether.

Exercice 1

The smooth setting.
Let M be a smooth manifold and $W \subset M$ an embedded closed sub-manifold. We denote by $\mathfrak{X}_W(M)$ the space of all compactly supported vector fields on M that are tangent to the sub-manifold W.

1. Check that $\mathfrak{X}_W(M)$ is a singular foliation on M. Give a set of local generators.
2. Describe the singular distribution $m \mapsto T_m(\mathfrak{X}_W(M))$.
3. Describe
   (a) the leaves of $\mathfrak{X}_W(M)$?
   (b) The set of all regular points.
   (c) The transverse singular foliation to each one of these leaves.
4. Compute
   (a) the isotropy Lie algebras of $\mathcal{F} = \mathfrak{X}_W(M)$ at every point.
   (b) The rank of $\mathfrak{X}_W(M)$ at every point .
5. Is $\mathfrak{X}_W(M)$ the image through the anchor map of a Lie algebroid ?
6. Describe the holonomy groupoid of $\mathfrak{X}_W(M)$.

Exercice 2

The complex setting.

Readjust Exercice 1 to the complex case.

Exercice 3

The algebraic setting.
Now, we will allow $W \subset M$ to be a singular subvariety. This mainly makes sense while working within the context of complex algebraic geometry (but also in the complex setting, as we shall explain).
More precisely, we will make the simplifying assumption that $M = \mathbb{C}^d$. And we define $W \subset \mathbb{C}^d$ to be an an affine variety, i.e. W is the zero locus of some polynomial functions $\varphi_1, \ldots, \varphi_r \in \mathbb{C}[x_1, \ldots, x_d]$. We denote by $\mathcal{I}_W$ the ideal of vanishing functions on W. Without any loss of generality, we can assume

that $\mathcal{I}_W$ is the ideal generated by $\varphi_1, \ldots, \varphi_r$, so we will make this assumption. Let $\mathcal{F} = \mathfrak{X}_W(M)$ be the space of all vector fields X on $M = \mathbb{C}^d$, with polynomial coefficients, that satisfy :

$$X[\mathcal{I}_W] \subset \mathcal{I}_W.$$

1. Explain and justify why it makes sense to define $\mathfrak{X}_W(M)$ as above.

   (For instance, consider $X \in \mathfrak{X}_W(M)$ as a complex vector field and show that its flow - when it exists - preserves W.)

2. Show that $\mathcal{F}$ is a singular foliation over the algebra of polynomials in d variables.

3. What is
   (a) $T_m\mathcal{F}$ for all $m \in M \setminus W$?

   (b) $T_m\mathcal{F}$ for m in the subset $W_{reg} \subset W$ of regular points of W.

   (c) $T_m\mathcal{F}$ for m an isolated singularity of W, i.e. a point m where $d_m\varphi_i = 0$ for $i = 1, \ldots, r$ and which is isolated among such points.

   (d) (Requires the notion of strata of an affine variety). Show that the tangent space of $\mathcal{F}$ at $m \in W$ is included into the tangent space of the strata of W at this point.

   Show that in the coming example, the latter inclusion is strict :

   $$W = \{(x, y, z) \in \mathbb{C}^3 \,|\, xy(x + y)(x + yz) = 0\}$$

   for any point in the straight line $x = y = 0$

   From now on, we assume that $r = 1$ and $\varphi := \varphi_1$ is a homogeneous polynomial[1] and admits an isolated singularity at zero.

   (a) Show that the complex singular foliation generated by $\mathcal{F}$ admits three leaves in this case : $M \setminus W$, $W \setminus \{0\}$ and $\{0\}$.

   We invite the reader to start with the weight homogeneous polynomial $\varphi(x, y, z) = xy - z^n$ with $n \geq 2$ (the weights of x, y, z being $n, n, 2$ respectively) in order to understand the logic of the construction.

   (b) Let $\overrightarrow{E}$ be the *Euler vector field* :

   $$\overrightarrow{E} := \sum_{i=1}^{d} n_i x_i \frac{\partial}{\partial x_i}.$$

   Show that the Euler vector field is in $\mathfrak{X}_W(M)$.

   (c) Show that any vector field of the form $P^{\#}(d\varphi)$ is in $\mathfrak{X}_W(M)$, with P a bivector field on $\mathbb{C}^d$, and $P^{\#}$ the corresponding $1-1$ tensor from T^*M to TM.

1. The variables may have non-negative weights $n_1, \ldots, n_d$

(d) Give a set of generators of $\mathcal{F}$.

(e) Compute the isotropy Lie algebras of $\mathcal{F}$ at the origin

i. when $\varphi(x_1, \ldots, x_d) = \sum_{i=1}^d x_i^2$,

ii. when $\varphi(x_1, \ldots, x_d) = \sum_{i=1}^d x_i^3$

(f) We consider the singular foliation

$$\mathcal{F}_\varphi := \{X \in \mathfrak{X}(V) \mid X[\varphi] = 0\}.$$

i. Give a set of generators of $\mathcal{F}_\varphi$.

ii. Give an almost algebroid structure $(A, [\,\cdot\,,\cdot\,]_A, \rho)$ for $\mathcal{F}_\varphi$.

Look for the notion of a "Koszul resolution", and show that k-vector fields with $k \geq 2$, equipped with the contraction by $d\varphi$ form a geometric resolution of $\mathcal{F}_\varphi$.

iii. The almost Lie algebroid is the beginning of a Lie ∞-algebroid structure on a geometric resolution : compute a 3-ary bracket.

(g) Apply the previous question to $\varphi(x_1, \ldots, x_d) = \sum_{i=1}^d x_i^3$.